VIRTUAL CRYSTALS AND KLEBER'S ALGORITHM

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ABSTRACT. Kirillov and Reshetikhin conjectured what is now known as the fermionic formula for the decomposition of tensor products of certain finite dimensional modules over quantum affine algebras. This formula can also be extended to the case of q-deformations of tensor product multiplicities as recently conjectured by Hatayama et al.. In its original formulation it is difficult to compute the fermionic formula efficiently. Kleber found an algorithm for the simply-laced algebras which overcomes this problem. We present a method which reduces all other cases to the simply-laced case using embeddings of affine algebras. This is the fermionic analogue of the virtual crystal construction by the authors, which is the realization of crystal graphs for arbitrary quantum affine algebras in terms of those of simply-laced type.

1. INTRODUCTION

In 1987 Kirillov and Reshetikhin [24] conjectured a formula, now known as the fermionic formula, for the decomposition of tensor products of certain finite dimensional representations over an untwisted quantum affine algebra $U_q(\mathfrak{g})$ into its $U_q(\overline{\mathfrak{g}})$ components, where $\overline{\mathfrak{g}}$ is the simple Lie algebra associated with the affine Kac-Moody algebra $\mathfrak{g}$. The conjecture is motivated by Bethe Ansatz studies. Recently, conjectures for fermionic formulas have been extended to q-deformations of tensor product multiplicities [7, 6]. In type $A_n^{(1)}$ this q-tensor multiplicity formula appeared in [25]. For a single tensor factor, the fermionic formula gives the $\overline{\mathfrak{g}}$-isotypical components of a $U_q(\mathfrak{g})$-module associated with a multiple of a fundamental weight. This conjecture was proven by Chari [3] in a number of cases. Recently, Nakajima [27] showed in the simply-laced case that the characters of such modules satisfy a certain system of algebraic relations (Q-system). Combining the result of [7], his result completes the proof of a "weak" version of the $q = 1$ fermionic formula in this case.

The term fermionic formula was coined by the Stony Brook group [18, 19], who interpreted fermionic-type formulas for characters and branching functions of conformal field theory models as partition functions of quasiparticle systems with "fractional" statistics obeying Pauli's exclusion principle.

Fermionic formulas are q-polynomials or q-series expressed as certain sums of products of q-binomial coefficients

$$\sum_{\{m_i^{(a)}\}} q^{\mathrm{cc}(\{m_i^{(a)}\})} \prod_{i,a} \begin{bmatrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{bmatrix},$$

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where $\begin{bmatrix} m+p \\ m \end{bmatrix} = (q)_{m+p}/(q)_m(q)_p$ is the q-binomial coefficient with $(q)_m = \prod_{i=1}^m (1-q^i)$, $\mathrm{cc}(\{m_i^{(a)}\})$ is some function of the summation variables $m_i^{(a)}$ and $p_i^{(a)}$ is the vacancy number (see (5.2)). The summation variables are subject to constraints (4.1). Those sets $\{m_i^{(a)}\}$ satisfying (4.1) are called admissible configurations. From (4.1) alone, it is computationally difficult to find the admissible configurations, making the evaluation of the fermionic formula intractable. For simply-laced algebras $\mathfrak{g}$, Kleber [16, 17] has given an efficient algorithm to determine the admissible configurations $\{m_i^{(a)}\}$. This algorithm generates a rooted tree with nodes labelled by dominant integral weights such that the tree nodes are in bijection with the admissible configurations. For non-simply laced algebras, the algorithm fails: some admissible nodes cannot be reached.

One of our goals in this paper is to modify Kleber's algorithm to work in all types. This is accomplished by using the well-known natural embeddings of any affine algebra into another of simply-laced type [9]:

$$\begin{array}{lcl} C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} & \hookrightarrow & A_{2n-1}^{(1)} \\ A_{2n-1}^{(2)}, B_n^{(1)} & \hookrightarrow & D_{n+1}^{(1)} \\ E_6^{(2)}, F_4^{(1)} & \hookrightarrow & E_6^{(1)} \\ D_4^{(3)}, G_2^{(1)} & \hookrightarrow & D_4^{(1)}. \end{array} \tag{1.1}$$

It is not hard to express the fermionic formula of the smaller algebra in terms of the larger; we call this the virtual fermionic formula. Our algorithm is an adaptation of Kleber's algorithm in the simply-laced affine algebra, which trims the tree so as not to generate nodes that cannot contribute to the virtual fermionic formula. This algorithm succeeds by using some nodes in the larger weight lattice that do not correspond to weights in the embedded weight lattice.

Fermionic formulas denoted M have crystal counterparts. Crystal bases were introduced by Kashiwara [12] and are bases of $U_q(\mathfrak{g})$-modules in the limit $q \to 0$. Let us denote the one-dimensional configuration sums, which are generating functions of highest weight elements in tensor products of finite dimensional crystals with energy statistics, by X. It was conjectured in [7, 6] that $X = M$.

In light of the embeddings of affine algebras (1.1), one might hope that such embeddings also exist for the quantized algebras. Unfortunately they do not. However we assert that such embeddings exist for all finite-dimensional affine crystals, and give a construction for them in terms of crystals of simply-laced type. A virtual crystal is such a realization of a crystal inside another of possibly different type. Perhaps the first instance of a virtual crystal is Kashiwara's embedding of a crystal of highest weight λ, into that of highest weight $k\lambda$ where k is a positive integer [14]. Extending Baker's work [2], in [29] we conjectured that finite dimensional crystals of type $C_n^{(1)}$, $A_{2n}^{(2)}$, and $D_{n+1}^{(2)}$ can be realized in terms of crystals of type $A_{2n-1}^{(1)}$. We proved this for crystals associated with single columns (i.e. fundamental weights).

In this paper we establish the correctness of the virtual crystal approach for crystals associated with single rows (that is, multiples of the first fundamental weight) for the two infinite families of embeddings.

The paper is organized as follows. In Section 2 we review the essentials of crystal theory. Virtual crystals are introduced in Section 3 and the characterization and validity of virtual crystals associated with single rows is proven. Sections 4 and

5 review the fermionic formulas conjectured in [7, 6] and the Kleber algorithm, respectively, and describe their virtual counterparts.

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2. Crystals

2.1. **Affine algebras.** We adopt the notation of [6]. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of affine type $X_N^{(r)}$, that is, one of the types $A_n^{(1)}(n \geq 1)$, $B_n^{(1)}(n \geq 3)$, $C_n^{(1)}(n \geq 2)$, $D_n^{(1)}(n \geq 4)$, $E_n^{(1)}(n = 6,7,8)$, $F_4^{(1)}$, $G_2^{(1)}$, $A_{2n}^{(2)}(n \geq 1)$, $A_{2n}^{(2)\dagger}(n \geq 1)$, $A_{2n-1}^{(2)}(n \geq 2)$, $D_{n+1}^{(2)}(n \geq 2)$, $E_6^{(2)}$ or $D_4^{(3)}$. The Dynkin diagram of $\mathfrak{g} = X_N^{(r)}$ is depicted in Figure 1 (Table Aff 1-3 in [10]). Its nodes are labelled by the set $I = \{0, 1, 2 \dots, n\}$. Let $\overline{I} = I \backslash \{0\}$.

Every affine algebra $\mathfrak{g}$ has a simple Lie subalgebra $\overline{\mathfrak{g}}$ obtained by removing the 0-node from the Dynkin diagram. This is summarized in the following table:

$$\begin{array}{c|ccccccc} \mathfrak{g} & X_n^{(1)} & A_{2n}^{(2)} & A_{2n}^{(2)\dagger} & A_{2n-1}^{(2)} & D_{n+1}^{(2)} & E_6^{(2)} & D_4^{(3)} \\ \hline \overline{\mathfrak{g}} & X_n & C_n & B_n & C_n & B_n & F_4 & G_2 \end{array} \tag{2.1}$$

Let α_i, h_i, Λ_i $(i \in I)$ be the simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$. Let δ and c denote the generator of imaginary roots and the canonical central element, respectively. Recall that $\delta = \sum_{i \in I} a_i \alpha_i$ and $c = \sum_{i \in I} a_i^\vee h_i$, where the Kac labels a_i are the unique set of relatively prime positive integers giving the linear dependency of the columns of the Cartan matrix A (that is, $A(a_0, \dots, a_n)^t = 0$).

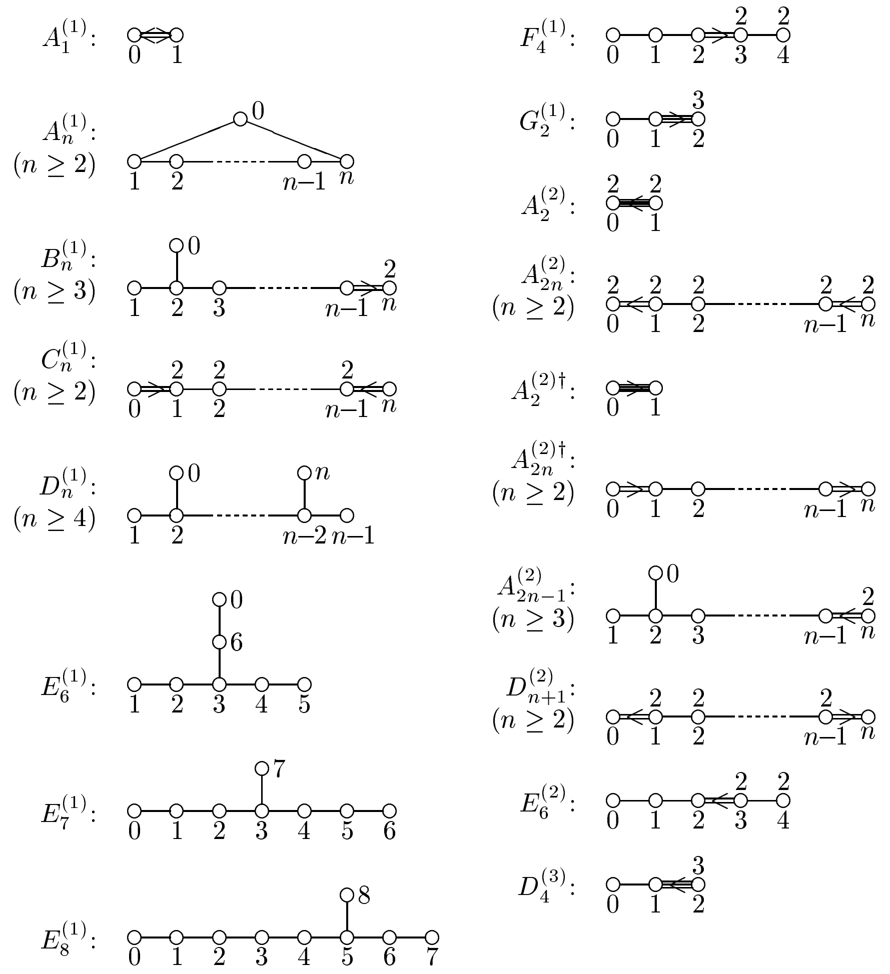

FIGURE 1. Dynkin diagrams for $X_N^{(r)}$. The enumeration of the nodes with $I = \{0, 1, \ldots, n\}$ is specified under or the right side of the nodes. In addition, the numbers t_i (resp. $t_i^\vee$) defined in (2.3) are attached *above* the nodes for $r = 1$ (resp. $r > 1$) if and only if $t_i \neq 1$ (resp. $t_i^\vee \neq 1$).

Explicitly,

$$
(2.2)\quad \delta = \begin{cases}
\alpha_0 + \cdots + \alpha_n & \text{if } \mathfrak{g} = A_n^{(1)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n & \text{if } \mathfrak{g} = B_n^{(1)} \\
\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = C_n^{(1)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_n^{(1)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 & \text{if } \mathfrak{g} = E_6^{(1)} \\
\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 & \text{if } \mathfrak{g} = E_7^{(1)} \\
\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 & \text{if } \mathfrak{g} = E_8^{(1)} \\
\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 & \text{if } \mathfrak{g} = F_4^{(1)} \\
\alpha_0 + 2\alpha_1 + 3\alpha_2 & \text{if } \mathfrak{g} = G_2^{(1)} \\
2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n}^{(2)} \\
\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + 2\alpha_n & \text{if } \mathfrak{g} = A_{2n}^{(2)\dagger} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = A_{2n-1}^{(2)} \\
\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } \mathfrak{g} = D_{n+1}^{(2)} \\
\alpha_0 + 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 & \text{if } \mathfrak{g} = E_6^{(2)} \\
\alpha_0 + 2\alpha_1 + \alpha_2 & \text{if } \mathfrak{g} = D_4^{(3)}.
\end{cases}
$$

The dual Kac label $a_i^\vee$ is the label a_i for the affine Dynkin diagram obtained by "reversing the arrows" of the Dynkin diagram of $\mathfrak{g}$, or equivalently, the coefficients giving the linear dependency of the rows of the Cartan matrix A.

Let $P = \bigoplus_{a\in I}\mathbb{Z}\Lambda_a \oplus \mathbb{Z}\delta$ be the weight lattice of $\mathfrak{g}$ and $P^+ = \bigoplus_{a\in I}\mathbb{Z}_{\geq 0}\Lambda_a$. Similarly, let $\overline{P} = \bigoplus_{a\in\overline{I}}\mathbb{Z}\overline{\Lambda}_a$ be the weight lattice of $\overline{\mathfrak{g}}$, $\overline{P}^+ = \bigoplus_{a\in\overline{I}}\mathbb{Z}_{\geq 0}\overline{\Lambda}_a$, $\overline{Q} = \bigoplus_{a\in\overline{I}}\mathbb{Z}\alpha_a$ the root lattice of $\overline{\mathfrak{g}}$ and $\overline{Q}^+ = \bigoplus_{a\in\overline{I}}\mathbb{Z}_{\geq 0}\alpha_a$ with simple roots and fundamental weights $\alpha_a, \overline{\Lambda}_a$ for $a\in\overline{I}$. For $\lambda,\mu\in\overline{P}$ write $\lambda \trianglerighteq \mu$ if $\lambda-\mu\in\overline{Q}^+$.

For $i\in I$ let

$$t_i = \max(\frac{a_i}{a_i^\vee}, a_0^\vee), \qquad t_i^\vee = \max(\frac{a_i^\vee}{a_i}, a_0). \tag{2.3}$$

The values t_i are given in Figure 1. We shall only use $t_i^\vee$ and t_i for $i\in\overline{I}$. For $a\in\overline{I}$ we have

$$t_a^\vee = 1 \ \text{ if } r=1, \qquad t_a = a_0^\vee \ \text{ if } r>1.$$

Let $(\cdot|\cdot)$ be the normalized invariant form on P [10]. It satisfies

$$(\alpha_i|\alpha_j) = \frac{a_i^\vee}{a_i}A_{ij} \tag{2.4}$$

for $i,j\in I$. In particular

$$(\alpha_a|\alpha_a) = \frac{2r}{a_0^\vee} \tag{2.5}$$

if α_a is a long root.

2.2. **Crystals.** The quantized universal enveloping algebra $U_q(\mathfrak{g})$ associated with a symmetrizable Kac–Moody Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [4] and Jimbo [8] in their study of two dimensional solvable lattice models in statistical mechanics. The parameter q corresponds to the temperature of the underlying model. Kashiwara [11] showed that at zero temperature or $q=0$ the representations of $U_q(\mathfrak{g})$ have bases, which he coined crystal bases, with a beautiful combinatorial structure and favorable properties such as uniqueness and stability under tensor products.

Let $\mathfrak{g}'$ be the derived subalgebra of $\mathfrak{g}$. Denote the corresponding quantized universal enveloping algebras of $\mathfrak{g}\supset\mathfrak{g}'\supset\overline{\mathfrak{g}}$ by $U_q(\mathfrak{g})\supset U_q'(\mathfrak{g})\supset U_q(\overline{\mathfrak{g}})$.

In [7, 6] it is conjectured that there is a family of finite-dimensional irreducible $U_q'(\mathfrak{g})$-modules $\{W_i^{(a)} \mid a\in\overline{I}, i\in\mathbb{Z}_{>0}\}$ which, unlike most finite-dimensional $U_q'(\mathfrak{g})$-modules, have crystal bases $B^{a,i}$. This family is conjecturally characterized in several different ways:

(1) Its characters form the unique solutions of a system of quadratic relations (the Q-system) [24].
(2) Every crystal graph of an irreducible integrable finite-dimensional $U_q'(\mathfrak{g})$-module, is a tensor product of the $B^{a,i}$.
(3) For $\lambda\in P$ let $V(\lambda)$ be the universal extremal weight module defined in [15, Section 3] and $B(\lambda)$ its crystal base, with unique vector $u_\lambda\in B(\lambda)$ of weight λ. Then the affinization of $B^{a,i}$ (in the sense of [22]) is isomorphic to the connected component of u_λ in $B(\lambda)$, for the weight $\lambda = i\overline{\Lambda}_a$.

In light of point (2) above, we consider the category of crystal graphs given by tensor products of the crystals $B^{a,i}$.

We introduce notation for tensor products of $B^{a,i}$. Let

$$B = \bigotimes_{(a,i)\in \overline{I}\times\mathbb{Z}_{>0}} (B^{a,i})^{\otimes L_i^{(a)}}, \tag{2.6}$$

where only finitely many $L_i^{(a)}$ are nonzero. In type $A_n^{(1)}$ this is the tensor product of modules, which, when restricted to A_n, are irreducible modules indexed by rectangular partitions. The set of classically restricted paths (or classical highest weight vectors) in B of weight $\lambda \in \overline{P}^+$ is by definition

$$\mathcal{P}(B,\lambda) = \{b \in B \mid \mathrm{wt}(b) = \lambda \text{ and } e_i b \text{ undefined for all } i \in \overline{I}\}.$$

Here e_i is given by the crystal graph. For $b, b' \in B^{a,i}$ we have $b' = e_i(b)$ if there is an arrow $b' \xrightarrow{i} b$ in the crystal graph; if no such arrow exists then $e_i(b)$ is undefined. Similarly, $b' = f_i(b)$ if there is an arrow $b \xrightarrow{i} b'$ in the crystal graph; if no such arrow exists then $f_i(b)$ is undefined. If B_1 and B_2 are crystals, then for $b_1 \otimes b_2 \in B_1 \otimes B_2$ the action of e_i is defined as

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & \text{if } \varepsilon_i(b_1) > \varphi_i(b_2), \\ b_1 \otimes e_i b_2 & \text{else,} \end{cases}$$

where $\varepsilon_i(b) = \max\{k \mid e_i^k b \text{ is defined}\}$ and $\varphi_i(b) = \max\{k \mid f_i^k b \text{ is defined}\}$. This is the opposite of the notation used by Kashiwara [11].

2.3. **Simple crystals.** Let W be the Weyl group of $\mathfrak{g}$, $\{s_i \mid i \in I\}$ the simple reflections in W. Let B be the crystal graph of an integrable $U_q(\mathfrak{g})$-module. Say that $b \in B$ is an extremal vector of weight $\lambda \in P$ provided that $\mathrm{wt}(b) = \lambda$ and there exists a family of elements $\{b_w \mid w \in W\} \subset B$ such that

(1) $b_w = b$ for $w = e$.
(2) If $\langle h_i\,, w\lambda\rangle \geq 0$ then $e_i(b_w) = \varnothing$ and $f_i^{\langle h_i\,, w\lambda\rangle}(b_w) = b_{s_i w}$.
(3) If $\langle h_i\,, w\lambda\rangle \leq 0$ then $f_i(b_w) = \varnothing$ and $e_i^{\langle h_i\,, w\lambda\rangle}(b_w) = b_{s_i w}$.

Following [1], say that a $U_q'(\mathfrak{g})$-crystal B is *simple* if

(1) B is the crystal base of a finite dimensional integrable $U_q'(\mathfrak{g})$-module.
(2) There is a weight $\lambda \in \overline{P}^+$ such that B has a unique vector (denoted $u(B)$) of weight λ, and the weight of any extremal vector of B is contained in $\overline{W}\lambda$ where $\overline{W}$ is the Weyl group of $\overline{\mathfrak{g}}$.

In the definition of simple crystal in [1], condition 1 is not present. However we always want to assume both conditions, so it is convenient to include condition 1 in the definition above.

Theorem 2.1 ([1])**.**

(1) *Simple crystals are connected.*
(2) *The tensor product of simple crystals is simple.*

For the $U_q'(\mathfrak{g})$-crystal B, define $\epsilon, \varphi : B \to P$ by

$$\epsilon(b) = \sum_{i\in I} \epsilon_i(b)\Lambda_i \qquad \text{and} \qquad \varphi(b) = \sum_{i\in I} \varphi_i(b)\Lambda_i.$$

Then the level of B is

$$\mathrm{lev}(B) = \min\{\langle c\,, \epsilon(b)\rangle \mid b \in B\}. \tag{2.7}$$

2.4. **Dual crystals.** The notion of a dual crystal is given in [13, Section 7.4]. Let B be a $U_q(\mathfrak{g})$-crystal. Then there is a $U_q(\mathfrak{g})$-crystal denoted $B^\vee$ obtained from B by reversing arrows. That is, $B^\vee = \{b^\vee \mid b \in B\}$ with

$$\begin{aligned} \mathrm{wt}(b^\vee) &= -\mathrm{wt}(b) \\ \epsilon_i(b^\vee) &= \varphi_i(b) \\ \varphi_i(b^\vee) &= \epsilon_i(b) \\ e_i(b^\vee) &= (f_i(b))^\vee \\ f_i(b^\vee) &= (e_i(b))^\vee. \end{aligned} \tag{2.8}$$

Proposition 2.2. [13] *There is an isomorphism* $(B_2 \otimes B_1)^\vee \cong B_1^\vee \otimes B_2^\vee$ *given by* $(b_2 \otimes b_1)^\vee \mapsto b_1^\vee \otimes b_2^\vee$.

2.5. **One dimensional sums.** In this section we recall the structure of a $U_q'(\mathfrak{g})$-crystal as a graded $U_q(\overline{\mathfrak{g}})$-crystal. The grading is given by the intrinsic energy function $D : B \to \mathbb{Z}$. For $b \in B$, one may define $D(b)$ as the minimum number of times e_0 occurs in a sequence of operators involving e_i, f_i for $i \in \overline{I}$ and e_0, leading from $u(B)$ to b. However we prefer to work with the following concrete definition when B is a tensor product of crystals of the form $B^{r,s}$. This definition essentially comes from [6], but it is useful to formulate it as follows [29].

Let B_1, B_2 be simple $U_q'(\mathfrak{g})$-crystals. It was shown in [22, Section 4] that there is a unique isomorphism of $U_q'(\mathfrak{g})$-crystals $R = R_{B_2,B_1} : B_2 \otimes B_1 \to B_1 \otimes B_2$, called the combinatorial R matrix. In addition there exists a function $H : B_1 \otimes B_2 \to \mathbb{Z}$ called the local energy function, that is unique up to a global additive constant, which is constant on $\overline{I}$ components and satisfies for all $b_2 \in B_2$ and $b_1 \in B_1$ with $R(b_2 \otimes b_1) = b_1' \otimes b_2'$

$$H(e_0(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } \epsilon_0(b_2) > \varphi_0(b_1) \text{ and } \epsilon_0(b_1') > \varphi_0(b_2') \\ 1 & \text{if } \epsilon_0(b_2) \le \varphi_0(b_1) \text{ and } \epsilon_0(b_1') \le \varphi_0(b_2') \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

We shall normalize the local energy function by the condition $H(u(B_2) \otimes u(B_1)) = 0$.

It was conjectured in [6] that

$$\varphi(b^\natural) = \mathrm{lev}(B^{r,s})\Lambda_0 \qquad \text{for a unique } b^\natural \in B^{r,s}. \tag{2.10}$$

For a given crystal $B^{r,s}$, denote this element also by $u^\natural(B^{r,s})$. Define the function $D_{B^{r,s}} : B^{r,s} \to \mathbb{Z}$ by

$$D_{B^{r,s}}(b) = H(b \otimes b^\natural) - H(u(B^{r,s}) \otimes b^\natural) \tag{2.11}$$

where $H = H_{B^{r,s},B^{r,s}}$ is the local energy function. In all cases in which the $U_q'(\mathfrak{g})$-module $W_s^{(r)}$ and its crystal base $B^{r,s}$ have been constructed, (2.10) holds and (2.11) agrees with the explicit grading on $B^{r,s}$ specified in a case-by-case manner in the appendices of [7, 6].

A graded simple crystal (B, D) is a simple crystal B together with a function $D : B \to \mathbb{Z}$. Let (B_j, D_j) be a graded simple $U_q'(\mathfrak{g})$-crystal and $u_j = u(B_j)$, for $1 \le j \le L$. Let $B = B_L \otimes \cdots \otimes B_1$. Following [28] define the energy function $E_B : B \to \mathbb{Z}$ by

$$E_B = \sum_{1 \le i < j \le L} H_i R_{i+1} R_{i+2} \cdots R_{j-1}, \tag{2.12}$$

where R_i is the combinatorial R-matrix and H_i is the local energy function, where the subscript i indicates that the operators act on the i-th and $(i+1)$-st tensor factors from the right. This given, define $D_B, D'_B : B \to \mathbb{Z}$ by

$$
\begin{aligned}
D'_B &= E_B + \sum_{j=1}^{L} D_j R_1 R_2 \cdots R_{j-1} \\
D_B(b) &= D'_B(b) - D'_B(u(B))
\end{aligned} \tag{2.13}
$$

where $D_j : B_j \to \mathbb{Z}$ acts on the rightmost tensor factor. Then we say that the graded simple crystal (B, D_B) is the tensor product of the graded simple crystals (B_j, D_j).

Theorem 2.3. [29] *Graded simple $U'_q(\mathfrak{g})$-crystals form a tensor category.*

Now suppose that for all j, B_j has the form $B^{r,s}$ and $D_j = D_{B^{r,s}}$ is the intrinsic energy as defined above. Then the function D_B is called the intrinsic energy of B. Let $b_j^\natural \in B_j$ be as in (2.10). Then conjecturally there is an element $b^\natural \in B$ such that $b_j^\natural$ is the leftmost tensor factor in $R_{L-1} \cdots R_{j+1} R_j b^\natural$.

Using the Yang-Baxter equation for R and the fact that $R_{B\otimes B}$ is the identity for any B, it follows that [6]

$$
D_B(b) = H(b \otimes b^\natural) - H(u(B) \otimes b^\natural). \tag{2.14}
$$

The one-dimensional sum $X(B, \lambda; q) \in \mathbb{Z}[q, q^{-1}]$ is the generating function of paths graded by the intrinsic energy

$$
X(B, \lambda; q) = \sum_{b \in \mathcal{P}(B,\lambda)} q^{D_B(b)}. \tag{2.15}
$$

2.6. **Crystals of type B_n, C_n, D_n.** In this and the next section we will describe the classical highest weight crystals $B(s\overline{\Lambda}_1)$ and the finite dimensional affine crystals $B^{1,s}$ for all nonexceptional types as weakly increasing words b in an alphabet $\mathcal{X}$. They are also determined by $x(b) = (x_i)_{i\in\mathcal{X}}$ where x_i is the number of i's in b. Whenever an operation yields a negative value for an x_i it will be undefined.

According to [23], the crystal of $B(\overline{\Lambda}_1)$ has underlying set

$$
\begin{aligned}
&\mathcal{X} = \{1 < 2 < \cdots < n < \circ < \bar{n} < \cdots < \bar{2} < \bar{1}\} && \text{for } B_n \\
&\mathcal{X} = \{1 < 2 < \cdots < n < \bar{n} < \cdots < \bar{2} < \bar{1}\} && \text{for } C_n \\
&\mathcal{X} = \{1 < 2 < \cdots < \begin{smallmatrix} n \\ \bar{n} \end{smallmatrix} < \cdots < \bar{2} < \bar{1}\} && \text{for } D_n.
\end{aligned}
$$

The crystal $B(s\overline{\Lambda}_1)$ is the set of weakly increasing words of length s in the alphabet $\mathcal{X}$ such that, in addition, for type B_n there is at most one $\circ$, and in type D_n, there are either no letters n or no letters $\bar{n}$.

The crystal operators e_i on $B(s\overline{\Lambda}_1)$ are given by

$$
e_i b = \begin{cases} (x_1, \dots, x_i + 1, x_{i+1} - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} > \bar{x}_{i+1} \\ (x_1, \dots, \bar{x}_{i+1} + 1, \bar{x}_i - 1, \dots, \bar{x}_1) & \text{if } x_{i+1} \le \bar{x}_{i+1} \end{cases} \tag{2.16}
$$

with the following exceptions:

$$
\begin{aligned}
&\text{Type } B_n: && e_n b = \begin{cases} (x_1,\dots,x_n,x_\circ+1,\bar{x}_n-1,\dots,\bar{x}_1) & \text{if } x_\circ = 0 \\ (x_1,\dots,x_n+1,x_\circ-1,\bar{x}_n,\dots,\bar{x}_1) & \text{if } x_\circ = 1. \end{cases} \\
&\text{Type } C_n: && e_n b = (x_1,\dots,x_n+1,\bar{x}_n-1,\dots,\bar{x}_1) \\
&\text{Type } D_n: && e_{n-1} b = \begin{cases} (x_1,\dots,x_{n-1}+1,x_n-1,\bar{x}_n,\dots,\bar{x}_1) & \text{if } x_n > 0 \\ (x_1,\dots,x_n,\bar{x}_n+1,\bar{x}_{n-1}-1,\dots,\bar{x}_1) & \text{if } x_n = 0 \end{cases} \\
& && e_n b = \begin{cases} (x_1,\dots,x_{n-1}+1,x_n,\bar{x}_n-1,\dots,\bar{x}_1) & \text{if } \bar{x}_n > 0 \\ (x_1,\dots,x_n+1,\bar{x}_n,\bar{x}_{n-1}-1,\dots,\bar{x}_1) & \text{if } \bar{x}_n = 0. \end{cases}
\end{aligned} \tag{2.17}
$$

2.7. **Affine crystals $B^{1,s}$.** We recall the crystals $B^{1,s}$ from [21] (and [20] for type $C_n^{(1)}$). The affine algebra $\mathfrak{g}$ has simple Lie subalgebra of type given in (2.1). There is an isomorphism of classical crystals

$$
B^{1,s} \cong \begin{cases} B(s\overline{\Lambda}_1) & \text{for types } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)} \\ \bigoplus_{s' \le s} B(s'\overline{\Lambda}_1) & \text{for types } A_{2n}^{(2)}, D_{n+1}^{(2)} \\ \bigoplus_{\substack{s' \le s \\ s-s' \in 2\mathbb{Z}}} B(s'\overline{\Lambda}_1) & \text{for type } C_n^{(1)}, A_{2n}^{(2)\dagger}. \end{cases} \tag{2.18}
$$

The crystal operators e_i for $1 \le i \le n$ are given in subsection 2.6. The operator e_0 is given by
(2.19)

$$
\begin{aligned}
&\text{Type } B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}: && e_0 b = \begin{cases} (x_1,x_2-1,\dots,\bar{x}_2,\bar{x}_1+1) & \text{if } x_2 > \bar{x}_2 \\ (x_1-1,x_2,\dots,\bar{x}_2+1,\bar{x}_1) & \text{if } x_2 \le \bar{x}_2 \end{cases} \\
&\text{Type } A_{2n}^{(2)}, D_{n+1}^{(2)}: && e_0 b = \begin{cases} (x_1-1,x_2,\dots,\bar{x}_2,\bar{x}_1) & \text{if } x_1 > \bar{x}_1 \\ (x_1,x_2,\dots,\bar{x}_2,\bar{x}_1+1) & \text{if } x_1 \le \bar{x}_1 \end{cases} \\
&\text{Type } C_n^{(1)}, A_{2n}^{(2)\dagger}: && e_0 b = \begin{cases} (x_1-2,x_2,\dots,\bar{x}_2,\bar{x}_1) & \text{if } x_1 \ge \bar{x}_1+2 \\ (x_1-1,x_2,\dots,\bar{x}_2,\bar{x}_1+1) & \text{if } x_1 = \bar{x}_1+1 \\ (x_1,x_2,\dots,\bar{x}_2,\bar{x}_1+2) & \text{if } x_1 \le \bar{x}_1. \end{cases}
\end{aligned}
$$

3. Virtual crystals

3.1. **Embeddings of affine algebras.** As given in (1.1), there are natural inclusions of the affine Lie algebras. These embeddings do not carry over to the corresponding quantum algebras. Nevertheless we expect that such embeddings exist for crystals. Note that every affine algebra can be embedded into one of type $A^{(1)}, D^{(1)}$ and $E^{(1)}$ which are the untwisted affine algebras whose canonical simple Lie subalgebra is simply-laced. Crystal embeddings $C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$ are studied in [29].

Consider one of the embeddings given in (1.1) of an affine algebra with Dynkin diagram X into one with diagram Y. We consider a graph automorphism σ of Y that fixes the 0 node. For type $A_{2n-1}^{(1)}$, $\sigma(i) = 2n - i \pmod{2n}$. For type $D_{n+1}^{(1)}$ the automorphism interchanges the nodes n and $n+1$ and fixes all other nodes. There is an additional automorphism for type $D_4^{(1)}$, namely, the cyclic permutation of the nodes 1,2 and 3. For type $E_6^{(1)}$ the automorphism exchanges nodes 1 and 5 and nodes 2 and 4.

Let I^X and I^Y be the vertex sets of the diagrams X and Y respectively, I^Y/σ the set of orbits of the action of σ on I^Y, and $\iota : I^X \to I^Y/\sigma$ a bijection which preserves edges and sends 0 to 0.

Example 3.1. If X is one of $C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)}$ and $Y = A_{2n-1}^{(1)}$, then $\iota(0) = 0$, $\iota(i) = \{i, 2n-i\}$ for $1 \le i < n$ and $\iota(n) = n$.

If $X = B_n^{(1)}$ or $A_{2n-1}^{(2)}$ and $Y = D_{n+1}^{(1)}$, then $\iota(i) = i$ for $i < n$ and $\iota(n) = \{n, n+1\}$.

If X is $E_6^{(2)}$ or $F_4^{(1)}$ and $Y = E_6^{(1)}$, then $\iota(0) = 0$, $\iota(1) = 1$, $\iota(2) = 3$, $\iota(3) = \{2,4\}$ and $\iota(4) = \{1,5\}$.

If X is $D_4^{(3)}$ or $G_2^{(1)}$ and $Y = D_4^{(1)}$, then $\iota(0) = 0$, $\iota(1) = 2$ and $\iota(2) = \{1,3,4\}$.

To describe the embedding we endow the bijection ι with additional data. For each $i \in I^X$ we shall define a multiplication factor γ_i that depends on the location of i with respect to a distinguished arrow (multiple bond) in X. Removing the arrow leaves two connected components. The factor γ_i is defined as follows:

(1) Suppose X has a unique arrow.
   (a) Suppose the arrow points towards the component of 0. Then $\gamma_i = 1$ for all $i \in I^X$.
   (b) Suppose the arrow points away from the component of 0. Then γ_i is the order of σ for i in the component of 0 and is 1 otherwise.

(2) Suppose X has two arrows, that is, $Y = A_{2n-1}^{(1)}$. Then $\gamma_i = 1$ for $1 \le i \le n-1$. For $i \in \{0, n\}$, $\gamma_i = 2$ (which is the order of σ) if the arrow incident to i points away from it and is 1 otherwise.

Example 3.2. For $X = B_n^{(1)}$ and $Y = D_{n+1}^{(1)}$ we have $\gamma_i = 2$ if $0 \le i \le n-1$ and $\gamma_n = 1$. For $X = A_{2n-1}^{(2)}$ and $Y = D_n^{(1)}$ we have $\gamma_i = 1$ for all i.

The embedding $\Psi : P^X \to P^Y$ of weight lattices is defined by

$$\Psi(\Lambda_i^X) = \gamma_i \sum_{j \in \iota(i)} \Lambda_j^Y.$$

As a consequence we have

$$\Psi(\alpha_i^X) = \gamma_i \sum_{j \in \iota(i)} \alpha_j^Y$$
$$\Psi(\delta^X) = a_0 \gamma_0 \, \delta^Y.$$

3.2. **Virtual crystals.** Suggested by the embeddings $X \hookrightarrow Y$ of affine algebras, we wish to realize crystals of type X using crystals of type Y.

Let $\widehat{V}$ be a Y-crystal. We define the virtual crystal operators $\widehat{e}_i, \widehat{f}_i$ for $i \in I^X$ as the composites of Y-crystal operators f_j, e_j given by

$$\widehat{f}_i = \prod_{j \in \iota(i)} f_j^{\gamma_i}$$
$$\widehat{e}_i = \prod_{j \in \iota(i)} e_j^{\gamma_i}.$$

These are designed to simulate X-crystal operators f_i, e_i for $i \in I^X$. The type Y operators on the right hand side, may be performed in any order, since distinct nodes $j, j' \in \iota(i)$ are not adjacent in Y and thus their corresponding raising and lowering operators commute.

A virtual crystal is a pair $(V, \widehat{V})$ such that:

(1) $\widehat{V}$ is a Y-crystal.
(2) $V \subset \widehat{V}$ is closed under $\widehat{e}_i, \widehat{f}_i$ for $i \in I^X$.
(3) There is an X-crystal B and an X-crystal isomorphism $\Psi : B \to V$ such that e_i, f_i correspond to $\widehat{e}_i, \widehat{f}_i$.

Sometimes by abuse of notation, V will be referred to as a virtual crystal.

Let $b \in \widehat{V}$ and $i \in I^X$. We say that b is i-aligned if

(1) $\varphi_j^Y(b) = \varphi_{j'}^Y(b)$ for all $j, j' \in \iota(i)$, and similarly for ε.
(2) $\varphi_j^Y(b) \in \gamma_i \mathbb{Z}$ for all $j \in \iota(i)$ and similarly for ε.

In this case

$$\varphi_i^X(b) = \frac{1}{\gamma_i} \varphi_j^Y(\Psi(b)) \qquad \text{for } j \in \iota(i), b \in B \tag{3.1}$$

and similarly for ε. Say that $b \in \widehat{V}$ is aligned if it is i-aligned for all $i \in I^X$ and a subset $V \subset \widehat{V}$ is aligned if all its elements are.

Proposition 3.3. [29] *Aligned virtual crystals form a tensor category.*

Say that $(V, \widehat{V})$ is simple if V and $\widehat{V}$ are simple crystals. For the rest of the definitions we assume that the virtual crystals are simple and aligned.

Let $(V, \widehat{V})$ and $(V', \widehat{V}')$ be virtual crystals.

Definition-Conjecture 3.4. *Define the virtual R-matrix $R^v : V \otimes V' \to V' \otimes V$ as the restriction of the type Y R-matrix $\widehat{R} : \widehat{V} \otimes \widehat{V}' \to \widehat{V}' \otimes \widehat{V}$.*

For this definition to make sense it needs to be shown that $\widehat{R}(V \otimes V') \subset V' \otimes V$. In this case, let $\Psi : B \cong V$ and $\Psi' : B' \cong V'$ be X-crystal isomorphisms. By the uniqueness of the R-matrix it follows that the diagram

$$\begin{array}{ccc} B \otimes B' & \xrightarrow{R} & B' \otimes B \\ {\scriptstyle \Psi \otimes \Psi'} \downarrow & & \downarrow {\scriptstyle \Psi' \otimes \Psi} \\ V \otimes V' & \xrightarrow{R^v} & V' \otimes V \end{array} \tag{3.2}$$

commutes.

Definition 3.5. Define the virtual energy function $H^v : V \otimes V' \to \mathbb{Z}$ by

$$H^v(b \otimes b') = \frac{1}{\gamma_0} H_Y(b \otimes b')$$

where $H_Y : \widehat{V} \otimes \widehat{V}' \to \mathbb{Z}$.

If Definition-Conjecture 3.4 holds, it follows that

$$H_X(b \otimes b') = H^v(\Psi(b) \otimes \Psi'(b')) \tag{3.3}$$

where $H_X : B \otimes B' \to \mathbb{Z}$ is the energy function.

Similarly, define $D^v : V \to \mathbb{Z}$ as

$$D^v(b) = \frac{1}{\gamma_0} D_{\widehat{V}}(b).$$

If (2.14) and Definition-Conjecture 3.4 hold then

$$D_X(b) = D^v(\Psi(b)) \qquad \text{for } b \in B. \tag{3.4}$$

where $D_X : B \to \mathbb{Z}$ is the intrinsic energy of B.

Finally, let $\lambda \in \overline{P}^+$ for the algebra X and

$$\mathcal{P}(V,\lambda) = \{b \in V \mid \mathrm{wt}(b) = \Psi(\lambda) \text{ and } \widehat{e}_i b = 0 \text{ undefined for } i \in \overline{I}^X\}.$$

Then let

$$X^v(V,\lambda) = \sum_{b \in \mathcal{P}(V,\lambda)} q^{D^v(b)}.$$

Let us define the Y-crystal

$$\widehat{V}^{r,s} = \bigotimes_{j \in \iota(r)} B_Y^{r,\gamma_r s}$$

except for $A_{2n}^{(2)}$ and $r = n$ in which case $\widehat{V}^{n,s} = B_Y^{n,s} \otimes B_Y^{n,s}$.

Definition 3.6. Let $V^{r,s}$ be the subset of $\widehat{V}^{r,s}$ generated from $u(\widehat{V}^{r,s})$ using the virtual crystal operators $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in I^X$.

Conjecture 3.7.

(V1) *The pair $(V^{r,s}, \widehat{V}^{r,s})$ is a simple aligned virtual crystal.*

(V2) *There is an isomorphism of X-crystals*

$$\Psi : B_X^{r,s} \cong V^{r,s}$$

such that e_i and f_i correspond to $\widehat{e}_i$ and $\widehat{f}_i$ respectively, for all $i \in I^X$.

(V3) *Let λ be a classical dominant weight for X, B a tensor product of X-crystals of the form $B^{r,s}$, and $(V, \widehat{V})$ the corresponding tensor product of virtual crystals $(V^{r,s}, \widehat{V}^{r,s})$. Then*

$$X^v(V,\lambda) = X(B,\lambda). \tag{3.5}$$

In [29] Conjecture 3.7 is proved for embeddings $C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$ and tensor factors of the form $B^{r,1}$.

Theorem 3.8. *Conjecture 3.7 holds when X is of nonexceptional affine type and B is a tensor product of crystals of the form $B^{1,s}$.*

This theorem is proven in subsection 3.3.

3.3. Virtual crystals $V^{1,s}$ for $A_{2n-1}^{(2)}, B_n^{(1)} \hookrightarrow D_{n+1}^{(1)}$.

Proposition 3.9. *For $X = B_n^{(1)}$ and $Y = D_{n+1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid x_i, \bar{x}_i \in 2\mathbb{Z} \text{ for } i < n,\ x_n + \bar{x}_n \in 2\mathbb{Z},\ x_{n+1} = \bar{x}_{n+1} = 0\}.$$

Moreover Theorem 3.8 holds.

Proof. The explicit form of $V^{1,s}$ follows from $u(\widehat{V}^{1,s}) = 1^{2s}$ and the definitions of the virtual crystal operators. It is easy to show that for $s = 1$ the map $B^{1,1} \to V^{1,1}$ defined by $i \mapsto ii$ and $\bar{i} \mapsto \overline{ii}$ for $1 \le i \le n$ and $\circ \mapsto n\bar{n}$, is the desired isomorphism for $s = 1$. Similarly, it is straightforward to show that for s arbitrary, the desired isomorphism $\Psi : B^{1,s} \to V^{1,s}$ is given by replacing each letter (which is an element of $B^{1,1}$) of a word in $B^{1,s}$ by the corresponding pair of letters as in the case $s = 1$. This proves (V1) and (V2).

For (V3) we need to check that

$$D^v(\Psi(b)) = D_B(b) \qquad \text{for } b \in B. \tag{3.6}$$

Since D_B is defined in terms of R, H and functions $D_{B^{1,s}}$, it suffices to verify (2.14) and Definition-Conjecture 3.4.

The element $u^\natural(B^{1,s})$ is given explicitly by $\bar{1}^s$. By the explicit computation of $H : B^{1,s} \otimes B^{1,s} \to \mathbb{Z}$ given in [5] it follows that (2.14) holds.

To check Definition-Conjecture 3.4 we consider the explicit expressions for the R-matrices of types $B_n^{(1)}$ and $D_{n+1}^{(1)}$ given in [5]. From this it suffices to show that the images of relations in the plactic monoid of type B_n are relations in the plactic monoid of type D_{n+1} [26]. This is straightforward. □

Proposition 3.10. *For $X = A_{2n-1}^{(2)}$ and $Y = D_{n+1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid x_{n+1} = \bar{x}_{n+1} = 0\}.$$

Moreover Theorem 3.8 holds.

Proof. The proof is similar to that of Proposition 3.9. In particular the bijection $B^{1,s} \to V^{1,s}$ is given by leaving a word unchanged. □

3.4. **Virtual crystals $V^{1,s}$ for $C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}$.** We require some preliminaries on crystals of type $A_{2n-1}^{(1)}$.

Consider $Y = A_{2n-1}^{(1)}$ and X one of $C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)}$. In all these cases $\widehat{V}^{1,s} = B_Y^{2n-1,s} \otimes B_Y^{1,s}$. We introduce the alphabets

$$\mathcal{Y} = \{1 < 2 < \cdots < 2n\} \qquad \mathcal{Y}^\vee = \{2n^\vee < (2n-1)^\vee < \cdots < 2^\vee < 1^\vee\}. \tag{3.7}$$

$\mathcal{Y}$ and $\mathcal{Y}^\vee$ are the sets of elements of $B_Y^{1,1}$ and $(B_Y^{1,1})^\vee \cong B_Y^{2n-1,1}$ respectively. The element $i^\vee \in B_Y^{2n-1,1}$ is the column of height $2n-1$ in the alphabet $\mathcal{Y}$ with the letter i missing. For $1 \le i \le 2n-1$, $f_i((2n+1-i)^\vee) = (2n-i)^\vee$ and $f_i(b)$ is undefined otherwise. $f_0(1^\vee) = (2n)^\vee$ and $f_0(b)$ is undefined otherwise. In this notation, $B_Y^{2n-1,s}$ consists of the weakly increasing words of length s in the alphabet $\mathcal{Y}^\vee$. For $b = b_1 \otimes b_2 \in \widehat{V}^{1,s}$, let y_i be the number of letters i in b_2 and $y_i^\vee$ the number of letters $i^\vee$ in b_1, for $1 \le i \le 2n$.

The R-matrix $R : B_Y^{1,1} \otimes B_Y^{2n-1,1} \to B_Y^{2n-1,1} \otimes B_Y^{1,1}$ is given by

$$i \otimes j^\vee \to \begin{cases} j^\vee \otimes i & \text{if } i \neq j \\ (i+1)^\vee \otimes (i+1) & \text{if } i = j < 2n \\ 1^\vee \otimes 1 & \text{if } i = j = 2n. \end{cases} \tag{3.8}$$

The R-matrix $R : B_Y^{1,s} \otimes B_Y^{2n-1,s} \to B_Y^{2n-1,s} \otimes B_Y^{1,s}$ is given by iterating the above R-matrix so that all of the elements of $\mathcal{Y}^\vee$ are commuted to the left. The element $1^\vee \otimes 1$ commutes with all elements of $B_Y^{1,1}$ and $B_Y^{2n-1,1}$.

To formulate the next propositions we also need an involution $* : B \to B$ on crystals of type $A_{2n-1}^{(1)}$ [29, Section 3.8]. Given a word u, let u^* be the word obtained by replacing each letter i by $2n+1-i$, and reversing the resulting word. Clearly if u is a column word then so is u^*. If $b = c_1 c_2 \dots c_s \in B^{r,s}$ where c_j is a column word for all j, then by definition $b^* = c_s^* \dots c_1^* \in B^{r,s}$, which is a sequence of column words. Under this map the crystal operators transform as follows:

$$\begin{aligned} f_i(b^*) &= e_{n-i}(b)^* \\ e_i(b^*) &= f_{n-i}(b)^* \\ \mathrm{wt}(b^*) &= w_0 \mathrm{wt}(b). \end{aligned}$$

Proposition 3.11. *For $X = C_n^{(1)}$ and $Y = A_{2n-1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid b^{\vee *} = R(b),\ \min(y_1, y_1^\vee), \min(y_{n+1}, y_{n+1}^\vee) \in 2\mathbb{Z}\} \tag{3.9}$$

Moreover Theorem 3.8 holds.

Proof. We first prove (3.9). By the definition of $V^{1,s}$, it suffices to show that the right hand side V' of (3.9) contains $u(\widehat{V}^{1,s})$, and every element of V' is reachable from $u(\widehat{V}^{1,s})$ using the virtual crystal operators $\widehat{e}_i, \widehat{f}_i$ for $i \in I^X$.

We first digress on the self-duality condition

$$b^{\vee *} = R(b). \tag{3.10}$$

By the proof of [29, Prop 6.8], in the set $\widehat{V}^{r,s}$, the condition (3.10) is preserved under e_0, e_n, $\widehat{e}_i$ for $1 \le i \le n-1$, and similarly for f. For $b \in \widehat{V}^{1,s}$, using (3.8) equation (3.10) is equivalent to

$$y_{2n+1-i} = y_i^\vee - \min(y_i, y_i^\vee) + \min(y_{i+1}, y_{i+1}^\vee) \tag{3.11}$$

$$y_{2n+1-i}^\vee = y_i - \min(y_i, y_i^\vee) + \min(y_{i+1}, y_{i+1}^\vee) \tag{3.12}$$

for $1 \le i \le 2n$, where $y_{2n+1} = y_1$ and $y_{2n+1}^\vee = y_1^\vee$.

We deduce two consequences of (3.10). Subtracting (3.11) and (3.12) we obtain

$$y_i + y_{2n+1-i} = y_i^\vee + y_{2n+1-i}^\vee \tag{3.13}$$

for $1 \le i \le 2n$. We also have $\varepsilon_0(b) = y_1 + y_{2n}^\vee - \min(y_{2n}, y_{2n}^\vee)$. By (3.12) with $i = 2n$ and (3.13) with $i = 1$, we have

$$\varepsilon_0(b) = 2y_1 - \min(y_1, y_1^\vee). \tag{3.14}$$

Now we show that $u = u(\widehat{V}^{1,s}) \in V'$. This element satisfies $y_i = s\delta_{i,1}$ and $y_i^\vee = s\delta_{i,2n}$ for $1 \le i \le 2n$. Comparing this with (3.11) and (3.12) it follows that u satisfies (3.10). It follows that $u \in V'$.

We next check that V' aligned. Let $b \in V'$ and $i \in I^X$. Since b satisfies (3.10) it is i-aligned if $1 \le i \le n-1$ by [29, Prop. 6.9]. For 0-alignedness, by (3.14) we see that $\varepsilon_0(b)$ is even since $\min(y_1, y_1^\vee)$ is. The proof that $\varphi_0(b)$ is even is similar. So b is 0-aligned. The proof that b is n-aligned, is similar as well. So V' is aligned.

Next it is shown that the set V' is closed under $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in I^X$. Let $b \in V'$. $\widehat{e}_i b$ is self-dual since b is. Note that the quantity $\min(y_1, y_1^\vee)$ is unchanged for $i \notin \{0,1\}$. We have $\varepsilon_1(b_1) = y_1^\vee$ and $\varphi_1(b_2) = y_1$. Hence by the tensor product rule, $\min(y_1, y_1^\vee)$ remains the same upon applying $\widehat{e}_1$. Let $i = 0$. Since $b \in V'$, b is 0-aligned, so that $\varepsilon_0(b) \in 2\mathbb{Z}$. Since $\varepsilon_0(\widehat{e}_0 b) = \varepsilon_0(b) - 2$ is even, by (3.14), the self-dual element $\widehat{e}_0 b$ has the property that $\min(y_1, y_1^\vee) \in 2\mathbb{Z}$. Thus $\widehat{e}_i b$ satisfies that property for all i. The property that $\min(y_{n+1}, y_{n+1}^\vee) \in 2\mathbb{Z}$ is satisfied for $\widehat{e}_i b$ is similar. Thus $\widehat{e}_i b \in V'$ for all $i \in I^X$. The proof that $\widehat{f}_i b \in V'$ for all $i \in I^X$ is again similar.

Let $b \in V'$. It suffices to find a sequence of operators $\widehat{e}_i$ and $\widehat{f}_i$ leading from b to u. We shall induct on the quantity $\min(y_1, y_1^\vee)$, which is invariant under $\widehat{e}_i$ and $\widehat{f}_i$ for $i \in I^X \backslash \{0\}$ by previous arguments. Suppose first that $\varepsilon_{j'}(b) > 0$ for some $j' \neq 0$. By alignedness it follows that we may apply a sequence of operators $\widehat{e}_i$ for $i \in I^X \backslash \{0\}$ to b, thereby passing to a classical highest weight vector of $\widehat{V}^{1,s}$. The classical highest weight vectors of $\widehat{V}^{1,s}$ are given explicitly by $u_k = (2n^\vee)^{s-k} 1^{\vee k} \otimes 1^s$, for $0 \le k \le s$. u_k satisfies $\min(y_1, y_1^\vee) = k$. By assumption $b = u_k$ for k even. If $k = 0$

then $b = u_0 = u$ and we are done. If $k > 0$ then $\widehat{f}_0 b$ satisfies $\min(y_1, y_1^\vee) = k - 2$, which is even. We are done by induction.

We have shown that (3.9) holds and that $V^{1,s}$ is aligned.

The bijection $\Psi : B^{1,s} \to V^{1,s}$ is given as follows. Let $b \in B^{1,s}$. In the case $s = 1$, the map $B^{1,1} \to V^{1,1}$ is given by $i \mapsto (2n+1-i)^\vee \otimes i$ and $\bar{i} \mapsto i^\vee \otimes (2n+1-i)$. The map $\Psi : B^{1,s} \to V^{1,s}$ is given by the composite map

$$B^{1,s} \hookrightarrow (B^{1,1})^{\otimes s} \to (B_Y^{2n-1,1} \otimes B_Y^{1,1})^{\otimes s} \to (B_Y^{2n-1,1})^{\otimes s} \otimes (B_Y^{1,1})^{\otimes s}. \tag{3.15}$$

It follows from (3.8) that the image of this map is contained in $B_Y^{2n-1,s} \otimes B_Y^{1,s}$. Computing this commutation explicitly and using the notation $x_i, \bar{x}_i$ to describe b for $1 \le i \le n$, and $y_i, y_i^\vee$ for $\Psi(b)$, we have

$$\begin{aligned}
y_1 &= x_1 - \min(x_1, \bar{x}_1) + s - \sum_{i=1}^n (x_i + \bar{x}_i) \\
y_1^\vee &= \bar{x}_1 - \min(x_1, \bar{x}_1) + s - \sum_{i=1}^n (x_i + \bar{x}_i) \\
y_i &= x_i - \min(x_i, \bar{x}_i) + \min(x_{i-1}, \bar{x}_{i-1}) \qquad \text{for } i > 1 \\
y_i^\vee &= \bar{x}_i - \min(x_i, \bar{x}_i) + \min(x_{i-1}, \bar{x}_{i-1}) \qquad \text{for } i > 1
\end{aligned} \tag{3.16}$$

To recover y_i and $y_i^\vee$ for $n+1 \le i \le 2n$ one may use (3.11) and (3.12), plus the fact that the total number of letters in either b_1 or b_2, is s.

The composite map given in (3.15) sends e_i to $\widehat{e}_i$ for $1 \le i \le n$ [2]. It is straightforward to check that e_0 goes to $\widehat{e}_0$ using (3.16), (3.11), and (3.12). Therefore Ψ is a morphism of X-crystals. It is clearly injective. The image is $V^{1,s}$ since $\Psi(u(B^{1,s})) = u(\widehat{V}^{1,s})$ and both $B^{1,s}$ and $V^{1,s}$ are connected. Therefore $\Psi : B^{1,s} \to V^{1,s}$ is an isomorphism of X-crystals. This completes the proof of (V1) and (V2). (V3) follows by [29, Section 6.6]. □

Proposition 3.12. *For $X = A_{2n}^{(2)}$ and $Y = A_{2n-1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid b^{\vee *} = R(b),\ \min(y_{n+1}, y_{n+1}^\vee) \in 2\mathbb{Z}\}. \tag{3.17}$$

Moreover Theorem 3.8 holds.

The proof is entirely similar to that of $C_n^{(1)}$.

Proposition 3.13. *For $X = D_{n+1}^{(2)}$ and $Y = A_{2n-1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid b^{\vee *} = R(b)\}. \tag{3.18}$$

Moreover Theorem 3.8 holds.

Proof. For $X = D_{n+1}^{(2)}$ most of the proof is similar to that of type $C_n^{(1)}$. Here the classical subalgebra of X is of type B_n, so the isomorphism $\Psi : B^{1,s} \to V^{1,s}$ is a bit different. It is given by $\circ \mapsto (n+1)^\vee \otimes (n+1)$, with the other letters mapped as in the $C_n^{(1)}$ case. The explicit map is given as in (3.16) except that

$$\begin{aligned}
y_1 &= x_1 - \min(x_1, \bar{x}_1) + s - x_\circ - \sum_{i=1}^n (x_i + \bar{x}_i) \\
y_1^\vee &= \bar{x}_1 - \min(x_1, \bar{x}_1) + s - x_\circ - \sum_{i=1}^n (x_i + \bar{x}_i).
\end{aligned} \tag{3.19}$$

□

Proposition 3.14. *For $X = A_{2n}^{(2)\dagger}$ and $Y = A_{2n-1}^{(1)}$,*

$$V^{1,s} = \{b \in \widehat{V}^{1,s} \mid b^{\vee *} = R(b), \min(y_1, y_1^\vee) \in 2\mathbb{Z}\}. \tag{3.20}$$

Moreover Theorem 3.8 holds.

The proof is similar.

4. Fermionic formula

4.1. **Review.** This subsection reviews definitions of [6, 7]. For this section we assume that $\mathfrak{g} \neq A_{2n}^{(2)\dagger}$; for that type we refer the reader to [29, Section 7.6]. Fix $\lambda \in \overline{P}^+$ and B a tensor product of crystals of the form $B^{r,s}$. Let $L_i^{(a)}$ be the number of tensor factors in B that are equal to $B^{a,i}$. Set $\tilde{\alpha}_a = \alpha_a$ for all $a \in \overline{I}$ except for type $A_{2n}^{(2)}$ in which case $\tilde{\alpha}_a$ are the simple roots of type B_n.

Let $\nu = (m_i^{(a)})$ be a matrix of nonnegative integers for $i \in \mathbb{Z}_{>0}$ and $a \in \overline{I}$. Say that ν is a (B,λ)-configuration if

$$\sum_{\substack{a\in\overline{I}\\ i\in\mathbb{Z}_{>0}}} i\, m_i^{(a)} \tilde{\alpha}_a = \sum_{\substack{a\in\overline{I}\\ i\in\mathbb{Z}_{>0}}} i\, L_i^{(a)} \overline{\Lambda}_a - \lambda \tag{4.1}$$

except for type $A_{2n}^{(2)}$. In this case the right hand side should be replaced by ι(r.h.s) where ι is a $\mathbb{Z}$-linear map from the weight lattice of type C_n to the weight lattice of type B_n such that

$$\iota(\overline{\Lambda}_a^C) = \begin{cases} \overline{\Lambda}_a^B & \text{for } 1 \le a < n \\ 2\overline{\Lambda}_a^B & \text{for } a = n. \end{cases}$$

Say that a configuration ν is admissible if

$$p_i^{(a)} \ge 0 \qquad \text{for all } a \in \overline{I} \text{ and } i \in \mathbb{Z}_{>0}, \tag{4.2}$$

where

$$p_i^{(a)} = \sum_{k\in\mathbb{Z}_{>0}} \left(L_k^{(a)} \min(i,k) - \frac{1}{t_a^\vee} \sum_{b\in\overline{I}} (\tilde{\alpha}_a|\tilde{\alpha}_b) \min(t_b i, t_a k)\, m_k^{(b)} \right). \tag{4.3}$$

Write $C(B,\lambda)$ for the set of admissible (B,λ)-configurations. Define

$$cc(\nu) = \frac{1}{2} \sum_{a,b\in\overline{I}} \sum_{j,k\in\mathbb{Z}_{>0}} (\tilde{\alpha}_a|\tilde{\alpha}_b) \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)}. \tag{4.4}$$

The fermionic formula is defined by

$$M(B,\lambda;q) = \sum_{\nu\in C(B,\lambda)} q^{cc(\nu)} \prod_{a\in\overline{I}} \prod_{i\in\mathbb{Z}_{>0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}_{q^{t_a^\vee}}. \tag{4.5}$$

The $X = M$ conjecture of [6, 7] states that

$$X(B,\lambda;q^{-1}) = M(B,\lambda;q). \tag{4.6}$$

The fermionic formula $M(B,\lambda)$ can be interpreted using combinatorial objects called rigged configurations. Denote by (ν, J) a pair where $\nu = (m_i^{(a)})$ is a matrix and $J = (J^{(a,i)})$ is a matrix of partitions with $a \in \overline{I}$ and $i \in \mathbb{Z}_{>0}$. Then a rigged configuration is a pair (ν, J) such that $\nu \in C(B,\lambda)$ and the partition $J^{(a,i)}$ is contained in a $m_i^{(a)}(\nu) \times p_i^{(a)}(\nu)$ rectangle for all a, i. The set of rigged (B,λ)-configurations for fixed λ and B is denoted by $\mathrm{RC}(B,\lambda)$. Then (4.5) is equivalent to

$$M(B,\lambda) = \sum_{(\nu,J)\in \mathrm{RC}(B,\lambda)} q^{\mathrm{cc}(\nu,J)}$$

where $\mathrm{cc}(\nu,J) = \mathrm{cc}(\nu) + |J|$ and $|J| = \sum_{(a,i)} t_a^\vee |J^{(a,i)}|$.

4.2. **Virtual fermionic formula.** We define virtual rigged configurations in analogy to virtual crystals.

Definition 4.1. Let X and Y be as in (1.1), and λ and B as in subsection 4.1 for type X. Let $(V,\widehat{V})$ be the virtual Y-crystal corresponding to B. Then $\mathrm{RC}^v(B,\lambda)$ is the set of elements $(\widehat{\nu},\widehat{J}) \in \mathrm{RC}(\widehat{V},\Psi(\lambda))$ such that:

(1) For all $i \in \mathbb{Z}_{>0}$, $\widehat{m}_i^{(a)} = \widehat{m}_i^{(b)}$ and $\widehat{J}_i^{(a)} = \widehat{J}_i^{(b)}$ if a and b are in the same σ-orbit in I^Y.
(2) For all $i \in \mathbb{Z}_{>0}$, $a \in \overline{I}^X$, and $b \in \iota(a) \subset I^Y$, we have $\widehat{m}_j^{(b)} = 0$ if $j \notin \gamma_a\mathbb{Z}$ and the parts of $\widehat{J}_i^{(b)}$ are multiples of γ_a.

Theorem 4.2. *There is a bijection* $\mathrm{RC}(B,\lambda) \to \mathrm{RC}^v(B,\lambda)$ *sending* $(\nu,J) \mapsto (\widehat{\nu},\widehat{J})$ *given as follows. For all* $a \in \overline{I}^X$, $b \in \iota(a) \subset \overline{I}^Y$, *and* $i \in \mathbb{Z}_{>0}$,

$$\widehat{m}_{\gamma_a i}^{(b)} = m_i^{(a)} \tag{4.7}$$
$$\widehat{J}_{\gamma_a i}^{(b)} = \gamma_a J_i^{(a)}, \tag{4.8}$$

except when $X = A_{2n}^{(2)}$ *and* $a = n$, *in which case*

$$\widehat{m}_i^{(n)} = m_i^{(n)}$$
$$\widehat{J}_i^{(n)} = 2J_i^{(n)}.$$

The cocharge changes by

$$\mathrm{cc}(\widehat{\nu},\widehat{J}) = \gamma_0\,\mathrm{cc}(\nu,J). \tag{4.9}$$

Proof. Let $\widehat{L}$ be to $\widehat{V}$ as L is to B as in subsection 4.1. For $a \in \overline{I}^X$, $b \in \iota(a)$, and $i \in \mathbb{Z}_{>0}$,

$$\widehat{L}_{\gamma_a i}^{(b)} = L_i^{(a)}$$
$$\widehat{L}_j^{(b)} = 0 \qquad \text{for } j \notin \gamma_a\mathbb{Z},$$

except when $X = A_{2n}^{(2)}$ and $a = n$, in which case $\widehat{L}_i^{(n)} = 2L_i^{(n)}$ for all i. Using (4.3) we have, for all $b \in \iota(a)$ and $i \in \mathbb{Z}_{>0}$,

$$\widehat{p}_{\gamma_a i}^{(b)} = \gamma_a p_i^{(a)},$$

except when $X = A_{2n}^{(2)}$ and $i = n$, in which case $\widehat{p}_i^{(n)} = 2p_i^{(n)}$. Therefore $(\nu,J) \mapsto (\widehat{\nu},\widehat{J})$ defines a bijection. Using (4.4) we see that (4.9) holds. $\square$

5. Algorithms for computing the fermionic formula

To compute the fermionic formula $M(B,\lambda)$, one must find the set of admissible (B,λ)-configurations $C(B,\lambda)$. One direct approach would be to test the admissibility conditions (4.2) on the set of (B,λ)-configurations (4.1) which consist of all possible n-tuples of partitions of sizes that depend on λ and B. This quickly becomes infeasible as B and λ grow.

In [16, 17] Kleber gives an efficient algorithm to compute the set of admissible configurations in the simply-laced types $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, and $E_8^{(1)}$. It generates a rooted tree $T(B)$ whose nodes are labelled by elements of $\overline{P}^+$. The tree $T(B)$ is constructed to have the property that the elements of $C(B,\lambda)$ are in bijection with the nodes of $T(B)$ labelled λ. If a node x labelled λ corresponds to a configuration ν, then ν can be recovered from the unique path in $T(B)$ from x to the root.

5.1. Kleber's algorithm.

We review Kleber's algorithm [16, 17]. Let X be the Dynkin diagram of an untwisted affine Lie algebra whose canonical simple subalgebra is of simply-laced type. Let B and L be as in subsection 4.1.

We define a tree $T(B)$ by the following algorithm. Each node x is labelled by an element $\mathrm{wt}(x)\in\overline{P}^+$ called its weight. It has the property that if x is a node and y is its child, then $\mathrm{wt}(x)\neq\mathrm{wt}(y)$ and $\mathrm{wt}(x)\trianglerighteq\mathrm{wt}(y)$. A tree edge (x,y) is labelled by the element $\mathrm{d}_{xy}=\mathrm{wt}(x)-\mathrm{wt}(y)\in\overline{Q}^+\setminus\{0\}$.

1. Let T_0 be the tree consisting of a single node of weight 0 and set $\ell=0$.
2. Add 1 to ℓ.
3. Let T'_ℓ be obtained from $T_{\ell-1}$ by adding $\sum_{a=1}^n\overline{\Lambda}_a\sum_{i\geq\ell}L_i^{(a)}$ to the weight of each node.
4. Let T_ℓ be obtained from T'_ℓ as follows. Let x be a node at depth $\ell-1$ of weight μ. Suppose there is a weight $\tau\in\overline{P}^+$ such that $\mu\neq\tau$, $\mu\trianglerighteq\tau$, and if x is not the root, $\nu-2\mu+\tau\in\overline{Q}^+$ where ν is the weight of the parent w of x. In every such case we attach to x a child y of weight τ. Note that if x is not the root, the condition $\nu-2\mu+\tau\in\overline{Q}^+$ is equivalent to $d_{wx}\trianglerighteq d_{xy}$.
5. If $T_\ell\neq T_{\ell-1}$ then go to step 2.
6. Otherwise set $T(B)=T_\ell$ and stop.

For large ℓ step 3 does not change the tree. For such ℓ, step 4 can only be applied finitely many times since there are finitely many elements of $\overline{P}^+$ dominated by a given element of $\overline{P}^+$. Hence the algorithm terminates.

There is a bijection from the nodes of $T(B)$ and the configurations $C(B)=\bigcup_{\lambda\in\overline{P}^+}C(B,\lambda)$ given as follows. Let x be a node at depth p in $T(B)$ of weight λ. Let $\lambda^{(0)},\lambda^{(1)},\dots,\lambda^{(p)}=\lambda$ be the weights of the nodes on the path from the root of $T(B)$ to x. Then the configuration $\nu\in C(B,\lambda)$ corresponding to x is defined by

$$m_i^{(a)}=(\lambda^{(i-1)}-2\lambda^{(i)}+\lambda^{(i+1)}\mid\overline{\Lambda}_a) \tag{5.1}$$

where we make the convention that $\lambda=\lambda^{(p+1)}=\lambda^{(p+2)}=\cdots$. The vacancy numbers are given by

$$p_i^{(a)}=-\sum_{j>i}L_i^{(a)}+(\lambda^{(i)}\mid\alpha_a). \tag{5.2}$$

Suppose we are only interested in finding $C(B,\lambda)$ for a particular $\lambda \in \overline{P}^+$. It is wasteful to generate the entire tree $T(B)$ and then select the nodes of weight λ. Because the weight of a node dominates that of any of its children, we can prune the tree as follows. In step 4, we only add a node of weight τ at depth ℓ if

$$\tau' := \tau + \sum_{j>\ell} \overline{\Lambda}_a L_j^{(a)} \trianglerighteq \lambda. \tag{5.3}$$

There is another condition under which we can prune. Suppose that in the absence of pruning, we would have added a node y of weight τ at depth ℓ in step 4, with parent x. Then we do not add y if there is an a such that $(\tau' - \lambda \mid \overline{\Lambda}_a) > 0$ and $(d_{xy} \mid \overline{\Lambda}_a) = 0$. For in this case, the condition in step 4 prevents one from reaching the weight λ as a descendant of τ.

Example 5.1. Let $B = B^{3,2} \otimes B^{2,1} \otimes B^{1,1} \otimes B^{1,1}$ of type $A_3^{(1)}$. The Kleber algorithm produces the tree $T(B)$ given in Figure 2. The corresponding configurations are given in the following diagram, where we represent ν as a sequence of partitions $\nu^{(a)}$ with $m_i^{(a)}$ rows of length i. The vacancy number $p_i^{(a)}$ is placed to the right of a row of length i in $\nu^{(a)}$.

λ	$\nu^{(1)}$	$\nu^{(2)}$	$\nu^{(3)}$
$2\overline{\Lambda}_1 + \overline{\Lambda}_2 + 2\overline{\Lambda}_3$	$\varnothing$	$\varnothing$	$\varnothing$
$2\overline{\Lambda}_2 + 2\overline{\Lambda}_3$	$\square$ 0	$\varnothing$	$\varnothing$
$\overline{\Lambda}_1 + 3\overline{\Lambda}_3$	$\square$ 1	$\square$ 0	$\varnothing$
$\overline{\Lambda}_1 + \overline{\Lambda}_2 + \overline{\Lambda}_3$	$\square$ 1	$\square$ 1	$\square$ 0
$2\overline{\Lambda}_3$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ 0	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ 0	$\square$ 1
	$\square\square$ 0	$\square\square$ 0	$\square$ 0
$3\overline{\Lambda}_1 + \overline{\Lambda}_3$	$\varnothing$	$\square$ 0	$\square$ 0
$\overline{\Lambda}_2$	$\square\square$ 0	$\square\square$ 1	$\square\square$ 0
	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ 0	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ 0	$\square\square$ 0
$2\overline{\Lambda}_1$	$\square$ 1	$\square\square$ 0	$\square\square$ 0

5.2. **Virtual Kleber algorithm.** Outside of the simply-laced case, Kleber's algorithm does not directly apply. However we use the embeddings of affine algebras into those of simply-laced type, where Kleber's algorithm can be applied. We call our method the virtual Kleber algorithm. Let X and Y be as in (1.1). Theorem 4.2 defines a bijection $C(B,\lambda) \cong C^v(B,\lambda)$ where $C^v(B,\lambda)$ consists of the configurations $\widehat{\nu} \in C(\widehat{V}, \Psi(\lambda))$ constrained as in Definition 4.1, or equivalently, the $\widehat{\nu}$ such that $(\widehat{\nu}, \widehat{J}) \in \mathrm{RC}^v(B,\lambda)$ for some $\widehat{J}$. A naive approach would be to run Kleber's algorithm to compute the set $C(\widehat{V}, \Psi(\lambda))$ and then to select the desired subset

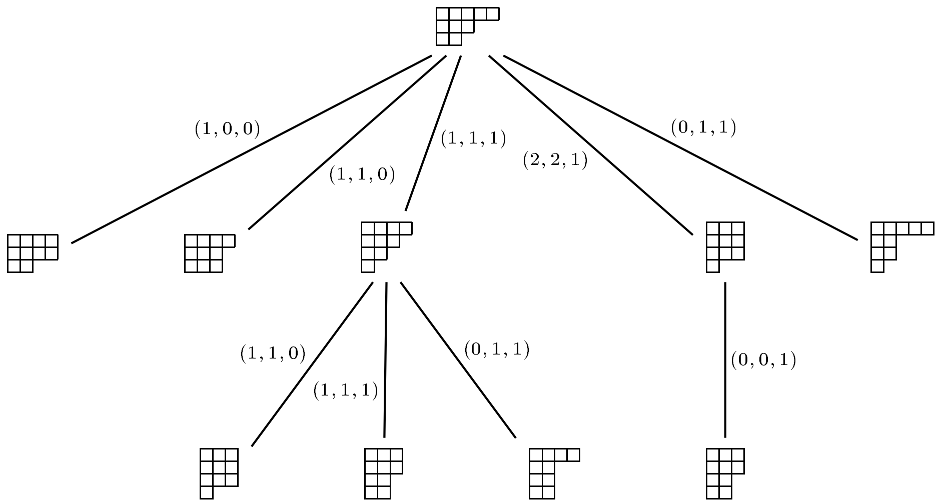

FIGURE 2. Tree $T(B)$

$C^v(B,\lambda)$. A more efficient way is to prune the branches that cannot contain elements of $C^v(B,\lambda)$. This results in a good algorithm to find $C^v(B,\lambda)$ and therefore $M(B,\lambda)$ for any affine type.

More precisely, one only adds the child y to the node x in step 4 at depth ℓ if:

(1) $(\mathrm{wt}(y) \mid \alpha_a) = (\mathrm{wt}(y) \mid \alpha_b)$ if a and b are in the same σ-orbit of I^Y.
(2) If $\ell - 1 \notin \gamma_a\mathbb{Z}$, then $d_{wx} = d_{xy}$ where w is the parent of x.

These conditions are equivalent to those in Definition 4.1.

Let $\widehat{T}(B)$ be the resulting tree. Let $\gamma = \max_a \gamma_a$. Then there is a bijection between $C^v(B,\lambda)$, and the set of nodes y of weight λ in $\widehat{T}(B)$ that satisfy either of the following conditions:

(1) y is at depth ℓ with $\ell \in \gamma\mathbb{Z}$, or
(2) $(d_{xy} \mid \overline{\Lambda}_a) = 0$ for every a such that $1 < \gamma = \gamma_a$, where x is the parent of y.

Observe that for $\ell \notin \gamma\mathbb{Z}$, there may be nodes at depth ℓ in T_ℓ whose weights are not in the image of the embedding $P^X \hookrightarrow P^Y$, but rather in a superlattice of index γ. These weights, which cannot appear in the final tree, are necessary as they allow the virtual Kleber algorithm to reach all of the desired weights.

Example 5.2. Let $X = C_2^{(1)}$, $Y = A_3^{(1)}$, $B = B^{1,2} \otimes B^{1,1} \otimes B^{2,1}$. The virtual Kleber algorithm produces the tree $\widehat{T}(B)$ given in Figure 3. The nodes corresponding to elements of $C^v(B,\lambda)$ are circled. We list the configurations corresponding to the circled nodes, ordered by increasing depth and then from left to right. Here we represent ν as a sequence of partitions $\nu^{(a)}$ with $m_i^{(a)}$ rows of length i. The vacancy

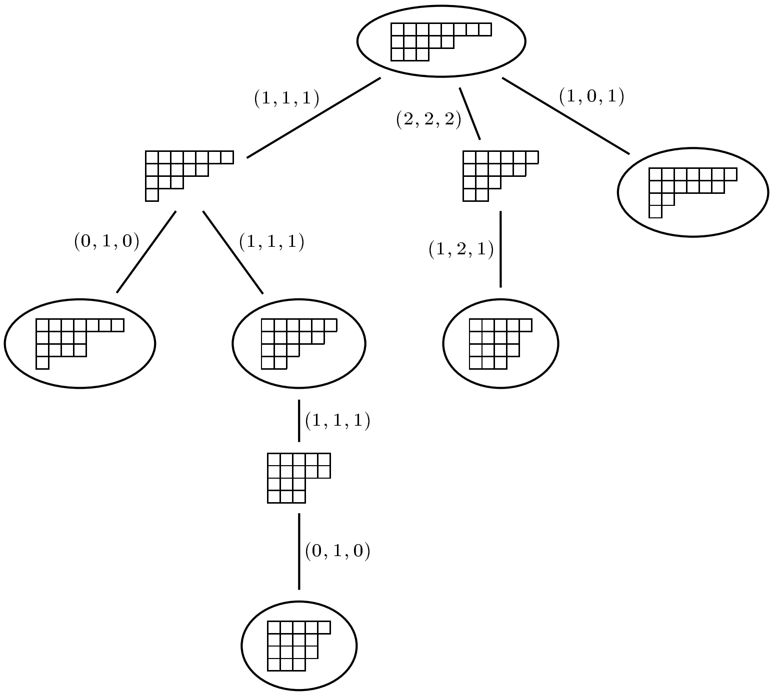

FIGURE 3. Tree $\widehat{T}(B)$

number $p_i^{(a)}$ is placed to the right of a row of length i in $\nu^{(a)}$.

λ	$\nu^{(1)}$	$\nu^{(2)}$	$\nu^{(3)}$
$3\overline{\Lambda}_1 + \overline{\Lambda}_2$	$\varnothing$	$\varnothing$	$\varnothing$
$\overline{\Lambda}_1 + 2\overline{\Lambda}_2$	$\square$ 0	$\varnothing$	$\square$ 0
$3\overline{\Lambda}_1$	$\square$ 1	$\square\square$ 0	$\square$ 1
$\overline{\Lambda}_1 + \overline{\Lambda}_2$	$\square\square$ 1	$\square\square$ 2	$\square\square$ 1
$\overline{\Lambda}_1$	$\square\square$ 1 $\square$ 1	$\square\square$ 0 $\square\square$	$\square\square$ 1 $\square$ 1
	$\square\square\square$ 0	$\square\square\square\square$ 0	$\square\square\square$ 0

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