On the Spectral Flow of Families of Dirac operators with constant symbol

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Abstract

We consider families of generalized Dirac operators D_t with constant principal symbol and constant essential spectrum such that the endpoints are gauge equivalent, i.e. $D_1 = W^* D_0 W$. The spectral flow in any gap in the essential spectrum we express as the Fredholm index of $1 + (W - 1)P$ where P is the spectral projection on the interval $[d, \infty)$ with respect to D_0 and d is in the gap. We reduce the computation of this index to the Atiyah-Singer index theorem for elliptic pseudodifferential operators. We find an invariant of the Riemannian geometry for odd dimensional spin manifolds estimating the length of gaps in the spectrum of the Dirac operator.

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1 Introduction

If D_t, $t \in [0,1]$, is a family of selfadjoint operators, $d \in res\ D_0 \cap res\ D_1$ and $d \notin \sigma_{ess}(D_t)$, $\forall t \in [0,1]$, then the spectral flow $sf(D_t)$ at d is defined as the number of families of eigenvalues $\lambda(t)$ of D_t crossing d from above minus the number of families of eigenvalues crossing d from below.

In the famous paper [1] the spectral flow of a closed loop of Dirac operators D_t on a compact manifold M at any d has been expressed as the index of a certain Dirac operator over $M \times S^1$. The η-invariant and K-theoretic arguments have been employed there. These techniques are well suited to compact manifolds.

In the present paper we propose a new more functional analytic approach to the spectral flow. It has the advantage that it allows to compute the spectral flow in the presence of essential spectrum, i.e. for families of Dirac operators on noncompact manifolds. The price we have to pay is the restriction to rather special families which we are going to describe now.

Let D be a generalized Dirac operator on a complete Riemannian manifold M associated to a Clifford bundle $E \to M$ and $W \in \Gamma(M, U(E))$ be a gauge transformation satisfying

Assumption 1 *1. $W - 1 \in C_c^\infty(M, End(E))$*

2. $R := W^*DW - D$ is of zero order.

Let us consider the family $D_t := D + tR$. Then $\sigma_{ess}(D_t) = \sigma_{ess}(D)$, $\forall t \in \mathbf{R}$, and if $d \in res(D)$ then the spectral flow of $\{D_t\}_{t=0}^1$ at d is expressed as the Fredholm index of $1 + (W-1)P$ where $P := E_D[d, \infty)$ and $E_D(.)$ denotes the family of spectral projections of D. The operator $1 + (W-1)P$ can be localized near the diagonal without changing the index. Then it will be approximated by an elliptic pseudodifferential operator with constant symbol at infinity [6]. The index computation of such operators can be reduced to an index problem on a compact manifold and hence to the Atiyah-Singer index theorem along the lines of [6], §4.1. If D is the twist of the usual Dirac operator on a Riemannian spin manifold M with some Hermitian vector bundle V and W comes from a gauge transformation of V then we end up with the formula

$$ind(1 + (W - 1)P) = (-1)^k \mathbf{CS}(W)\hat{\mathbf{A}}(M)[M] \tag{1}$$

where $dim M = n = 2k+1$ is odd, $\mathbf{CS}(W) \in H_c^{odd}(M, \mathbf{Q})$ is the Chern-Simons class repesented by the compactly supported form

$$\begin{aligned} CS(W) &:= \frac{\imath}{2\pi} Tr\, W^* \nabla W \int_0^1 exp\left(\frac{\imath(t - t^2)}{2\pi} W^* \nabla W W^* \nabla W\right) dt \quad (2) \\ CS(W)_{2r-1} &= 2\frac{\imath^r}{(2\pi)^r}\frac{r!}{(2r)!} Tr([W^* \nabla W]^{2r-1}), \quad (3) \end{aligned}$$

$\hat{\mathbf{A}}(M)$ is the $\hat{A}$-genus and $[M]$ is the fundamental cycle of M. If the dimension of M is even then

$$ind(1 + (W - 1)P) = 0. \tag{4}$$

Because of (1) the spectral flow

$$sf\{D_t\} = ind(1 + (W - 1)P) \tag{5}$$

does not depend on the gap containing d.

There are many odd-dimensional examples where (1) does not vanish. In this case one has in fact proven the completeness of $(D, R, [0, 1])$ in the notation of [8]. Namely for every $\lambda \in \mathbf{R}$ there exists a $t \in [0, 1]$ such that $\lambda \in \sigma(D + tR)$.

Another application is very similar to the ideas of [17]. Let $\omega_M \in H_c^n(M, \mathbf{Q})$ be the dual of the fundamental cycle. We define the Riemannian invariant for odd-dimensional spin manifolds M

$$G(M)^2 := \inf \sup_M tr(W^* D_V W - D_V)^2 \tag{6}$$

where the infimum is taken over all trivial Hermitian vector bundles with the flat connection V and gauge transformations W of V satisfying Assumption

1 and $\mathbf{CS}(W) = c\omega_M$ with some $c \neq 0$. Here D is the Dirac operator associated to the spinor bundle $S(M)$ and D_V is associated to the tensor product $S(M) \otimes V$. We find out that $G(M)$ estimates the length of gaps in the spectrum of D from above.

On one hand if $G(M) = 0$ then we have $\sigma(D) = \mathbf{R}$. By geometric arguments we can show that $G(M) = 0$ if M is simply connected and has nonpositive sectional curvature. Hence in this case there is no gap in the spectrum of the Dirac operator.

On the other hand, if there is a gap of length l in $\sigma(D)$ then $G(M) \geq l$. In particular, if M has nonnegative scalar curvature s then

$$G(M)^2 \geq \frac{n\, s_0}{n-1} \tag{7}$$

where $n := dim\ M$ is odd and $s_0 = \inf_{m \in M} s(m)$. Here we employ a lower bound of the spectrum of D^2 obtained by the method of [13]. On compact manifolds there holds $G(M) > 0$ since the spectrum of D is pure point and has, of course, gaps. It seems interesting to study the invariant $G(M)$ in more detail.

For completeness we will discuss the even-dimensional case simultaneously.

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2 Generalized Dirac operators

Let (M, g) be a complete Riemannian manifold and $E \to M$ be a Clifford bundle, i.e. E is a Hermitian vector bundle together with a compatible connection ∇^E and a Clifford multiplication $TM \otimes E \to E$ such that

1. $XX\phi = -|X|^2\phi \quad \forall m \in M, X \in T_mM, \phi \in E_m$
2. $< X\phi, \psi > = - < \phi, X\psi > \quad \forall m \in M, X \in T_mM, \psi, \phi \in E_m$
3. $\nabla^E_X(Y\psi) = (\nabla_X Y)\psi + Y\nabla^E_X\psi \quad \forall X, Y \in \Gamma(M, TM), \psi \in \Gamma(M, E)$

where ∇ is the Levi-Civita connection. The Dirac operator $D : \Gamma(M, E) \to \Gamma(M, E)$ is the first order elliptic differential operator given by the composition

$$\Gamma(M,E) \xrightarrow{\nabla^E} \Gamma(M, T^*M \otimes E) \xrightarrow{g} \Gamma(M, TM \otimes E) \to \Gamma(M, E) \tag{8}$$

where g is the identification $TM \cong T^*M$ by the Riemannian metric and the last map is induced by the Clifford multiplication. Standard examples are $d + \delta$ associated to $E := \Lambda^* T^* M$ and the Dirac operator associated to the spinor bundle on a Riemannian spin manifold. If E is a Clifford bundle and $V \to M$ a Hermitian vector bundle with compatible connection ∇^V then we can build a new Clifford bundle $E \otimes V$ equiped with the product connection. The Clifford multiplication is extended trivially to the second factor.

Let us consider the Hilbert space $L^2(M, E)$. The Dirac operator D is essentially selfadjoint on the domain $dom\ D = C_c^\infty(M, E)$, i.e. $\bar{D} = D^*$ is the unique selfadjoint extension. By $H^1(M, E)$ we denote the Sobolev space $dom\ \bar{D}$ with the norm $\|\psi\|^2_{H^1} = \|\psi\|^2 + \|\bar{D}\psi\|^2$.

3 Perturbation theory

Let $R \in C_c^\infty(M, End(E))$ be a selfadjoint bundle endomorphism of E. We consider the family $D_t = D + tR$, $t \in \mathbf{R}$. Then D_t is essentially selfadjoint on $dom\ D_t = C_c^\infty(M, E)$. By the decomposition principle [11], [9] we have $\sigma_{ess}(D_t) = \sigma_{ess}(D)$, $\forall t \in \mathbf{R}$.

Let us fix $d \in res(D_t)$ and let $P(t) := E_{D_t}[d, \infty)$ be the spectral projection of D_t on the interval $[d, \infty)$.

Lemma 3.1 *For $\phi \in dom\ \bar{D}_t$ there holds*

$$\begin{aligned} P(t)\phi &= \frac{\phi}{2} + \lim_{S\to\infty} \frac{1}{2\pi} \int_{-S}^{S} (D_t - d + \imath\lambda)^{-1} \phi\, d\lambda & (9) \\ &= \frac{\phi}{2} + \frac{1}{\pi} \int_0^\infty (D_t - d)[(D_t - d)^2 + \lambda^2]^{-1} \phi\, d\lambda. & (10) \end{aligned}$$

The integrals are strongly convergent.

Proof: The assertion is a consequence of the identity

$$\theta(x) = \frac{1}{2} + \lim_{S\to\infty} \frac{1}{2\pi} \int_{-S}^{S} (x + \imath\lambda)^{-1} d\lambda \quad \text{if } x \neq 0 \tag{11}$$

and of the spectral theorem. □

Lemma 3.2 *1. Let $d \in res(D_t) \cap res(D_s)$. Then $P(t) - P(s)$ is compact.*

2. If $d \in \cap_{t\in[a,b]} res(D_t)$ then the function $t \in [a,b] \to P(t)$ is operator norm continuous.

Proof: We have

$$P(t) - P(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[(D_t - d + \imath\lambda)^{-1} - (D_s - d + \imath\lambda)^{-1}\right] d\lambda \tag{12}$$

where the integral converges in the operator norm. In fact by the resolvent identity

$$(D_t - d + \imath\lambda)^{-1} - (D_s - d + \imath\lambda)^{-1} = (D_t - d + \imath\lambda)^{-1}(s-t)R(D_s - d + \imath\lambda)^{-1}. \tag{13}$$

Hence the norm of the integrand is bounded by $C|s-t|\|R\|(1+|\lambda|)^{-2}$. Moreover, by Rellich's theorem the composition

$$L^2(M,E) \stackrel{(D_s-d+\imath\lambda)^{-1}}{\longrightarrow} H^1(M,E) \stackrel{R}{\to} L^2(M,E) \tag{14}$$

is compact. This proves the first assertion. The second follows from the estimate

$$\|P(t) - P(s)\| \le C|s-t|. \tag{15}$$

□

Let $d \in res(D)$ and $P := P(0)$. If M is compact then P is a pseudodifferential operator of zero order. Hence it can be localized near the diagonal modulo compact operators. The next proposition shows that this remains true in some sense also for complete M. Let $K_1 \subset K \subset M$ be compact subsets such that K contains a neighbourhood of K_1 and $\chi, \chi_1 \in C^\infty(M)$, $supp(\chi_1) \subset K_1$, $supp(\chi) \subset M \setminus K$.

Lemma 3.3 *The composition $\chi_1 P \chi$ is compact.*

Proof: Using $\chi_1\chi = 0$, the second representation of P in Lemma 3.1 and

$$(D-d)[(D-d)^2 + \lambda^2]^{-1} = \int_0^\infty (D-d)e^{-t[(D-d)^2+\lambda^2]}dt \tag{16}$$

we find

$$\chi_1 P\chi = \frac{1}{\pi}\left(\int\int_{I_1} + \int\int_{I_2}\right)\chi_1(D-d)e^{-t[(D-d)^2+\lambda^2]}\chi dt d\lambda \tag{17}$$

where

$$\begin{aligned} I_1 &:= \{(t,\lambda)\in[1,\infty)\times[0,\infty)\} && (18)\\ I_2 &:= \{(t,\lambda)\in[0,1]\times[0,\infty)\}. && (19)\end{aligned}$$

There are constants $C_1, C_3 < \infty$, $c_2, c_4 > 0$ such that

$$\begin{aligned} \|\chi_1(D-d)e^{-t[(D-d)^2+\lambda^2]}\chi\| &\leq C_1 e^{-c_2 t[1+\lambda^2]} \quad \forall(t,\lambda)\in I_1 && (20)\\ \|\chi_1(D-d)e^{-t[(D-d)^2+\lambda^2]}\chi\| &\leq C_3 e^{-t\lambda}e^{-c_4/t} \quad \forall(t,\lambda)\in I_2. && (21)\end{aligned}$$

The last esimate is an off-diagonal estimate obtained by the finite propagation speed method of [7]. Thus (17) converges with respect to the operator norm. Since the heat operator is smoothing and χ_1 restricts to a compact set, the integrand is compact by Rellich's theorem. This proves the Lemma. □.

Let $\chi_2 \in C_c^\infty(M)$ such that $supp\chi_2 \subset K$. Then using a parametrix construction one can find a pseudodifferential operator (in the local sense) A approximating $\chi_2 P\chi$ modulo compact operators. For the principal symbol of A we get $\sigma_A(x,\xi) = \chi_2(x)p(x,\xi)\chi(x)$ where $p(x,\xi)$ is the projection onto the positive spectral subspace of the Clifford multiplication with $\imath\xi$. Note that it does not depend on d.

4 Relative index of projections

Lemma 3.2 indicates when the difference $P(t) - P(s)$ of projections is compact. This is exactly the situation where one can define the relative index [2],[16].

Let P, Q be projections on a Hilbert space H such that $P - Q$ is compact. We define the relative index of P and Q by

$$I(P,Q) := ind_{ImQ\to ImP}PQ. \tag{22}$$

In order to verify that PQ is Fredholm (from $Im\, Q$ to $Im\, P$) we give parametrices. Modulo compact operators

$$PQQP = PQP \sim PPP = P \tag{23}$$

$$QPPQ = QPQ \sim QQQ = Q. \tag{24}$$

Lemma 4.1 *Let $Q(t)$ be a norm continuous path of projections such that $P - Q(t)$ is compact for $t \in [0,1]$. Then $I(P,Q(0)) = I(P,Q(1))$.*

Proof: Note that the domain of $PQ(t)$ depends on t. Thus the argument is not standard.

By [4], 4.3.3, there is a norm continuous family $U(t)$ of unitaries such that

$$U^*(t)Q(t)U(t) = Q(0). \tag{25}$$

Then

$$I(P,Q(t)) = ind_{ImU(t)Q(0)\to ImP} PU(t)Q(0)U(t)^* = ind_{ImQ(0)\to ImP} PU(t)Q(0). \tag{26}$$

But now $t \to PU(t)Q(0)$ is a norm continuous family of Fredholm operators between fixed subspaces of H. Hence $I(P,Q(t))$ does not depend on t. This proves the Lemma. □

Let R be a finite projection, $dim\ R = n$, $QR = 0$.

Lemma 4.2 $I(P,Q+R) = I(P,Q) + n$

Proof: If $RP = 0$ then the assertion is obvious. For the general case we use the double $H \oplus H$ and extend R, Q, P by 0 on the second factor. Let

$$R(t) = \begin{pmatrix} cos(t) & sin(t) \\ -sin(t) & cos(t) \end{pmatrix} \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix}. \tag{27}$$

Then $QR(t) = 0$, $PR(\pi/2) = 0$ and $I(P, Q + R(0)) = I(P, Q + R(\pi/2)) = I(P,Q) + n$ by Lemma 4.1. □

Lemma 4.3 *Let P, Q be related by an unitary W,i.e $P = W^*QW$. Then $I(P,Q) = ind(1 + (W-1)Q)$.*

Proof: Note that $[Q,W] = W(P-Q)$ is compact. Hence

$$\begin{aligned} I(P,Q) &= ind_{ImQ\to ImP} PQ \\ &= ind_{ImQ\to ImW^*Q} W^*QWQ \\ &= ind_{ImQ\to ImQ} QWQ \\ &= ind(1 - Q + QWQ) \\ &= ind(1 + (W-1)Q + [Q,W]Q) \\ &= ind(1 + (W-1)Q) \end{aligned} \tag{28}$$

□

5 Spectral flow

Let us return to families of Dirac operators. Let W be a gauge transformation satisfying Assumption 1, $R := W^*DW - D$ and $D_t := D + tR$. Then D_t satisfies the assumptions made in section 3. We are going to introduce the spectral flow of $\{D_t\}_{t=0}^1$.

First note that D_t is a holomorphic family of type (A) in the sense of [15], i.e. $dom\ \bar{D}_t = dom\ \bar{D}$, $\forall t \in \mathbf{R}$, and $t \to D_t\psi$ is holomorphic for all $\psi \in dom\ \bar{D}$. Let $[a,b] \subset \mathbf{R} \setminus \sigma_{ess}(D)$ be a gap in the essential spectrum and choose $d \in [a,b] \cap res\ D$ and $\epsilon > 0$ such that $[d-\epsilon, d+\epsilon] \subset res\ D$. Then the eigenvalues $\lambda_\alpha(t) \in [d-\epsilon, d+\epsilon]$, $\alpha \in J$ (J beeing some index set), form holomorphic families with Lipschitz constant bounded by $\|R\|$. It is easy to see that J must be finite. Every familiy is defined on a maximal interval $[s_\alpha, t_\alpha]$. The spectral flow of $\{D_t\}_{t=0}^1$ is defined by

$$sf(D_t) = \sharp\{\alpha \in J | \lambda_\alpha(s_\alpha) > d, \lambda_\alpha(t_\alpha) < d\} - \sharp\{\alpha \in J | \lambda_\alpha(s_\alpha) < d, \lambda_\alpha(t_\alpha) > d\} \tag{29}$$

where the families are counted according to their multiplicities. The main result of this section is

Proposition 5.1 $sf\{D_t\} = ind(1 + (W-1)P)$

Proof: Choose a finite partition of $[0,1]$:

$$0 = s_0 < t_0 = s_1 < t_1 \ldots = s_N < t_N = 1 \tag{30}$$

and d_i, $i = 0, \ldots, N$, such that $d_i \in res\ D_t$, $\forall t \in [s_i, t_i]$, $d_0 = d_N = d$. Let $P(t) = E_{D_t}[d_i, \infty)$ for $t \in [s_i, t_i)$. Then by Lemma 3.2 $P(t)$ is continuous on every interval $[s_i, t_i)$ and

$$\lim_{t \to t_i} P(t) = P(s_{i+1}) + sign(d_{i+1} - d_i) R_{i+1} \tag{31}$$

where R_{i+1} is the finite projection onto the eigenspaces according to the eigenvalues of $D_{s_{i+1}}$ between d_i and d_{i+1}. Hence $R_{i+1}P(s_{i+1}) = 0$. Moreover $P(t) - P(1)$ is compact since there is no essential spectrum in $[a,b]$. Hence by Lemma 4.2

$$I(P(1), P(s_i)) = I(P(1), P(s_{i+1})) + sign(d_{i+1} - d_i) dim\ R_{i+1}. \tag{32}$$

Summing up it follows $I(P(1), P(0)) = sf\{D_t\}$. Finally apply Lemma 4.3. $\square$

6 Computation of the spectral flow

Let D_t be as in the last section. We have shown that the spectral flow at d is given by

$$sf\{D_t\} = ind(1 + (W-1)P) \tag{33}$$

where $P = E_D[d,\infty)$. We want to compute this index along the lines of [6]. Let $K_1 = supp(W_1)$ and $K \subset M$ be compact containing a neigbourhood of K_1. Choose $\chi \in C_c^\infty(M)$ such that $\chi_{|K} = 1$.

Lemma 6.1 *Let $\tilde{A} := (W-1)P\chi + 1$. Then $\tilde{A} - (1 + (W-1)P)$ is compact and $sf\{D_t\} = ind\,\tilde{A}$.*

Proof: We have

$$(1 + (W-1)P) - \tilde{A} = (W-1)P(1-\chi). \tag{34}$$

Since $(W-1)(1-\chi) = 0$ we can apply Lemma 3.3. □

$\tilde{A}$ has a pseudodifferential approximation A modulo compact operators as described at the end of section 3. A is an elliptic operator with constant symbol at infinity. In [6] the index of such operators was computed.

Theorem 6.2 *Let M be a complete Riemannian spin manifold, V be a Hermitian vector bundle with compatible connection and D_V be the twisted Dirac operator on $S(M)\otimes V$. Moreover let W be a gauge transformation of V with $supp(1-W)$ compact and $D_t := (1-t)D_V + tW^*D_VW$. Choose $d \in res(D_V)$ (i.e. d is in a gap of $\sigma_{ess}(D_V)$). Then the spectral flow of D_t at d is 0 if $dim\,M$ is even and*

$$sf\{D_t\} = (-1)^k \hat{\mathbf{A}}(M)\mathbf{CS}(W)[M] \tag{35}$$

if $dim\,M = 2k+1$ where $\hat{\mathbf{A}}(M)$ is the $\hat{A}$-genus, $[M]$ is the fundamental cycle of M and $\mathbf{CS}(W) \in H_c^{odd}(M,\mathbf{Q})$ is represented by

$$CS(W) := \frac{\imath}{2\pi} Tr\, W^*\nabla W \int_0^1 exp\left(\frac{\imath(t-t^2)}{2\pi} W^*\nabla W W^*\nabla W\right) dt \tag{36}$$

$$CS(W)_{2r-1} = 2\frac{\imath^r}{(2\pi)^r}\frac{r!}{(2r)!} Tr([W^*\nabla W]^{2r-1}). \tag{37}$$

Proof: Use the formula for the index of A given in [6]. □

7 Application to spectral theory

Let M be a complete Riemannian spin manifold with the spinor bundle $S(M)$ and $V := M \times \mathbf{C}^N$ be the flat Hermitian vector bundle. The twisted Dirac operator D_V is isomorphic to the direct sum of N copies of the Dirac operator D associated to $S(M)$. A gauge transformation $W \in C_c^\infty(M, U(V))$ represents an element $[W] \in K_1(C_0(M)) \cong K_c^1(M)$. Here $K_1(C_0(M))$ is the K_1-group of the algebra of continuous functions vanishing at infinity and $K_c^1(M)$ is the K-theory with compact support. Note that the Chern-Simons class gives an isomorphism of groups

$$K_c^1(M)_{\mathbf{Q}} \overset{CS}{\to} H_c^{odd}(M, \mathbf{Q}). \tag{38}$$

Let $dim\, M = 2r-1$ be odd. Then there is a class $\omega_M \in H_c^{2r-1}(M, \mathbf{Q})$ dual to $[M]$ and there is a class $[W] \in K_c^1(M)$ with $\mathbf{CS}([W]) = l\omega_M$ for some $l \neq 0$. Moreover there is a $N \in \mathbf{N}$ such that $[W]$ is represented by a gauge transformation of the bundle V. We introduce the following Riemannian invariant: Let $R(W) := W^* D_V W - D_V \in \Gamma(M, End(S(M) \otimes V))$. Then we define

$$G(M)^2 := \inf \sup_M trR(W)^2 \tag{39}$$

where the infimum is taken over all trivial Hermitian vector bundles V with the flat connection and gauge transformations satisfying Assumption 1 with $\mathbf{CS}(W) = l\omega_M$ for some $l \neq 0$. We have $\|R(W)\|^2 = \|R^2(W)\| = \sup_M |R(W)^2| \leq \sup_M trR(W)^2$.

Theorem 7.1 *Let $dim\, M$ be odd. $G(M)$ is an upper bound of the length of gaps in the spectrum of D.*

Proof: Note that the spectrum of D_V coincides (as a set) with the spectrum of D. For every $\epsilon > 0$ there is a family $\{D_t\}_{t=0}^1$ with $D_0 = D_V$, having nonvanishing spectral flow and $\|D_1 - D_0\| \leq G(M) + \epsilon$. Assume that there is an interval $[a, b] \in \mathbf{R}$ with $\sigma(D_V) \cap [a, b] = \emptyset$ and $b - a > G(M)$. According to the sign of the spectral flow there is at least one family of eigenvalues $\lambda(t)$ crossing a from below or b from above and not crossing back the same boundary. But then for ϵ small enough $\lambda(1) \in [a, b]$ and $\lambda(1) \in \sigma(D_V)$. This is a contradiction. $\Box$.

Corollary 7.2 *If $G(M) = 0$ then $\sigma_{ess}(D) = \mathbf{R}$.*

Let us now discuss the even-dimensional analog of $G(M)$. On even-dimensional M one employs the relative index theorem [14],[10] and certain splittings of trivial bundles of compact support. Let $V := M \times \mathbf{C}^N$ be a trivial Hermitian vector bundle. A splitting of compact support is a decomposition $V := U \oplus W$ where W, U are Hermitian vector bundles with compatible connections beeing trivial and with the flat connection outside of a compact subset of M. Then $\mathbf{ch}(W), \mathbf{ch}(U) \in H_c^{ev}(M, \mathbf{Q})$. In this case we set $R := D_V - D_U \oplus D_W \in \Gamma(M, End(S(M) \otimes V))$ and define

$$G(M)^2 := \inf \sup_M \frac{1}{2} trR^2 \tag{40}$$

where the infimum is taken over all V and splittings $U \oplus W$ of compact support such that $\mathbf{ch}(U) = l\omega_M$ for some $l \neq 0$. Such splittings exist. Again $\|R\|^2 \leq \frac{1}{2} trR^2$ (see [3]).

Theorem 7.3 *If $dim\, M$ is even then $[-G(M), G(M)] \cap \sigma(D) \neq \emptyset$.*

Proof: Apply the relative index theorem. The argument is similar to those used in [17],[3],[5]. □

In fact the relative index theorem for Dirac operators [14] has been introduced for a similar application. The result in the even-dimensional case is much weaker than that in the odd-dimensional case since only the gap at zero can be estimated.

8 The invariant $G(M)$

In this section we study the invariant $G(M)$ in more detail. Let M be a complete spin manifold.

Lemma 8.1 *If M^n has nonnegative scalar curvature s then*

$$G(M)^2 \geq \frac{s_0 n}{(n-1)} \quad \text{if } n \text{ is odd} \tag{41}$$

$$G(M)^2 \geq \frac{s_0 n}{4(n-1)} \quad \text{if } n \text{ is even} \tag{42}$$

where $s_0 := \inf_{m \in M} s(m)$.

Proof: Using the Weizenböck formula of [13] it is easy to show that $\sigma(D^2) \subset [\frac{s_0 n}{4(n-1)}, \infty)$. The Lemma follows from Theorem 7.1 and 7.3. □.

Of course for M compact $G(M) > 0$ in any case.

Now we want to find upper bounds for $G(M)$ in geometric terms. We say that a Riemannian manifold (M^n, g) dominates another Riemannian manifold $(\tilde{M}^n, \tilde{g})$ if there is a map $f : M \to \tilde{M}$ of nonvanishing degree such that $g \geq f^*\tilde{g}$. If $\tilde{M}$ is compact then f has to be constant outside of a compact subset of M. If $\tilde{M}$ is noncompact then we require f to be proper.

Lemma 8.2 *If M^n dominates $\tilde{M}^n$ then $G(M) \leq G(\tilde{M})$.*

Proof: f is used to pull back the gauge transforma/tions in the odd-dimensional and the splittings in the even-dimensional case. We consider first the odd-dimensional case. A local computation shows that

$$R(W) = \sum_{i=1}^{n} \tilde{e}_i \otimes W^* dW(\tilde{e}_i) \tag{43}$$

where $\{\tilde{e}_i\}_{i=1}^n$ is a local orthonormal repere of $T\tilde{M}$. Hence

$$trR^2(W) = -2^{[n/2]} \sum_{i=1}^{n} trW^* dW(\tilde{e}_i) W^* dW(\tilde{e}_i) \tag{44}$$

where we have used $tr\tilde{e}_i\tilde{e}_j = -2^{[n/2]}\delta_{ij}$. Let $m \in M$ and $f(m) =: \tilde{m}$. Then at m

$$R(f^*W) = \sum_{i=1}^{n} e_i \otimes W^* dW(df(e_i)) \tag{45}$$

where $\{e_i\}_{i=1}^n$ is a local orthonormal repere of TM around m and W, dW is taken at $\tilde{m}$. Hence

$$trR^2(f^*W) = -2^{[n/2]} \sum_{i=1}^{n} trW^* dW(df(e_i)) W^* dW(df(e_i)). \tag{46}$$

Let $df(e_i) = A_{ij}\tilde{e}_j$ and $B_{ij} := -trW^*dW(\tilde{e}_i)W^*dW(\tilde{e}_j)$. The condition $f^*\tilde{g} \leq g$ translates to $AA^* \leq 1$ and hence $\|A\| \leq 1$. Moreover $B \geq 0$ since $(W^*dW(X))^* = -W^*dW(X)$ for all $X \in T\tilde{M}$. It follows

$$\begin{aligned} trR^2(f^*W) &= 2^{[n/2]} trABA^* \\ &\leq 2^{[n/2]} trB \\ &= trR^2(W). \end{aligned} \tag{47}$$

This proves the Lemma in the odd-dimensional case.

Now we consider the even-dimensional case. If $V = U \oplus W$ is a splitting and ∇^U, ∇^W are compatible connections we set $Q := \nabla^V - \nabla^U \oplus \nabla^W \in \Gamma(\tilde{M}, T^*\tilde{M} \otimes End(V))$. Locally

$$R := D_V - D_U \oplus D_W = \sum_{i=1}^{n} \tilde{e}_i \otimes Q(\tilde{e}_i). \tag{48}$$

To prove the Lemma we use the same argument as above replacing W^*dW by Q. Note that for the pulled back splitting one gets f^*Q and that $Q(X)^* = -Q(X)$ for all $X \in T\tilde{M}$. □

Lemma 8.3 *$G(S^n) \leq 2^{n/2}\sqrt{n}$ for n odd and $G(S^n) \leq 2^{\frac{n-3}{2}}\sqrt{n}$ for n even.*

Proof: If n is even this follows from [3]. If n is odd we construct explicitely a generator of $\pi_n(U(2^{(n+1)/2}))$. Let Δ^{n+2} be the spinor representation of $Spin(n+2)$. The Clifford algebra $Cliff(\mathbf{R}^{n+2})$ acts on Δ^{n+2} in the usual way. Let Δ^{n+1}_{+} be the $-\imath$-eigenspace of the endomorphism $N := (0, \ldots, 0, 1) \in \mathbf{R}^{n+2}$. We consider the embedding $S^n \subset S^{n+1}$ as the equator such that N is the north pole of S^{n+1}. Let $e_1 := (1, 0, \ldots, 0)$. Then the gauge transformation W is given by $\eta \in S^n \to W(\eta) := -\imath e_1 \eta \in U(\Delta^{n+1}_{+})$. One can show that $\mathbf{CS}(W)[S^n] = 1$. For this gauge transformation we have $tr(W^* D_V W - D_V)^2 = 2^n n$. □

For the sphere of sectional curvature K we have

$$G(S^n(K)) = \sqrt{K} G(S^n(1)).$$

It was shown e.g. in [3] that every complete Riemannian manifold dominates some sphere.

Lemma 8.4 *Let $K = \inf_{p \in M} K(p)$ with*

$$K(p) = \sup_{m \in B(p, r_{inj}(p))} K(m) \wedge \frac{\pi^2}{r^2_{inj}(p)} \tag{49}$$

where $r_{inj}(p)$ is the injectivity radius of M at $p \in M$ and $K(m)$ is the maximal sectional curvature at $m \in M$. There is a function $\alpha(\epsilon)$ with $\lim_{\epsilon \to 0} \alpha(\epsilon) = 0$ such that for every small $\epsilon > 0$ the manifold $(M^n, (1 + \alpha(\epsilon))g)$ dominates $S^n(K + \epsilon)$. Hence $G(M) \leq G(S^n(K)) \leq \sqrt{K} G(S^n)$.

Proof: The required maps $f_\epsilon : M \to S^n(K+\epsilon)$ are constructed in [3]. □

Corollary 8.5 *If M^n is a complete Riemannian spin manifold of odd dimension with nonpositive sectional curvature then $\frac{\pi}{r_0}2^{n/2}\sqrt{n}$ is an upper bound for the length of gaps in the spectrum of the Dirac operator. Here $r_o := \sup_{m\in M} r_{inj}(m)$. In particular if M has a pole (i.e. M is simply connected) then $\sigma(D) = \mathbf{R}$.*

Corollary 8.6 *If M^n is a complete Riemannian spin manifold of even dimension with nonpositive sectional curvature then there is spectrum of the Dirac operator D in the interval $[-\frac{\pi}{r_0}2^{\frac{n-3}{2}}\sqrt{n}, \frac{\pi}{r_0}2^{\frac{n-3}{2}}\sqrt{n}]$. If M has a pole then 0 is in the spectrum of D.*

In [14] manifolds which dominate spheres of arbitrary large radius are called hyperspherical. In that paper the property $0 \in \sigma(D)$ was employed as an obstruction against the existence of a metric of positive scalar curvature on hyperspherical manifolds.

9 The number of eigenvalues

In this section we will refine the results of the last section taking into account the actual value of the spectral flow or the index respectively. Let (M^n, g) be a Riemannian spin manifold, $U \subset M$ be some compact and $K(U) := \sup_{m\in U} K(m) \vee 0$ where $K(m)$ is the maximal sectional curvature at m. We set

$$r_0(U) := \inf_{p\in U} r_{inj}(p) \wedge \frac{\pi}{\sqrt{K(U)}} \tag{50}$$

and for $r \le r_0(U)$

$$N_U(r) := \{\text{maximum number of disjoint balls of radius } r \text{ in } U\}. \tag{51}$$

By the results of [3] there exist maps

$$f_\epsilon : M^n \to S^n((\pi/r)^2 + \epsilon) \tag{52}$$

of degree $N(r)$ with $|df| \le 1 + \alpha(\epsilon)$ where $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$. Pulling back gauge transformations and splittings respectively we obtain families of twisted Dirac operators the spectral flow of which is $N_U(r)$ in the odd-dimensional case and twisted Dirac operators with index $N_U(r)$ in the even-dimensional case. Setting

$$\mu(r) := \frac{\pi\sqrt{n}}{r}\left\{\begin{array}{ll} 2^{n/2} & \text{if } n \text{ is odd} \\ 2^{(n-3)/2} & \text{if } n \text{ is even} \end{array}\right\} \tag{53}$$

we have

$$\inf_{\epsilon\to 0} \|R_\epsilon\| \le \mu(r) \tag{54}$$

where $R_\epsilon := R(f_\epsilon^* W)$ if n is odd. In the even-dimensional case R_ϵ is given by (48) where the splitting is the pull-back by f_ϵ of the splitting of the spinor bundle of S^n (see [3]).

Theorem 9.1 *There are at least $\{N_U(r)/2^{\{\frac{n-1}{2}\}}\}$ eigenvalues of D in the interval $[-\mu(r), \mu(r)]$ as long as $r \le r_0(U)$ and $\mu(r) < \inf|\sigma_{ess}(D)|$. Here $\{x\}$ denotes the smallest integer greater or equal x.*

Proof: One applies essentially the same argument as for Theorem 7.1. Note that the dimension of the twisting trivial bundle is $dim\, V = 2^{[\frac{n-1}{2}]+1}$. □

Define

$$\begin{aligned} C(n) &:= \liminf_{r\to 0} \frac{\{\frac{N_U(r)}{2^{\{\frac{n-1}{2}\}}}\}}{Vol(U)\mu^n(r)} & (55)\\ &= \frac{\gamma_n}{Vol(B^n)n^{n/2}\pi^n}\left\{\begin{array}{ll} 2^{n(1-n/2)} & \text{if } n \text{ is even} \\ 2^{\frac{1-n^2-n}{2}} & \text{if } n \text{ is odd} \end{array}\right\} & (56)\end{aligned}$$

where γ_n is the optimal constant arising in the sphere-packing problem in dimension n and $Vol(B^n)$ is the volume of the unit ball in $\mathbf{R}^n$.

Corollary 9.2 *For every $\epsilon > 0$ there is a T_0 such that for all T satisfying $\inf|\sigma_{ess}(D)| > T \ge T_0$ holds*

$$\sharp\{\lambda \in \sigma(D) \mid |\lambda| \le T\} \ge Vol(U)(C(n) - \epsilon)T^n. \tag{57}$$

A similar result could have been obtained by the Dirichlet-comparison method. But we do not know a reference for explicite values of the constant. This Corollary is interesting since there are examples of Dirac operators on complete Riemannian spin manifolds without essential spectrum [12]. These manifolds may have finite as well as infinite volume. If $\sigma_{ess}(D) = \emptyset$ then

$$\liminf_{T\to 0} \frac{\sharp\{\lambda \in \sigma(D) \mid |\lambda| \le T\}}{T^n} \ge Vol(M)C(n). \tag{58}$$

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