# REPRESENTATIONS OF FINITE GROUPS GENERATE TOPOLOGICAL FIELD THEORIES

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Abstract. We prove that any complex (respectively real) representation of finite group naturally generates a Open-Closed (respectively Klein) Topological Field Theory over complex numbers. We relate the 1-point correlator for the projective plane with the Frobenius-Schur indicator.

## Introduction

Topological Field Theories were introduced by Segal [15], Atiyah [5] and Witten [16]. In this paper we concentrate on Open-Closed and Klein Topological Field Theories. These are same generalization of two-dimensional Topological Field Theories inspired by the String Theory, where particles as one-dimensional objects [8]. Open-Closed and Klein Topological Field Theories appear also and in purely geometrical problems, for example, in theory of Hurwitz numbers [7, 1, 2, 3, 4].

In section 1 we reproduce definitions of Closed, Open-Closed, and Klein Topological Field Theories in useful for us form [1]. We recall also theorems [1] that categories of these theories are equivalent to categories of Frobenius pairs, Cardy-Frobenius algebras and Equipped Cardy-Frobenius algebras respectively (similar theorems for more complicated Topological Field Theories are proved in [13, 14]). Therefore constructions of a topological field theories are reduced to constructions of (Equipped) Cardy-Frobenius algebras.

In section 2 we prove that the group algebra and the center of group algebra of any finite group $G$ form a semi-simple Equipped Cardy-Frobenius algebra over any number field. We call it Regular. Later we present full description of Regular complex algebras of a groups.

In section 3 we prove that the center of group algebra and the intertwining algebra of any representation of $G$ generate a Cardy-Frobenius algebra that is Equipped if the representation is real. For representations, that appear from group actions, we relate this construction with proposed in [4].

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## 1. Topological Field Theories and related algebras

1.1. **Closed Topological Field Theories.** The simplest variant of Topological Field Theory is Closed Topological Field Theory [5], [6]. In this case we consider oriented closed surfaces without boundary. Also we fix a finite-dimensional vector space $A$ over a field $\mathbb{K}$ with basis $\alpha_1, \dots, \alpha_N$ and correspond a number $\langle a_1, a_2, ..., a_n\rangle_\Omega$ to each system of vectors $a_1, a_2, ..., a_n \in A$ situated at a set of points $p_1, p_2, ..., p_n$ on a surface $\Omega$ (Figure 1.).

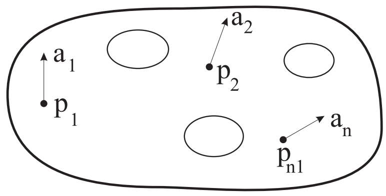


Figure 1

We assume that the numbers $\langle a_1, a_2, ..., a_n\rangle_\Omega$ are invariant with respect to any homeomorphisms of surfaces with marked points. Moreover, we postulate that the system $\{\langle a_1, a_2, ..., a_n\rangle_\Omega\}$ consist of multilinear forms and satisfies a non-degenerate and a cut axioms.

The *non-degenerate axiom* says that the matrix $\left(\langle \alpha_i, \alpha_j\rangle_{S^2}\right)_{1\le i,j\le N}$ is non-degenerate. By $F_A^{\alpha_i,\alpha_j}$ denote the inverse matrix.

The *cut axioms* describes evolution of $\langle a_1, a_2, ..., a_n\rangle_\Omega$ by cutting and collapsing along contours $\gamma \subset \Omega$. Indeed, there are two cut axioms related to different topological types of contours. If $\gamma$ decompose $\Omega$ on $\Omega'$ and $\Omega''$ (Figure 2.)

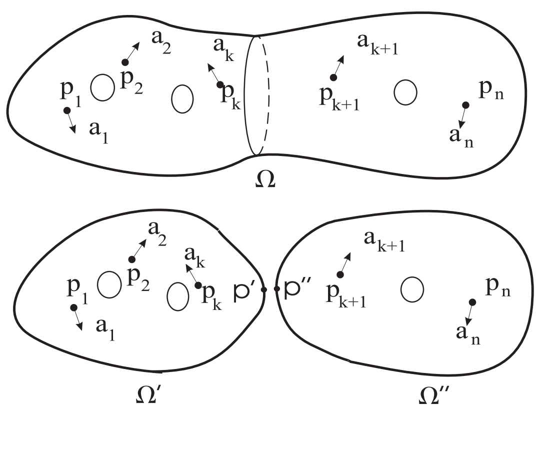


Figure 2

then

$$\langle a_1, a_2, ..., a_n\rangle_\Omega = \sum_{i,j} \langle a_1, a_2, ..., a_k, \alpha_i\rangle_{\Omega'} F_A^{\alpha_i,\alpha_j} \langle \alpha_j, a_{k+1}, a_{k+2}, ..., a_n\rangle_{\Omega''}.$$

If $\gamma$ does not decompose $\Omega$ (Figure 3.)

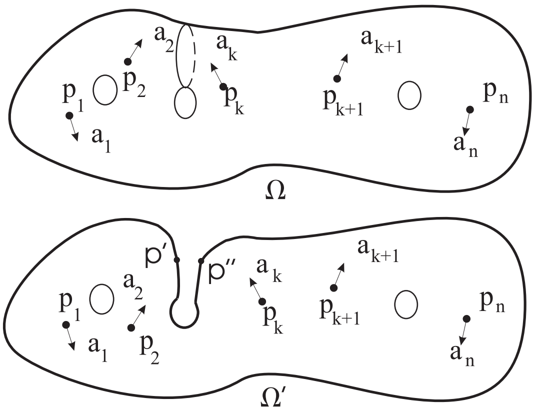


FIGURE 3

then

$$\langle a_1, a_2, ..., a_n \rangle_\Omega = \sum_{i,j} \langle a_1, a_2, ..., a_n, \alpha_i, \alpha_j \rangle_{\Omega'} F_A^{\alpha_i,\alpha_j}.$$

The first consequence of the Topological Field Theory axioms is a *structure of algebra* on $A$. Namely, the multiplication is defined by $\langle a_1 a_2,\, a_3 \rangle_{S^2} = \langle a_1,\, a_2,\, a_3 \rangle_{S^2}$, so the numbers $c_{ij}^k = \sum_s \langle \alpha_i,\, \alpha_j,\, \alpha_s \rangle_{S^2} F_A^{\alpha_s,\alpha_k}$ are structure constants for this algebra. The cut axiom gives (Figure 4.)

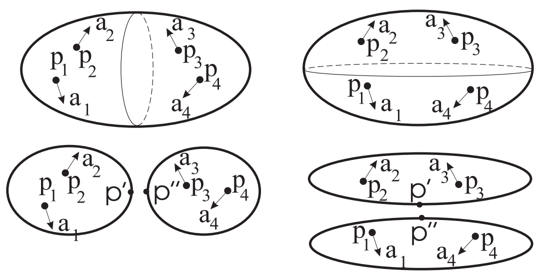


FIGURE 4

$$\sum_{i,j} \langle a_1, a_2, \alpha_i \rangle_{S^2} F_A^{\alpha_i,\alpha_j} \langle \alpha_j, a_3, a_4 \rangle_{S^2} =$$

$$= \langle a_1, a_2, a_3, a_4 \rangle_{S^2} =$$

$$\sum_{i,j} \langle a_2, a_3, \alpha_i \rangle_{S^2} F_A^{\alpha_i,\alpha_j} \langle \alpha_j, a_4, a_1 \rangle_{S^2}.$$

Therefore $\sum_{s,t} c_{ij}^s c_{sk}^t = \sum_{s,t} c_{jk}^s c_{si}^t$ and thus $A$ is an associative algebra. The vector $\sum_i \langle \alpha_i \rangle_{S^2} F_A^{\alpha_i,\alpha_j} \alpha_j$ is the unit of the algebra $A$. The linear form $l(a) =< a >_{S^2}$ is a co-unit, also it defines the non-degenerate invariant bilinear form $(a_1, a_2)_A = l(a_1 a_2) = \langle a_1, a_2 \rangle_{S^2}$ on $A$. The topological invariance makes all marked points $p_i$ equivalent and, therefore, $A$ is a commutative algebra.

Thus, $A$ is a Frobenius algebra [9], that is an algebra with a unit and an invariant non-degenerate scalar multiplication. Moreover, the construction gives a functor $\mathcal{F}$ from the category of Closed Topological Field Theories to the category of **Frobenius pairs** $(A, l_A)$, that is a Frobenius algebra $A$ and a linear form $l_A : A \to \mathbb{K}$ providing a non-degenerated invariant bilinear form.

**Theorem 1.1.** [6] *The functor $\mathcal{F}$ is equivalence between categories Closed Topological Field Theories and commutative Frobenius pairs.*

The Frobenius structure gives an explicit formula for correlators:

$$\langle a_1, a_2, ..., a_n \rangle_\Omega = l_A(a_1 a_2 ... a_n \, (K_A)^g),$$

where $K_A = \sum_{ij} F_A^{\alpha_i,\alpha_j} \alpha_i \alpha_j$ and $g$ is genus of $\Omega$.

1.2. **Open-Closed Topological Field Theories.** More complicated variant of Topological Field Theory is Open-Closed Topological Field Theory [11],[12],[1]. In this case we admit oriented compact surfaces $\Omega$ with boundary $\partial\Omega$ and some marked points on $\partial\Omega$. Let us note the interior marked points and vectors as before by $p_1, p_2, ..., p_n$ and $a_1, a_2, ..., a_n \in A$. But we endow a special numeration $q_i^j$ for the boundary marked points, where $i = 1, \dots, s$ corresponds to a connected component of $\partial\Omega$ (that is a boundary contour of $\Omega$). The numeration $j$ is individual for any boundary contour, it counts the points consequently on the circle, following the direction determined by the orientation of $\Omega$. The vectors $b_i^j$ attached to $q_i^j$ belong to a different vector space $B$ over $\mathbb{K}$ with basis $\beta_1, \dots, \beta_M$ (Figure 5.).

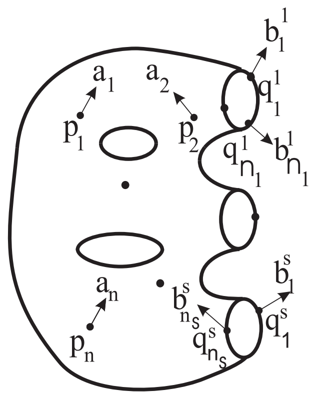


FIGURE 5

To keep in mind this picture, let us denote the corresponding correlation function by $\big\langle a_1, \dots, a_n, (b_1^1, \dots, b_{n_1}^1), \dots, (b_1^s, \dots, b_{n_s}^s) \big\rangle_\Omega$. Note that diffeomorphisms of $\Omega$ can induce any permutation of $a_i$, but only cyclic permutations in each group $b_1^i, \dots b_{n_i}^i$.

We suppose that topological invariance axiom and all axioms of Closed Topological Field Theory are fulfilled for interior marked points and cut-contours. Thus Open-Closed Topological Field Theory also generates a commutative Frobenius pair $(A, l_A)$. Also we impose an additional non-degenerate axiom and cut axioms related to the boundary.

*The additional non-degenerate axiom* says that for any disk $D$ with two marked boundary points the matrix $\left(\langle\beta_i,\beta_j\rangle_D\right)$, where $\beta_1,\beta_2,\ldots$ is a basis of $B$, is non-degenerate. By $F_B^{\beta_i,\beta_j}$ denote the inverse matrix. It play for "segment-cuts" the same "gluing role" that $F_A^{\alpha_i,\alpha_j}$ for "contour-cuts".

In Open-Closed Topological Field Theory we consider cuts by simple segments $[0,1]\to\Omega$ such that the image of 0 and 1 belongs to the boundary. Then there are three topological types of such cuts (Figure 6.).

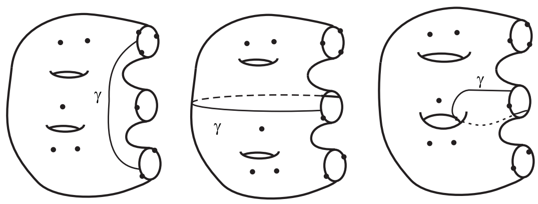


FIGURE 6

Using such cuts one can reduce any market oriented surface to elementary marked surfaces from next list (Figure 7.).

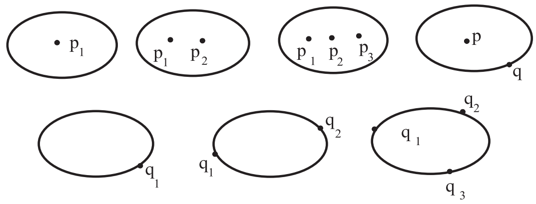


FIGURE 7

Three topological types of segments provide three new cut axioms. For example, the axiom for the cut of type 2 (Figure 8.) is

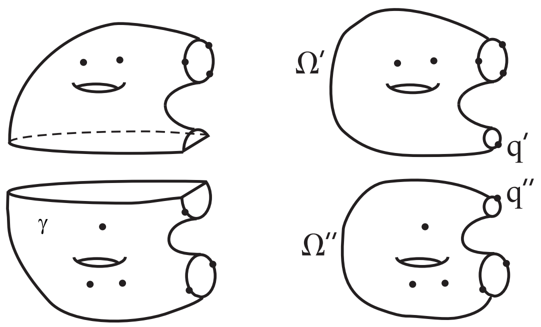


FIGURE 8

$$\left\langle a_1,a_2,\ldots,a_n,(b_1^1,\ldots,b_{n_1}^1),\ldots,(b_1^s,\ldots,b_{n_s}^s)\right\rangle_\Omega =$$

$$\sum_{i,j} \left\langle a_1, a_2, \dots, a_k, (b^1_1, \dots, b^1_{n_1}), \dots, (b^t_1, \dots, b^t_{n'_t}, \beta_i) \right\rangle_{\Omega'} F_B^{\beta_i,\beta_j}$$
$$\left\langle a_{k+1}, \dots, a_n, (\beta_j, b^t_{n'_t+1}, \dots, b^t_{n_t}), \dots, (b^s_1, \dots, b^s_{n_s}) \right\rangle_{\Omega''} .$$

The correlators for the disk $D$ with up to three boundary points $\langle (b_1) \rangle_D$, $\langle (b_1, b_2) \rangle_D$ and $\langle (b_1, b_2, b_3) \rangle_D$ give us a *Frobenius pair* $(B, l_B)$ with structure constants defined in a usual way: $d^k_{ij} = \sum_s \langle (\beta_i, \beta_j, \beta_s) \rangle_D F_B^{\beta_s,\beta_k}$. The associativity of $B$ follows from the picture below (Figure 9.)

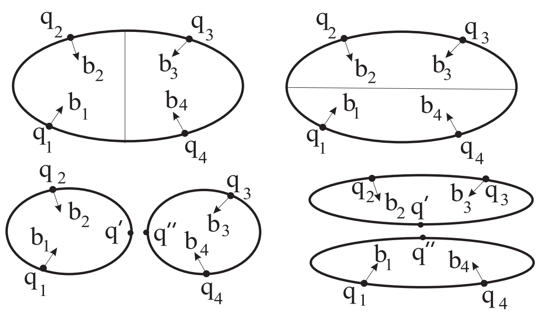


FIGURE 9

Thus $\langle (b_1, b_2, b_3, b_4) \rangle_D$ is equal both to $\sum_i \langle (b_1, b_2, \beta_i) \rangle_D F_B^{\beta_i,\beta_j} \langle (\beta_j, b_3, b_4) \rangle_D$ as well as to $\sum_i \langle (b_2, b_3, \beta_i) \rangle_D F_B^{\beta_i,\beta_j} \langle (\beta_j, b_4, b_1) \rangle_D$. However the algebra $B$ is not commutative in general, because there is no homeomorphisms of disk that interchanging $q_1$ with $q_2$ and preserving $q_3$.

The correlator $\langle a, (b) \rangle_D : A \times B \to \mathbb{C}$ together with non-degenerate bilinear forms $\langle a_1, a_2 \rangle_{S^2} : A \times A \to \mathbb{C}$, $\langle b_1, b_2 ) \rangle_D : B \times B \to \mathbb{C}$ generates two *homomorphisms of vector spaces* $\phi : A \to B$ and $\phi^* : B \to A$.

Let us deduce some consequences from additional topological axioms.

**Proposition 1.1.** *We have*

*1)* $\phi$ *and* $\phi^*$ *are homomorphisms,*

*2)* $\phi(A)$ *belong to center of* $B$*,*

*3)* $(\phi^*(b'), \phi^*(b''))_A = \operatorname{Tr} W_{b'b''}$, *where the operator* $W_{b'b''} : B \to B$ *is* $W_{b'b''}(b) = b'bb''$.

Last condition has name *Cardy condition* because appear 20 years ago in work of J. Cardy about strings.

Thus, we construct a functor $\mathcal{F}$ from the category of Open-Closed Topological Field Theory to a category of **Cardy-Frobenius algebras** $((A, l_A), (B, l_B), \phi)$, that is:

1)commutative Frobenius pair $(A, l_A)$;

2) arbitrary Frobenius pair $(B, l_B)$;

3) a homomorphism $\varphi : A \to B$ such that $\phi(A)$ belong to center of $B$ and $(\phi_*(b'), \phi_*(b''))_A = \operatorname{Tr} W_{b'b''}$.

**Theorem 1.2.** [1] *The functor* $\mathcal{F}$ *is equivalence between categories Open-Closed Topological Field Theories and Cardy-Frobenius algebras.*

The structure of Cardy-Frobenius algebra provides an explicit formula for correlators:

$$\left\langle a_1, a_2, \ldots, a_n, (b_1^1, \ldots, b_{n_1}^1), \ldots, (b_1^s, \ldots, b_{n_s}^s)\right\rangle_\Omega =$$
$$l_B\left(\phi(a_1 a_2 \ldots a_n K_A^g)\; b_1^1 \ldots b_{n_1}^1\; V_{K_B}(b_1^2 \ldots b_{n_2}^2) \ldots V_{K_B}(b_1^s \ldots b_{n_s}^s)\right),$$

where the operator $V_{K_B} : B \to B$ is given by $V_{K_B}(b) = F_B^{\beta_i,\beta_j}\beta_i b \beta_j$, and $g$ is genus of $\Omega$.

1.3. **Klein Topological Field Theories.** The orientability restriction is indeed avoidable, the corresponding settings were introduced in [1] as Klein Topological Field Theory. It is an extension of Open-Closed Topological Field Theory to arbitrary compact surfaces (possible non-orientable and with boundary) equipped by a finite set of marked points with local orientation of their vicinities.

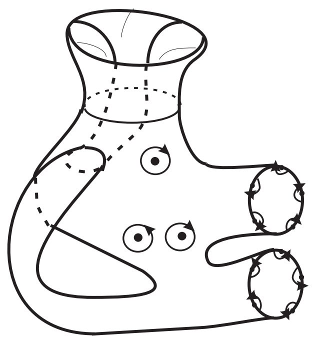

FIGURE 10

In the same way as in Open-Closed Topological Field Theory in order to calculate a correlator we attach vectors from a space $A$ (respectively $B$) to interior (resp. boundary) marked points on the surface.

We assume that topological invariance axiom and all axioms of Open-Closed Topological Field Theory are fulfilled for cuts that belong to any orientable part of the surface. Thus Klein Topological Field Theory also generates a Cardy-Frobenius algebra $((A, l_A), (B, l_B), \phi)$.

Non-orientable surfaces gives 4 new types of cuts (2 types of cuts by segments and 2 types of cuts by contours)(Figure 11.).

Full system of cuts give possible to reduce any marked non-orientable surface to marked surfaces from list on Figure 7 and the projective plane with one marked point $P$. Let $l_U(a) = \langle a\rangle_P : A \to \mathbb{K}$ be the corresponding linear functional, by $U \in A$ denote the dual vector defined by $l_A(Ua) = l_U(a)$.

Four new topological type of cuts give 4 new topological axioms. The axiom for the cut of type 2 is, for example,

$$\left\langle a_1, a_2, \ldots, a_n, (b_1^1, \ldots, b_{n_1}^1), \ldots, (b_1^s, \ldots, b_{n_s}^s)\right\rangle_\Omega =$$
$$= \left\langle a_1, a_2, \ldots, a_n, U, (b_1^1, \ldots, b_{n_1}^1), \ldots, (b_1^s, \ldots, b_{n_s}^s)\right\rangle_{\Omega'}.$$

Now suppose that there are linear involutions $\star : A \to A$ and $\star : B \to B$, such that applying $\star$ inside the correlator gives the same answer as changing the local orientation

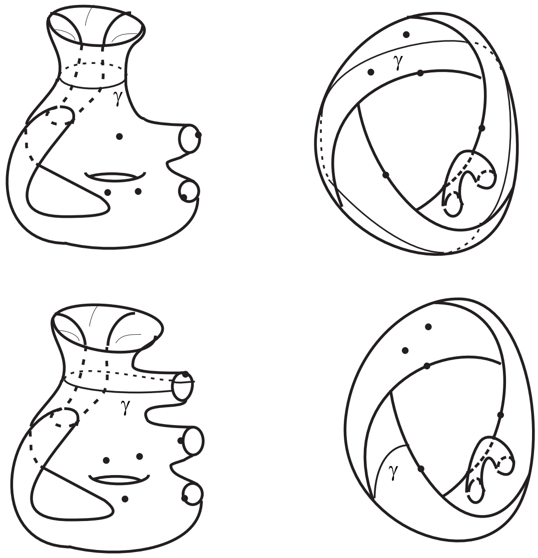


FIGURE 11

around corresponding points. Sometimes we will write $c^\star = \star(c)$. Let us describe the algebraic consequences of these assumptions.

**Proposition 1.2.** *We have*

*1) the involution* $\star : A \to A$ *is automorphism, the involution* $\star : B \to B$ *is anti-automorphism (that is* $(b_1b_2)^\star = b_2^\star b_1^\star$*),*

*2)* $l_A(x^\star) = l_A(x)$, $l_B(x^\star) = l_B(x)$, $\phi(x^\star) = \phi(x)^\star$,

*3)* $U^2 = K_A^\star = F_A^{\alpha_i,\alpha_j}\alpha_i\alpha_j^\star$,

*4)* $\phi(U) = K_B^\star = F_B^{\beta_i,\beta_j}\beta_i\beta_j^\star$.

Thus, we constructed a functor $\mathcal{F}$ from the category of Klein Topological Field Theory to a categories of **Equipped Cardy-Frobenius algebras** $((A, l_A), (B, l_B), \phi, U, \star)$, that is:

1) a Cardy-Frobenius algebra $((A, l_A), (B, l_B), \phi)$;

2) anti-automorphisms $\star : A \to A$ and $\star : B \to B$ such that $l_A(x^\star) = l_A(x)$, $l_B(x^\star) = l_B(x)$, $\phi(x^\star) = \phi(x)^\star$;

3) an element $U \in A$ such that $U^2 = K_A^\star$ and $\phi(U) = K_B^\star$.

**Theorem 1.3.** [1] *The functor* $\mathcal{F}$ *is equivalence between categories of Klein Topological Field Theories and Equipped Cardy-Frobenius algebras.*

The Equipped Cardy-Frobenius algebra provides an explicit formula for correlators on non-orientable surfaces:

$$\left\langle a_1, a_2, \dots, a_n, (b_1^1, \dots, b_{n_1}^1), \dots, (b_1^s, \dots, b_{n_s}^s) \right\rangle_\Omega =$$

$$l_B\left(\phi(a_1a_2\dots a_nU^{2g})\ b_1^1\dots b_{n_1}^1\ V_{K_B}(b_1^2\dots b_{n_2}^2)\dots V_{K_B}(b_1^s\dots b_{n_s}^s)\right),$$

where $g$ is geometrical genus of $\Omega$, that is, $g = a+1$, if $\Omega$ is a Klein bottle with $a$ handles and $g = a + \frac{1}{2}$, if $\Omega$ is a projective plane $a$ handles.

## 2. Regular Cardy-Frobenius algebra of finite group

2.1. **Construction of Regular algebra.** In this section we present a construction that corresponds an Equipped Cardy-Frobenius algebra and, therefore, a Klein Topological Field Theory to any finite group $G$.

By $|M|$ denote cardinality of a finite set $M$. Let $\mathbb{K}$ be any field such that $\operatorname{char}\mathbb{K}$ is not a divisor of $|G|$. By $B = \mathbb{K}[G]$ denote the group algebra. It can be defined as the algebra, formed by linear combinations of elements of $G$ with the natural multiplication as well as the algebra of $\mathbb{K}$-valued functions on $G$ with multiplication defined by convolution. It has a natural structure of a Frobenius pair with $l_B(f) = f(1)$. Note that $l_B(f) = \operatorname{Tr}_{\mathbb{K}[G]} f/|G|$.

The center $A = Z(B)$ with the functional $l_A(f) = f(1)/|G|$ forms a Frobenius pair as well. Take $U = \sum_{g\in G} g^2 \in A$.

Let $\phi$ be the natural inclusion from $A$ to $B$. Let $\star : B \to B$ be the antipode map, sending $g$ to $g^{-1}$. This map preserves the center, so we have a map $\star : A \to A$, compatible with the inclusion $\phi$.

**Theorem 2.1.** *The data above form a semi-simple Equipped Cardy-Frobenius algebra over $K$.*

*Proof.* The arguments here are the same as in [4]. First let us show that $U^2 = K_A^\star$. For a conjugation class $\alpha \subset G$ let $E_\alpha = \sum_{g\in\alpha} g$. Then $E_\alpha$ form a basis of $A$.

Note that $E_\alpha^\star = E_\alpha$ and $(E_\alpha, E_\alpha)_A = |\alpha|$, so we have

$$K_A^\star = \frac{1}{|G|}\sum_\alpha \frac{E_\alpha^2}{|\alpha|} = \frac{1}{|G|}\sum_\alpha \sum_{g,g'\in\alpha} \frac{gg'}{|\alpha|} =$$

$$= \frac{1}{|G|}\sum_\alpha \sum_{g\in\alpha,\, h\in G} \frac{g\, h^{-1}gh}{|\alpha|}\frac{|\alpha|}{|G|} = \sum_{g,h\in G} gh^{-1}gh = \sum_{a,b\in G} a^2b^2 = U^2,$$

where $a = gh^{-1}$, $b = h$.

Also we have $K_B^\star = \sum_{x,y\in G} l_B(xy^{-1})xy = \sum_{g\in G} g^2 = U$. It remains to prove the Cardy condition $l_A(\phi^*(x)\phi^*(y)) = \operatorname{Tr} W_{x,y}$.

We have $\operatorname{Tr} W_{x,y} = |\{g|xgy = g\}| = |\{g|y = g^{-1}x^{-1}g\}|$. This number is zero if $x^{-1}$ and $y$ are in different conjugation classes, and it is $\frac{|G|}{\gamma}$ if $x^{-1}$ belongs to the conjugation class $\gamma$ of $y$. On the other hand, $\phi^*(y) = \sum_{g\in G} g^{-1}yg = \frac{|G|}{|\gamma|}\sum_{h\in\gamma} h = \frac{|G|}{|\gamma|}E_\gamma$. Thus the number $l_A(\phi^*(x)\phi^*(y))$ is exactly the same. □

We denote the constructed algebra by $H_{\mathbb{K}}^G$ and call it *the regular algebra of* $G$. This algebra is semi-simple due to semi-simplicity of the group algebra. Our next aim is full description of $H_{\mathbb{R}}^G$ and $H_{\mathbb{C}}^G$.

2.2. **Classification of complex semi-simple Equipped Cardy-Frobenius algebras.** We call a complex Cardy-Frobenius algebra $((A, l_A), (B, l_B), \phi)$ *pseudoreal* if $A = A_R \otimes \mathbb{C}$, $B = B_R \otimes \mathbb{C}$ and $\phi = \phi_R \otimes \mathbb{C}$ where $A_R$, $B_R$ are real algebras and $\phi_R : A_R \to B_R$ is an homomorphism. It appears that any equipped complex Cardy-Frobenius algebra is pseudoreal.

Let $\mathbb{D}$ be a division algebra over $\mathbb{R}$, that is, $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. For each $\mathbb{D}$ introduce a family of real semi-simple Equipped Cardy-Frobenius algebras.

Namely, let $n$ be an integer, $\mu \in \mathbb{C}$, put $d = \dim_{\mathbb{R}} \mathbb{D}$. Introduce

$$B_R = \mathrm{Mat}_n(\mathbb{D}), \qquad l_{B_R}(x) = \mu\, \Re e \mathrm{Tr}\,(x) \text{ for } x \in \mathrm{Mat}_n(\mathbb{D}),$$

$$A_R = Z(\mathbb{D}), \qquad l_{A_R}(a) = \mu^2 \Re e(a)/d \text{ for } a \in Z(\mathbb{D}), \qquad \phi_R(a) = a\mathrm{Id} \in Z(B_R).$$

For $z \in \mathbb{D}$ by $\overline{z}$ denote the conjugated element. The involution $\star_R$ is defined by $a^{\star_R} = \overline{a}$ for $a \in A_R$, $x^{\star_R} = \overline{x}^t$ for $x \in B_R$, where $^t$ means transposition of a matrix. Now take

$$U_R = \frac{2-d}{\mu} \in A_R. \tag{1}$$

Denote this set $((A_R, l_{A_R}), (B_R, l_{B_R}), \phi_R, U_R, \star_R)$ by $H^{\mathbb{D}}_{n,\mu}$.

**Proposition 2.1.** *The $H^{\mathbb{D}}_{n,\mu}$ is a semi-simple real Equipped Cardy-Frobenius algebra.*

*Proof.* Introduce a natural projection $Z : \mathbb{D} \to Z(\mathbb{D})$ sending $x \in \mathbb{D}$ to $x$ for $\mathbb{D} = \mathbb{R}, \mathbb{C}$ and to $\Re e(x)$ for $\mathbb{D} = \mathbb{H}$. We have $\phi_R^*(x) = Z(\mathrm{Tr}\,(x))d/\mu$. On the other hand, a direct calculation in the standard basis shows that

$$\mathrm{Tr}\, W_{x,y} = \mathrm{Tr}\, W^{\mathbb{D}}_{\mathrm{Tr}\,(x), \mathrm{Tr}\,(y)} = d \Re e \left( Z(\mathrm{Tr}\,(x)) Z(\mathrm{Tr}\,(y)) \right),$$

so $(\phi^*(x), \phi^*(y)) = \mathrm{Tr}\, W_{x,y}$.

Another observation is that $K^{\star}_{B_R} = K_{\mathbb{D}}\mathrm{Id}/\mu$, where $K_{\mathbb{D}}$ is the Casimir element of $\mathbb{D}$ with respect to the form $(a, b) = a\overline{b}$. We have $K_{\mathbb{D}} = (2 - d)$, so $\phi(U_R) = K_{\mathbb{D}}$. At last, $K_{A_R} = d/\mu^2$ for $A_R = \mathbb{R}$ and $K_{A_R} = 0$ for $A_R = \mathbb{C}$, so $U_R^2 = K_{A_R}$. ☐

**Theorem 2.2.** [1] *Any semi-simple Equipped Cardy-Frobenius algebra $((A, l_A), (B, l_B), \phi, U, \star)$ over $\mathbb{C}$ is a direct sum of $H^{\mathbb{D}_i}_{n_i,\mu_i} \otimes \mathbb{C}$ and* $\mathrm{Ker}(\phi)$*.*

To identify $H^{\mathbb{D}_i}_{n_i,\mu_i} \otimes \mathbb{C}$ with the algebras introduced in [1] let us describe $H^{\mathbb{D}_i}_{n_i,\mu_i} \otimes \mathbb{C}$ in detail.

- If $\mathbb{D} = \mathbb{R}$, then $A \cong \mathbb{C}$ equipped with identical involution $\star$ and linear form $l_A(z) = \mu^2 z$, $U = \frac{1}{\mu} \in A$; $B \cong \mathrm{Mat}_n(n, \mathbb{C})$ equipped with involutive anti-automorphism $\star : X \mapsto X^t$, and linear form $l_B(X) = \mu \mathrm{Tr}\, X$. The homomorphism $\phi : A \to B$ sends the unit to the identity matrix;
- If $\mathbb{D} = \mathbb{C}$, then $A \cong \mathbb{C} \oplus \mathbb{C}$ with the involution $(x, y)^\star = (y, x)$ for $(x, y) \in \mathbb{C} \oplus \mathbb{C}$ and the linear form by formula $l_A(x, y) = \mu^2(x + y)/4$, $U = 0$; $B \cong \mathrm{Mat}_n(n, \mathbb{C}) \oplus \mathrm{Mat}_n(n, \mathbb{C})$ with a linear form $l_B(X, Y) = \mu(\mathrm{Tr}\, X + \mathrm{Tr}\, Y)/2$ and involutive anti-automorphism $\star : (X, Y) \mapsto (Y^t, X^t)$. The homomorphism $\phi : A \to B$ is given by the equality $\phi(x, y) = (xE, yE)$.
- If $\mathbb{D} = \mathbb{H}$, then $(A, l_A, \star)$ is the same as for $\mathbb{D} = \mathbb{R}$, but $U = -\frac{2}{\mu} \in A$; $B \cong \mathrm{Mat}_{2m}(\mathbb{C})$ with a linear form $l_B(X) = \mu \mathrm{Tr}\, X/2$. A matrix $X \in B$ we may present in block form as $X = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$. Then the involute anti-automorphism $\star : X \mapsto X^\tau$ is given by the formula $X^\tau = \begin{pmatrix} m_{22}^t & -m_{12}^t \\ -m_{21}^t & m_{11}^t \end{pmatrix}$, in other words, $X^\tau$ is the matrix adjoint to $X$ with respect to a natural symplectic form. The homomorphism $\phi : A \to B$ sends the unit to the identity matrix.

**Remark 2.1.** *There are real equipped Cardy-Frobenius algebras not isomorphic to* $H^{\mathbb{D}}_{n,\mu}$*. A simplest example is a generalization of* $H^{\mathbb{R}}_{n,\mu}$ *with the same Cardy-Frobenius structure, but the involution* $\star$ *defined as the transposition with respect to a bi-linear form with non-trivial signature.*

*Now we think that equipped Cardy-Frobenius algebras over an arbitrary field* $\mathbb{K}$ *with* $\operatorname{char}\mathbb{K} \neq 2$ *can be classified in terms of Brauer group of* $\mathbb{K}$.

2.3. **Description of complex Regular algebra.** Let us denote complex representations of $G$ by capital Latin letters (such as $V$) and real representation by Greek letters (such as $\pi$). Any irreducible real representations $\pi$ is one of next [10]:

- Real type: $\operatorname{End}(\pi) = \mathbb{R}$ and $\pi \otimes_{\mathbb{R}} \mathbb{C}$ is irreducible;
- Complex type: $\operatorname{End}(\pi) = \mathbb{C}$, $\pi \otimes_{\mathbb{R}} \mathbb{C} \cong V^+ \oplus V^-$, $V^+$ is not isomorphic to $V^-$ (but $V^+ \cong (V^-)^*$);
- Quaternionic type: $\operatorname{End}(\pi) = \mathbb{H}$, $\pi \otimes_{\mathbb{R}} \mathbb{C} \cong V^0 \oplus V^0$.

Denote by let $Ir_{\mathbb{D}}(G)$ be the set of isomorphism classes of corresponding irreducible real representations.

**Theorem 2.3.** *We have*

$$H^{\mathbb{R}}_G \cong \bigoplus_{\mathbb{D}=\mathbb{R},\mathbb{C},\mathbb{H}} \bigoplus_{\pi \in Ir_{\mathbb{D}}(G)} H^{\mathbb{D}}_{\frac{\dim \pi}{\dim \mathbb{D}}, \frac{\dim \pi}{|G|}}, \tag{2}$$

$$H^{\mathbb{C}}_G \cong \bigoplus_{\mathbb{D}=\mathbb{R},\mathbb{C},\mathbb{H}} \bigoplus_{\pi \in Ir_{\mathbb{D}}(G)} H^{\mathbb{D}}_{\frac{\dim \pi}{\dim \mathbb{D}}, \frac{\dim \pi}{|G|}} \otimes \mathbb{C} \tag{3}$$

*Proof.* Let us show (2), then (3) follows because $\mathbb{C}[G] \cong \mathbb{R}[G] \otimes \mathbb{C}$. By the Wedderburn theorem

$$\mathbb{R}[G] \cong \bigoplus_{\mathbb{D}} \bigoplus_{\pi \in Ir_{\mathbb{D}}(G)} \operatorname{Mat}_{\frac{\dim \pi}{\dim \mathbb{D}}}(\mathbb{D}), \tag{4}$$

where the map $\mathbb{R}[G] \to \operatorname{Mat}_{\frac{\dim \pi}{\dim \mathbb{D}}}(\mathbb{D})$ is the action of $\mathbb{R}[G]$ on $\pi \cong \mathbb{D}^{\frac{\dim \pi}{\dim \mathbb{D}}}$. Due to the classification theorem it is enough to identify the map $\star$ and the constant $\mu$ on each summand with the same in $H^{\mathbb{D}}_{n,\mu}$.

Concerning $\star$, choose an invariant scalar product on $\pi$. As $\pi$ is irreducible, this invariant bilinear form is unique up to a scalar. Then $\star$ is just the conjugation with respect to this form. By the orthogonalization process we can suppose that $\pi \cong \mathbb{D}e_1 \oplus \cdots \oplus \mathbb{D}e_m$, where $\{e_l\}$ is the set of orthogonal vectors.

Note that $\operatorname{Mat}_{\frac{\dim \pi}{\dim \mathbb{D}}}(\mathbb{D})$ is the tensor product of its subalgebras $\operatorname{Mat}_{\frac{\dim \pi}{\dim \mathbb{D}}}(\mathbb{R})$ and $\mathbb{D}$. In the basis $\{e_l\}$ we identify the action of $\star$ on $\operatorname{Mat}_{\frac{\dim \pi}{\dim \mathbb{D}}}(\mathbb{R})$ with the matrix transposition. For $\mathbb{D}$ note that it also acts by right multiplication, an this action commutes with the action of $\mathbb{R}[G]$, hence this right action preserves the bilinear form up to a scalar. Then it follows that the set $e_l$, $ie_l$ (for $\mathbb{D} \supset \mathbb{C}$), $je_l$ and $ke_l$ (for $\mathbb{D} = \mathbb{H}$) form an orthogonal basis. In this basis imaginary elements of $\mathbb{D}$ act by skew-symmetric matrices, so $\star$ acts on $\mathbb{D}$ as the standard conjugation.

It remains to find $\mu$ for a summand corresponding to each irreducible real representation $\pi$. Let $e_\pi \in A$ be the idempotent corresponding to $\pi$. It acts on $\mathbb{R}[G]$ by projection onto the corresponding summand in (4). Then from the definition of the regular algebra we have $l_B(\phi(e_\pi)) = \mathrm{Tr}_{\mathbb{R}[G]} e_\pi/|G| = \frac{(\dim\pi)^2}{|G|\dim\mathbb{D}}$. But from the definition of $H^{\mathbb{D}}_{n,\mu}$ we have $l_B(\phi(e_\pi)) = \mu n = \mu \dim\pi/\dim\mathbb{D}$. So $\mu = \frac{\dim\pi}{|G|}$. □

**Corollary 2.1.** *(cf. [10]) Let $\pi \in Ir_{\mathbb{D}}(G)$ be an irreducible real representation of $G$. Then* $\mathrm{Tr}\,(U)$ *on $\pi$ is equal to* $(2-\dim\mathbb{D})|G|$

*Proof.* The element $U$ acts on $\pi \in Ir_{\mathbb{D}}(G)$ by the same scalar as on $H^{\mathbb{D}}_{\frac{\dim\pi}{\dim\mathbb{D}},\frac{\dim\pi}{|G|}}$. Substituting the definition (1) for $U$ and multiplying by $\dim\pi$, we obtain the proposed formula. □

Such an element $U$ is known as *Frobenius-Schur indicator* (see [10]). It provides an easy way to determine type of $\pi$.

**Remark 2.2.** *Note that Corollary 2.1 is applicable to a complex representation in the same way. Indeed any irreducible complex representation $V$ can be obtained as a summand in $\pi\otimes\mathbb{C}$ for a real irreducible representation $\pi$. Then the action of $U$ on $V$ also determines type of $\pi$.*

## 3. Cardy-Frobenius algebras of representations

3.1. **Cardy-Frobenius algebra of a complex representation.** Let $V$ be a complex representation (possibly reducible) of a finite group $G$. Put $A = Z(\mathbb{C}[G])$ with $l_A$ as above, and let $B = \mathrm{End}_G(V)$ be the algebra of intertwining operators on $V$ with $l_B(x) = \mathrm{Tr}_V x/|G|$. As the center of $\mathbb{C}[G]$ acts on $V$ by intertwining operators, we have a natural map $\phi : A \to B$.

**Theorem 3.1.** *The data above form a semi-simple complex Cardy-Frobenius algebra.*

*Proof.* The algebra $A$ is generated by orthogonal idempotents $\{e_i\}$, corresponding to irreducible complex representations $V_i$. Note that $e_i$ as a function on $G$ coincides with character of $V_i^*$ multiplied by $\dim V_i/|G|$ (see [10]), so $l_A(e_i) = (\frac{\dim V_i}{|G|})^2$. Therefore we have $(e_i, e_j)_A = \delta_{ij}(\frac{\dim V_i}{|G|})^2$.

If $V = \sum V_i^{\oplus m_i}$ then $B = \oplus_{i=1}^s \mathrm{Mat}_{n_i}(\mathbb{C})$. Note that for $x_i \in \mathrm{Mat}_{n_i}(\mathbb{C}) \subset B$ we have $\mathrm{Tr}_V x_i = (\dim V_i)\,\mathrm{Tr}\, x_i$, thus we obtain $\phi^*(x) = |G|\sum_{i=1}^s e_i \frac{\mathrm{Tr}\, x_i}{\dim V_i}$ , and $(\phi^*(x), \phi^*(y))_A = \sum_i \mathrm{Tr}\, x_i \mathrm{Tr}\, y_i$ for $x = (x_1, \ldots, x_s)$, $y = (y_1, \ldots, y_s) \in B$. On the other hand, for such elements we have $\mathrm{Tr}\, W_{x,y} = \sum_i \mathrm{Tr}\, x_i \mathrm{Tr}\, y_i$. So $(\phi^*(x), \phi^*(y))_A = \mathrm{Tr}\, W_{x,y}$. □

3.2. **Equipped Cardy-Frobenius algebra of a real representation.** Now suppose $V = \rho \otimes_{\mathbb{R}} \mathbb{C}$ is a complexification of a real representation $\rho$.

So there is a non-degenerate symmetric invariant bilinear form on $V$ obtained from the scalar product on $\rho$. Therefore for any operator $x \in \mathrm{End}(V)$ there exists a unique adjoint operator $x^\tau \in \mathrm{End}(V)$. The map sending $x$ to $x^\tau$ is an anti-involution of $\mathrm{End}(V)$, preserving the subalgebra $\mathrm{End}_G(V)$. Thus we obtain a map $\star : \mathrm{End}_G(V) \to \mathrm{End}_G(V)$.

As before, the involution on $A = \mathbb{C}[G]$ is defined by sending $g \to g^{-1}$, and $U = \sum_{g\in G} g^2 \in A$.

**Theorem 3.2.** *The data above form a semi-simple complex Equipped Cardy-Frobenius algebra $H^\rho$. Moreover, we have $\rho \cong \bigoplus_{\pi\in Ir(G)} n_\pi \pi$ and*

$$H^\rho \cong \bigoplus_{\mathbb{D}=\mathbb{R},\mathbb{C},\mathbb{H}} \bigoplus_{\pi\in Ir_{\mathbb{D}}(G)} H^{\mathbb{D}}_{n_\pi, \frac{\dim\pi}{|G|}} \otimes \mathbb{C}.$$

*Proof.* The decomposition of $\rho$ is given by Maschke theorem. The involution $\star$ on $A$ is compatible with the involution $\star$ on $B$ because sending $g \to g^{-1}$ corresponds to the action on the dual representation, and this action can be expressed by adjoint operators with respect to an invariant bilinear form.

We already know that $U^2 = K_A^*$, and it follows from Corollary 2.1 that $\phi(U) = K_B^*$.

The summands in this decompositions can be identified similarly to Theorem 2.3. Here $l_B(\phi(e_\pi)) = \mathrm{Tr}\,_V e_\pi/|G| = \frac{n_\pi \dim\pi}{|G|\dim\mathbb{D}}$. □

3.3. **Group action case.** A particular case of this construction was already discovered in [3]. Suppose that the group $G$ acts on a finite set $X$. Let $\pi_X = \mathbb{R}X$ be the real representation of $G$ in the vector space formed by formal linear combinations of the elements of $X$.

Let $H^{\pi_X} = ((A, l_A), (B, l_B), \phi, U, \star)$. Then an explicit construction of $B$ is proposed in [3].

The group $G$ acts on $X^n = X \times \cdots \times X$ by formula $g(x_1, \dots, x_n) = (g(x_1), \dots, g(x_n))$. Let $\mathcal{B}_n = X^n/G$. By $\mathsf{Aut}\,\bar{x}$ denote the stabilizer of element $\bar{x} \in X_n$. Indeed for $\bar{x} = (x_1, \dots, x_n)$ we have $\mathsf{Aut}\,\bar{x} = \cap_i \mathsf{Aut}\, x_i$. Cardinality of this subgroup $|\mathsf{Aut}\,\bar{x}|$ depends only on the orbit of $\bar{x}$, so we consider it as a function on $\mathcal{B}_n$.

By $B_X$ denote the vector space generated by $\mathcal{B}_2$. The involution $(x_1, x_2) \mapsto (x_2, x_1)$ generates the involution $\star_X : B_X \to B_X$. Introduce a bi-linear and a three-linear form on $B_X$ as follows:

$$(b_1, b_2)_X = \frac{\delta_{b_1, b_2^\star}}{|\mathsf{Aut}\, b_1|} \qquad (b_1, b_2, b_3)_X = \sum_{(x_1,x_2)\in b_1,\ (x_2,x_3)\in b_2,\ (x_3,x_1)\in b_3} \frac{1}{|\mathsf{Aut}(x_1, x_2, x_3)|}.$$

Define a multiplication on $B_X$ by $(b_1 b_2, b_3)_X = (b_1, b_2, b_3)_X$. The element $e = \sum_{x\in X}(x, x)$ is a unit of $B_X$. At last, let $l_{B_X}(b) = (b, e)_X$.

**Theorem 3.3.** *We have an isomorphism $B \cong B_X$ identifying $l_B$ with $l_{B_X}$ and $\star$ with $\star_X$.*

*Proof.* Essentially, it was done in [3]. Elements $(x_1, x_2) \in X \times X$ enumerates matrix units $E_{x_1,x_2} \in \mathrm{End}(\pi_X)$, so to any orbit $b \in \mathcal{B}_2$ we correspond the operator $\sum_{(x_1,x_2)\in b} E_{x_1,x_2} \in \mathrm{End}_G(\pi_X)$. One can check by a direct computation that this map is an algebra homomorphism and that the trace $l_B$ can be written as $l_{B_X}$. The operator $\sum_{g\in G} g$ on $\mathrm{End}(\pi_X)$ is the projection to the subspace of invariants, so we have $B \cong B_X$. At last, the involution $\star_X$ corresponds to transposition of a matrix in the natural orthonormal basis of $\pi_X$, hence it corresponds to $\star$. □

**Remark 3.1.** *This construction defines a structure of real equipped Cardy-Frobenius algebra on the Hecke algebra $H \setminus G/H$ for an arbitrary subgroup $H \subset G$. To this end one can take $X$ to be the left coset $G/H$ with the natural action of $G$.*

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