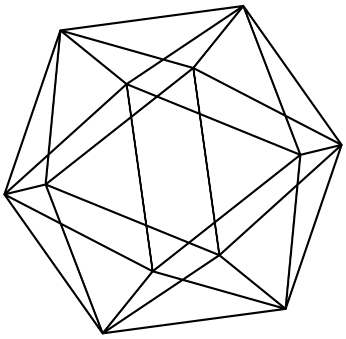
Max-Planck-Institut für Mathematik
Bonn

Partial holomorphic semiconjugacies between rational functions

by

Vladlen Timorin



Max-Planck-Institut für Mathematik
Preprint Series 2010 (95)

Partial holomorphic semiconjugacies between rational functions

Vladlen Timorin

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Faculty of Mathematics
Higher School of Economics
7 Vavilova St.
112312 Moscow
Russia

MPIM 10-95

PARTIAL HOLOMORPHIC SEMICONJUGACIES BETWEEN RATIONAL FUNCTIONS

V. TIMORIN

Abstract. We establish a general result on the existence of partially defined semiconjugacies between rational functions acting on the Riemann sphere. The semiconjugacies are defined on the complements to at most one-dimensional sets. They are holomorphic in a certain sense.

1. Introduction

Let $A \subseteq \mathbb{C}P^1$ be a subset of the Riemann sphere, not necessarily open. A map $\Phi : A \to \mathbb{C}P^1$ is said to be *holomorphic* if there is a sequence of holomorphic maps $\Phi_n : A_n \to \mathbb{C}P^1$ such that $A_n \supseteq A$ are open subsets of $\mathbb{C}P^1$, and Φ_n converge to Φ uniformly on A. If a map Φ is holomorphic in our sense, then it is continuous, its restriction to the interior of A is holomorphic in the usual sense, and probably not much more than that. E.g. there is the following theorem of Mergelyan (1951): let A be a compact subset of $\mathbb{C}$ such that $\mathbb{C} - A$ is connected. Then, every continuous function $f : A \to \mathbb{C}$ such that the restriction of f to the interior of A is holomorphic can be approximated uniformly on A with polynomials. Still, holomorphic maps on arbitrary subsets in $\mathbb{C}P^1$ as defined above provide a convenient language.

Recall that a *real semi-algebraic subset* in a real algebraic variety is a set given by any boolean combination of real algebraic equations and inequalities. The main result of this paper is the following

Main Theorem. *Suppose that $R : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a hyperbolic rational function with a finite postcritical set P_R, and $Q : \mathbb{C}P^1 \to \mathbb{C}P^1$ is a rational function such that the diagram*

$$\begin{tikzcd}
\mathbb{C}P^1 \arrow[r, "R"] \arrow[d, "\tilde{\eta}"'] & \mathbb{C}P^1 \arrow[d, "\eta"] \\
\mathbb{C}P^1 \arrow[r, "Q"] & \mathbb{C}P^1
\end{tikzcd}$$

Partially supported by the Deligne scholarship, RFBR grant 10-01-00540-a and MPIM research grant.

is commutative, where η *and* $\tilde{\eta}$ *are homeomorphisms that coincide on* $R(P_R)$ *and are isotopic relative to* $R(P_R)$*. If* P_R *has at least three points, then there exists a countable union* Z *of real semi-algebraic sets of dimension* ≤ 1 *backward invariant under* Q *and a holomorphic map* $\Phi : \mathbb{C}P^1 - Z \to \mathbb{C}P^1$ *such that* $R \circ \Phi = \Phi \circ Q$ *on* $\mathbb{C}P^1 - Z$ *and the restriction of* Φ *to some neighborhood of* $\eta(R(P_R))$ *conjugates* Q *with* R*.*

Recall that the postcritical set of a rational function $R : \mathbb{C}P^1 \to \mathbb{C}P^1$ is defined as the closure of the set $\{R^{\circ n}(c)\}$, where c runs through all critical points of R, and n runs through all positive integers. A rational function R is called *critically finite* if its postcritical set is finite. The relation between R and Q resembles Thurston equivalence (in which we require that η and $\tilde{\eta}$ coincide on P_R, map P_R onto the postcritical set of Q, and be isotopic relative to P_R) but is in fact much weaker. If Q has at least one superattracting cycle of period > 1, then, as a rule, there are infinitely many different functions R that satisfy the assumptions of the theorem. Note that the assumptions of the Main Theorem imply the existence of at least one super-attracting cycle of Q, namely, $\eta(C)$, where C is a super-attracting cycle of R in $R(P_R)$. The relation between R and Q can be stated in the language of A. Epstein's deformation spaces.

The map Φ from the Main Theorem semiconjugates the restriction of Q to $\mathbb{C}P^1 - Z$ with a certain restriction of R. Note that the set $\mathbb{C}P^1 - Z$ is forward invariant under Q since Z is backward invariant. The set Z can be constructed explicitly, and in many different ways. The Main Theorem is only useful in combination with the knowledge of what Z is. In fact, Z is very flexible and can be tailored to specific needs. Semi-algebraicity is only one possible application of this flexibility. We could have replaced semi-algebraicity with many other nice properties.

The main theorem is closely related to the *regluing surgery* of [T], although we will not use it explicitly. Many of the ideas used in this paper are inspired by works of M. Rees (see e.g. [R]). In Section 2, we briefly describe some particular applications of the Main Theorem, from which these relations may become clear. To prove the Main Theorem, we will use a version of Thurston's algorithm [DH93]. This version is described in Section 3. In Section 4, we introduce a relevant function space and prove the convergence of Thurston's algorithm in this function space. The main theorem will follow.

A bigger part of this work has been done during my visit at Max Planck Institute for Mathematics, Bonn (January–April 2010). I would like to thank the institute for providing inspiring working conditions.

2. SUPPORTIVE REAL SEMI-ALGEBRAIC SETS

Recall that the *support* of a homeomorphism $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$ is defined as the closure of the set of all points $x \in \mathbb{C}P^1$ such that $\sigma(x) \neq x$. Let P be a finite subset of $\mathbb{C}P^1$ and $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$ a homeomorphism. We say that a closed subset $Z_0 \subset \mathbb{C}P^1$ is *supportive* for (σ, P) if, for every open neighborhood U of Z_0, there exists a homeomorphism $\tilde{\sigma}$ with the following properties:

- the homeomorphisms σ and $\tilde{\sigma}$ coincide on P;
- they are isotopic relative to P;
- the support of $\tilde{\sigma}$ is contained in U.

Proposition 2.1. *For every orientation-preserving homeomorphism* $\sigma : \mathbb{C}P^1 \to \mathbb{C}P^1$ *and every finite set* $P \subset \mathbb{C}P^1$*, there exists a closed real semi-algebraic set of positive codimension supportive for* (σ, P)*.*

Proof. Consider a continuous one-parameter family σ_t of homeomorphisms connecting σ with the identity: $\sigma_0 = \sigma$, $\sigma_1 = id$. For every point $x \in P$ define a continuous path $\beta_x : [0,1] \to \mathbb{C}P^1$ by the formula $\beta_x(t) = \sigma_t(x)$. We can assume that the curves $\beta_x[0,1]$ are real semi-algebraic and the paths β_x are smooth. This can be arranged by a small perturbation of the family σ_t, e.g. using the Weierstrass approximation theorem.

Take any open neighborhood U of the closed real semi-algebraic set

$$Z_0 = \bigcup_{x \in P} \beta_x[0,1].$$

Define a vector field v_t on $\mathbb{C}P^1$ depending smoothly on t and having the following properties:

- at the point $\beta_x(t)$, the vector field v_t is equal to $d\beta_x(t)/dt$;
- $v_t = 0$ outside of a small neighborhood of $\beta_x(t)$ contained in U.

Let now g^t be the time $[0,t]$ flow of the non-autonomous differential equation $\dot{z}(t) = v_t$. Clearly, $g^t(\sigma(x)) = \beta_x(t)$ for every $x \in P$, and the support of g^t is contained in U.

Now define the following isotopy: $\tilde{\sigma}_t = (g^t)^{-1} \circ \sigma_t$, $t \in [0,1]$. We have $\tilde{\sigma}_0(x) = \sigma(x)$. On the other hand, the support of $\tilde{\sigma}_1 = (g_1)^{-1}$ is contained in U. Therefore, Z_0 is supportive for (σ, P). $\square$

Remark 2.2. It also follows from the proof of Proposition 2.1 that there exists a continuous one-parameter family of homeomorphisms connecting $\tilde{\sigma}_1$ with the identity such that the supports of all these homeomorphisms are contained in U. To obtain such a family, we can apply the procedure, described in the proof of Proposition 2.1, to homeomorphisms σ_t rather than σ (reparameterize the family $t' \mapsto \sigma_{t'}$, $t' \leq t$, so that the new parameter runs from 0 to 1, and apply the argument of Proposition 2.1 to the new family, with parameter t replaced by the new parameter everywhere).

Proposition 2.3. *Let rational functions Q and R be as in the statement of the Main Theorem. Set $\sigma = \tilde{\eta} \circ \eta^{-1}$, so that $\sigma \circ Q = \tilde{\eta} \circ R \circ \tilde{\eta}^{-1}$. There exists a closed real semi-algebraic set Z_0 of dimension ≤ 1 that is supportive for $(\sigma, \eta(P_R))$ and that is disjoint from $\eta(R(P_R))$.*

Proof. Note that $\sigma = id$ on $\eta(R(P_R))$, and σ is homotopic to the identity relative to the set $\eta(R(P_R))$, by the assumptions of the Main Theorem. In the proof of Proposition 2.1, we can therefore assume that $\sigma_t(x) = x$ for all $x \in \eta(R(P_R))$ and all $t \in [0, 1]$. For these x, we do not consider the curves β_x. The rest of the proof works as before. ☐

The main theorem will follow from

Theorem 2.4. *Suppose that Q and R are rational functions as in the statement of the Main Theorem, and Z_0 is the set from Proposition 2.3. Define the set*

$$Z = \bigcup_{n=1}^{\infty} Q^{-n}(Z_0).$$

There exists a holomorphic map $\Phi : \mathbb{C}P^1 - Z \to \mathbb{C}P^1$ such that $R \circ \Phi = \Phi \circ Q$ on $\mathbb{C}P^1 - Z$.

Clearly, Z is a countable union of real semi-algebraic sets of positive codimension, namely, of the iterated preimages of Z_0 under Q.

We now describe some particular applications of Theorem 2.4. Let $\mathcal{R}_2$ be the set of Möbius conjugacy classes of quadratic rational functions with marked critical points. Following M. Rees [R] and J. Milnor [M], consider the slice $Per_k(0) \subset \mathcal{R}_2$ defined by the condition that the second critical point is periodic of period k. The slices $Per_k(0)$ form a natural sequence of parameter curves starting with $Per_1(0)$, the plane of quadratic polynomials. We say that a critically finite rational function $R \in Per_k(0)$ is of *type C* if the first critical point is eventually mapped to the second (periodic) critical point but does not belong to the cycle of the second critical point. Let $R \in Per_k(0)$ be any type C critically finite rational function, and $Q \in Per_k(0)$ be almost any function. There are only finitely many exceptions, and all exceptional maps are critically finite. Then, as M.Rees has shown in [R], there is a homeomorphism $\sigma_\beta : \mathbb{C}P^1 \to \mathbb{C}P^1$, whose support is contained in an arbitrarily small neighborhood of a simple path $\beta : [0, 1] \to \mathbb{C}P^1$ such that $\sigma_\beta(\beta(0)) = \beta(1)$, and $\sigma_\beta \circ Q$ is a critically finite branched covering Thurston equivalent to R. It is a simple exercise to check that

Q and R satisfy the assumptions of the Main Theorem, and that we can take $Z_0 = \beta[0,1]$. In this way, we obtain a partial semiconjugacy between almost any function from $Per_k(0)$ and any type C critically finite function. It will be defined on the complement to all pullbacks of the simple curve Z_0 under Q. E.g. for Q we can take a quadratic polynomial $z \mapsto z^2 + c$, whose critical point 0 is periodic of period k. Then the Main Theorem implies, in particular, the topological models for *captures* of Q introduced in [R].

3. THURSTON'S ALGORITHM

In the proof of Theorem 2.4, we will use *Thurston's algorithm* (see [DH93]). We now briefly recall how it works (in a slightly more general setting than usual). Let X be a topological space and $f : X \to X$ be a continuous map. Suppose that there is a topological semiconjugacy between f and a rational function acting on the Riemann sphere. Thurston's algorithm serves to find this semi-conjugacy. It starts with a surjective continuous map $\phi_0 : X \to \mathbb{C}P^1$. Assume that there is a rational function R_0 and a continuous map $\phi_1 : X \to \mathbb{C}P^1$ that make the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\phi_1"'] & X \arrow[d, "\phi_0"] \\
\mathbb{C}P^1 \arrow[r, "R_0"] & \mathbb{C}P^1
\end{tikzcd}$$

This is always the case if the map ϕ_0 is a homeomorphism (in particular, X is a topological sphere) and f a branched covering. Indeed, we can arrange that f and ϕ_0 be smooth by small deformations preserving the critical values of $\phi_0 \circ f$. Then we consider the pullback κ of the complex structure on $\mathbb{C}P^1$ under the map $\phi_0 \circ f$. We can integrate κ, i.e. there is a homeomorphism $\phi_1 : X \to \mathbb{C}P^1$ taking the complex structure κ on X to the standard complex structure on $\mathbb{C}P^1$. Clearly, $R_0 = \phi_0 \circ f \circ \phi_1^{-1}$ preserves the standard complex structure, hence it is a rational function (provided it preserves the orientation; this can be always arranged, perhaps replacing ϕ_1 with $\overline{\phi_1}$). If f or ϕ_0 were not smooth, then R_0 constructed for smooth deformations of f and ϕ_0 will also work for f and ϕ_0, i.e. ϕ_1 can be defined as a branch of $R_0^{-1}(\phi_0 \circ f)$. Note that R_0 is only defined up to precomposition with an automorphism of $\mathbb{C}P^1$, and ϕ_1 is only defined up to post-composition with an automorphism of $\mathbb{C}P^1$.

The transition from ϕ_0 to ϕ_1 is the main step of Thurston's algorithm. Doing this step repeatedly, we obtain a sequence of maps ϕ_n. We want that ϕ_n converge to a semiconjugacy between f and some rational function.

We will now use notation from Proposition 2.3 and Theorem 2.4. Let us consider Thurston's algorithm for $\tilde{\sigma}\circ Q$, where $\tilde{\sigma}$ is a homeomorphism isotopic to σ relative to the set $\eta(P_R)$. Note that the branched covering $\tilde{\sigma}\circ Q$ is Thurston equivalent to R, and $P=\tilde{\eta}(P_R)$ is the postcritical set of this branched covering. Indeed, all critical values of Q are contained in $\eta(P_R)$ (note that the critical points of R must map to the critical points of Q under $\tilde{\eta}$, and to the critical values of Q under $Q\circ\tilde{\eta}=\eta\circ R$); the images of these critical values under $\tilde{\sigma}$ are contained in $\tilde{\eta}(P_R)=\tilde{\sigma}\circ\eta(P_R)$; the further images under $\tilde{\sigma}\circ Q$ are contained in $P'=\eta(R(P_R))$ because the action of $\tilde{\sigma}\circ Q$ on $\tilde{\eta}(P_R)$ coincides with the action of Q, and $Q(\tilde{\eta}(P_R))=P'$. By Proposition 2.3, we can assume that the support of $\tilde{\sigma}$ is contained in an arbitrarily small neighborhood U of Z_0. We choose this neighborhood so that it is disjoint from the set P'.

Set $\hat{f}=\tilde{\sigma}\circ Q$. Thurston's algorithm for $\hat{f}$ yields an infinite commutative diagram

$$\begin{tikzcd}
 & {} \arrow[r] & \mathbb{C}P^1 \arrow[r, "\hat{f}"] \arrow[d, "\hat{\phi}_3"'] & \mathbb{C}P^1 \arrow[r, "\hat{f}"] \arrow[d, "\hat{\phi}_2"'] & \mathbb{C}P^1 \arrow[r, "\hat{f}"] \arrow[d, "\hat{\phi}_1"'] & \mathbb{C}P^1 \arrow[d, "\hat{\phi}_0"] \\
\dots & {} \arrow[r] & \mathbb{C}P^1 \arrow[r, "R_2"] & \mathbb{C}P^1 \arrow[r, "R_1"] & \mathbb{C}P^1 \arrow[r, "R_0"] & \mathbb{C}P^1
\end{tikzcd}$$

We can set $\hat{\phi}_0=id$. The classes of $\hat{\phi}_n$ in the Teichmüller space of $(\mathbb{C}P^1,P)$ are well defined. They depend only on the Thurston equivalence class of $\hat{f}$ and not on a particular choice of the homeomorphism $\tilde{\sigma}$. However, the maps $\hat{\phi}_n$ are only defined up to post-composition with conformal automorphisms of $\mathbb{C}P^1$. To make a definite choice of $\hat{\phi}_n$, we introduce the following normalization. Let P_0 be any 3-point subset of P. Note that the sets $Q^{\circ n}(P_0)$ are disjoint from U for all $n>0$ since they lie in P'. We require that the restriction of every $\hat{\phi}_n$ to P_0 be the identity. This normalization makes the maps $\hat{\phi}_n$ uniquely defined. However, the maps $\hat{\phi}_n$ depend on the choice of $\tilde{\sigma}$. The rational functions R_n are uniquely defined by the classes of $\hat{\phi}_n$ in the Teichmüller space of $(\mathbb{C}P^1,P)$ and the normalization $\hat{\phi}_n|_{P_0}=id$. Therefore, they do not depend on the choice of $\tilde{\sigma}$.

Proposition 3.1. *Set U_n to be the union of $Q^{-i}(U)$ for $i=1,\dots,n$. The values of $\hat{\phi}_n$ at points $z\notin U_n$ do not depend on a particular choice of a homeomorphism $\tilde{\sigma}$ with support in U.*

Proof. Indeed, different homeomorphisms $\tilde{\sigma}_0$ and $\tilde{\sigma}_1$ are isotopic relative to P. Let $\tilde{\sigma}_t$, $t \in [0,1]$ be an isotopy. We can assume that the support of $\tilde{\sigma}_t$ is contained in U for every t, see Remark 2.1. Let $\hat{\phi}_{n,t}$ be the maps that correspond to $\hat{\phi}_n$ as we replace $\tilde{\sigma}$ with $\tilde{\sigma}_t$. Take $z \notin U_n$. Suppose by induction that $\hat{\phi}_{n-1,t}(\hat{f}(z))$ does not depend on t (note that $\hat{f}(z) = Q(z) \notin U_{n-1}$). Then $\hat{\phi}_{n,t}(z)$ is a continuous path such that $R_{n-1} \circ \hat{\phi}_{n,t}(z) = \hat{\phi}_{n-1}(z) \circ \hat{f}$. Hence the values of this path lie in the finite set $R^{-1}_{n-1}(\hat{\phi}_{n-1}(z) \circ \hat{f})$. It follows that the path is constant. ☐

We can include the maps $\hat{\phi}_n$ into a continuous family of homeomorphisms $\hat{\phi}_t : \mathbb{C}P^1 \to \mathbb{C}P^1$ defined for all real non-negative values of t. This is done in the following way. By Remark 2.2, there is a continuous one-parameter family of homeomorphisms $\tilde{\sigma}_t$, $t \in [0,1]$ connecting id with $\tilde{\sigma}$ such that the supports of all $\tilde{\sigma}_t$ are contained in an arbitrarily small neighborhood U of Z_0, and $\tilde{\sigma}_t(\eta(P_R - R(P_R))) \subset Z_0$ (we use the notation of Proposition 2.3). Consider the first step of Thurston's algorithm for $\hat{f}_t = \tilde{\sigma}_t \circ Q$:

$$\begin{tikzcd}
\mathbb{C}P^1 \arrow[r, "\hat{f}_t"] \arrow[d, "\hat{\phi}_t"'] & \mathbb{C}P^1 \arrow[d, "id"] \\
\mathbb{C}P^1 \arrow[r, "R_{t-1}"] & \mathbb{C}P^1
\end{tikzcd}$$

Note that the critical values of $\hat{f}_t$ (hence also of R_{t-1}) different from the critical values of Q lie in Z_0. Normalize $\hat{\phi}_t$ by requiring that their restrictions to P_0 be the identity. Note that, for $t = 1$, we obtain the same $\hat{\phi}_1$ and R_0 as before. We have $\hat{f}_1 = \hat{f}$. We can now start Thurston's algorithm with $\hat{\phi}_t$, $t \in (0,1)$ rather than starting it with $\hat{\phi}_0 = id$, but we do *the same* algorithm for all $\hat{\phi}_t$, namely, the algorithm associated with $\hat{f}$!. In this way, we obtain a continuous path of rational functions R_t and a continuous path of homeomorphisms $\hat{\phi}_t$ defined for all real nonnegative t and satisfying the identity $R_t \circ \hat{\phi}_{t+1} = \hat{\phi}_t \circ \hat{f}$ for $t \geq 0$.

Proposition 3.2. *The rational functions R_t converge to a rational function Möbius conjugate to R.*

In the sequel, we will always assume that R_t converge to R, since nothing changes in the statement of the Main Theorem if we replace the function R by its Möbious conjugate.

Proof. We know that $\hat{f}$ is Thurston equivalent to R. From Thurston's Characterization Theorem [DH93] (in fact, from its easy part) it follows that the classes of $\hat{\phi}_n$ converge to the class of some homeomorphism $\hat{\phi}_\infty : \mathbb{C}P^1 \to \mathbb{C}P^1$ in the Teichmüller space of $(\mathbb{C}P^1, P)$. The path $t \mapsto [\hat{\phi}_t]$, $t \in [0, \infty)$ converges in the Teichmüller space of $(\mathbb{C}P^1, P)$ as well. This follows from the convergence of $[\hat{\phi}_n]$ and the contraction property of Thurston's pullback map.

Since the class $[\hat{\phi}_\infty]$ of $\hat{\phi}_\infty$ in the Teichmüller space coincides with the class $[M \circ \hat{\phi}_\infty]$ for every Möbius transformation M, we can assume that $\hat{\phi}_\infty = id$ on P_0. Convergence of $[\hat{\phi}_t]$ to $[\hat{\phi}_\infty]$ means that there is a family of quasiconformal homeomorphisms $h_t : \mathbb{C}P^1 \to \mathbb{C}P^1$ such that the quasiconformal constant of h_t tends to 1, and the equality $\hat{\phi}_t = h_t \circ \hat{\phi}_\infty$ holds on P and holds on $\mathbb{C}P^1$ up to isotopy relative to P. The maps $\hat{\phi}_t$ and $\hat{\phi}_\infty$ are the identity on P_0, hence so is h_t. It follows that h_t converge uniformly to the identity.

Note that the branched covering $h_t^{-1} \circ R_t \circ h_{t+1}$ is homotopic to $\hat{\phi}_\infty \circ \hat{f} \circ \hat{\phi}_\infty^{-1}$ relative to the set $\hat{\phi}_\infty(P)$ through branched coverings. Since $h_t \to id$, any partial limit of R_t as $t \to \infty$ is a rational function homotopic to $\hat{\phi}_\infty \circ \hat{f} \circ \hat{\phi}_\infty^{-1}$ relative to the set $\hat{\phi}_\infty(P)$ through branched coverings (in particular, this rational function is critically finite, hyperbolic and Thurston equivalent to $\hat{f}$). By Thurston's Uniqueness Theorem, such rational function is unique. $\square$

Recall that, by our assumptions, $P \cap Z = \varnothing$, where Z as in Theorem 2.4. Set Z_t to be the union of $Q^{-i}(Z_0)$ for i running from 1 to the smallest integer that is greater than or equal to t. We will now define a family of holomorphic maps $\Phi_t : \mathbb{C}P^1 - Z_t \to \mathbb{C}P^1$ with the following properties:

$$R_t \circ \Phi_{t+1} = \Phi_t \circ Q, \quad \Phi_t|_P = \hat{\phi}_t|_P,$$

where the rational functions R_t are the same as before. For $z \notin U_n$, where $n \geq t$, we set $\Phi_t(z) = \hat{\phi}_t(z)$. This value is well-defined by the construction of $\hat{\phi}_t$ and the same argument as in Proposition 3.1. On the other hand, for every $z \notin Z_t$, we can choose U and n such that $z \notin U_n$, so that the definition applies. The holomorphy of Φ_t follows from the fact that locally near $z \notin Z_t$, the function $\Phi_t(z)$ is a branch of $R_{t-1}^{-1} \circ \Phi_t \circ Q$.

The key lemma is the following:

Lemma 3.3. *The maps $\Phi_n : \mathbb{C}P^1 - Z \to \mathbb{C}P^1$, $n = 1, 2, \dots$, converge uniformly.*

Proof of Theorem 2.4 assuming Lemma 3.3. Every map Φ_n is a restriction of a holomorphic function defined on the complement to a real semi-algebraic set of dimension ≤ 1. By definition of a holomorphic function on $\mathbb{C}P^1 - Z$, it follows that the uniform limit Φ of Φ_n is holomorphic on $\mathbb{C}P^1 - Z$. $\square$

We will need to consider a certain compactification of the space $\mathbb{C}P^1 - Z$. Let X_n be the Caratheodory compactification of U_n (i.e. the space of all prime ends of U_n). We have natural continuous maps $\hat{Q}_n : X_{n+1} \to X_n$ induced by Q. Let $f : X \to X$ be the inverse limit of this system. Denote by $\pi : X \to \mathbb{C}P^1$ the inverse limit of the natural projections $\pi_n : X_n \to \mathbb{C}P^1$ mapping the prime ends of U_n to the corresponding impressions.

Since $P \cap Z = \varnothing$, the map π restricted to $\pi^{-1}(P)$ is one-to-one. Set $P_f = \pi^{-1}(P)$. There exists a one-parameter family of continuous maps $\phi_t : X \to \mathbb{C}P^1$ such that $\phi_t = \Phi_t \circ \pi$ on $\pi^{-1}(\mathbb{C}P^1 - Z)$. Indeed, every Φ_t extends to the Caratheodory compactification of $\mathbb{C}P^1 - Z_t$. The maps ϕ_t make the following diagram commutative:

$$\begin{tikzcd}
(X, f^{-1}(P_f)) \arrow[r, "f"] \arrow[d, "\phi_{t+1}"'] & (X, P_f) \arrow[d, "\phi_t"] \\
(\mathbb{C}P^1, R_t^{-1}(P_t)) \arrow[r, "R_t"] & (\mathbb{C}P^1, P_t)
\end{tikzcd} \tag{$*$}$$

where $P_t = \phi_t(P_f)$. The restrictions of the maps $\hat{\phi}_t$ to P_f converge. This follows from the convergence of $\hat{\phi}_t$ in the Teichmüller space of $(\mathbb{C}P^1, P)$ (more precisely, from the convergence of the projections of $\hat{\phi}_t$ to the moduli space of $\mathbb{C}P^1 - P$). Hence, the restrictions of ϕ_t to P_f converge as well. It is clear that, for every critical value v of $\hat{f}$, the limit of $\hat{\phi}_t(v)$ is a critical value of R. It follows that the limit of $\hat{\phi}_t(z)$ is in P_R for every $z \in P$. It also follows that the limit of $\phi_t(x)$ is in P_R for every $x \in P_f$.

Lemma 3.4. *There exists a map $\iota : f^{-1}(P_f) \to R^{-1}(P_R)$ such that the following diagram is commutative*

$$\begin{tikzcd}
f^{-1}(P_f) \arrow[r, "f"] \arrow[d, "\iota"'] & P_f \arrow[d, "\iota"] \\
R^{-1}(P_R) \arrow[r, "R"] & P_R
\end{tikzcd}$$

and the restrictions of ϕ_t to the set $f^{-1}(P_f)$ converge to ι.

Proof. Consider a point $x \in f^{-1}(P_f)$. We have proved that the points $\phi_t \circ f(x) \in P_t$ converge to some point $a \in P_R$ as $t \to \infty$. Let $t_n \to \infty$ be any sequence such that $\phi_{t_n}(x)$ converges; denote the limit by b. Passing to the limit in both sides of the equation $R_{t_n - 1} \circ \phi_{t_n}(x) = \phi_{t_n - 1} \circ f(x)$, we obtain that $R(b) = a$. It follows that the entire ω-limit set of the family $\phi_t(x)$ is contained in the finite set $R^{-1}(a)$. As the ω-limit set is connected, this implies that $\phi_t(x)$ converges to b as $t \to \infty$. Set $\iota(x) = b$.

The commutative diagram in the statement of the lemma is obtained by passing to the limit as $t \to \infty$ in the diagram $(*)$. ☐

Lemma 3.3, and hence also the Main Theorem, is now reduced to

Theorem 3.5. *The maps $\phi_n : X \to \mathbb{C}P^1$ converge uniformly.*

If this holds, then the maps $\Phi_n = \phi_n \circ \pi^{-1}$ on $\mathbb{C}P^1 - Z$ also converge uniformly. The remaining part of the paper contains the proof of Theorem 3.5. This is a statement about uniform convergence of Thurston's algorithm. As such, it is perhaps not surprising, although we state it for a topological space X that is not S^2 (actually, we need nothing from the space X except that it is locally compact and that some neighborhood of P_f in X has a structure of a Riemann surface; however, specific properties of ϕ_n will be used, e.g. that the restrictions of ϕ_n to P_f converge and that ϕ_n are holomorphic near P_f). Thurston's algorithm is generally expected to converge uniformly, and theorems to this effect have been proved in a variety of contexts. E.g. a general theorem about uniform convergence of Thurston's algorithm has appeared in [CT]. I am grateful to Tan Lei for showing me a draft of this work.

The underlying ideas of the proof of Theorem 3.5 can be traced back to [DH84]. Very roughly, it is an application of the contraction principle to a certain lifting map on a certain functional space. It is even possible to state a general theorem of this sort but many fine details would make its statement too cumbersome. The things are not complicated but they are not straightforward either.

4. THE SPACE $\mathcal{C}$

Notation: for a topological space X and a metric space Y, we denote by $C(X, Y)$ the set of all continuous maps from X to Y. We will always equip this set with the partially defined uniform metric (note that the uniform distance between two elements of $C(X, Y)$ may well be infinite).

In this section, we start the proof of Theorem 3.5. We first set up a suitable function space. In the next section, we prove the convergence in this space. Consider the hyperbolic critically finite rational function R from the statement of the Main Theorem. Recall that P_R denotes the postcritical set of R.

Lemma 4.1 (Expanding metric on $\mathbb{C}P^1 - P_R$)**.** *There exists a piecewise smooth metric on* $\mathbb{C}P^1 - P_R$ *equal to a constant multiple of* $|d\xi|/|\xi|$ *near every point* $z \in P_R$ *for some local holomorphic coordinate* ξ *with* $\xi(z) = 0$ *and such that the map* $R : \mathbb{C}P^1 - R^{-1}(P_R) \to \mathbb{C}P^1 - P_R$ *is uniformly expanding with respect to this metric.*

Proof. It follows from hyperbolicity that there exists a neighborhood V_0 of the Julia set $J(R)$ of R and a Riemannian metric $g_0 = \sigma(z)|dz|$ on V_0 such that R is uniformly expanding with respect to g_0 i.e.

$$\sigma \circ R(z)|dR(z)| \geq E_0\sigma(z)|dz| \tag{1}$$

for some $E_0 > 1$ and all $z \in V_0 \cap R^{-1}(V_0)$. We can assume that V_0 is bounded by smooth curves and that $V_0 = R^{-1}(R(V_0))$. Now extend the metric g_0 to the set $R(V_0) - V_0 - P_R$ by the formula

$$g_0(z) = E_0 \cdot \max_i g(S_i(z)),$$

or, equivalently,

$$\sigma(z) = E_0 \cdot \max_i \left\{ \sigma(S_i(z)) \left| \frac{dS_i(z)}{dz} \right| \right\},$$

where $S_i(z)$ are all local branches of R^{-1} near z. They are well defined since all critical values of R belong to P_R. With this definition, inequality (1) holds also in $R(V_0) - P_R$. Using the same formula, we can extend the metric g_0 to $R^{\circ m}(V_0) - P_R$ for every $m > 0$, hence to the complement of an arbitrarily small neighborhood of P_R. The extended metric is piecewise smooth and satisfies inequality (1) provided that g_0 is defined at both z and $R(z)$.

Now let V_1 be a small neighborhood of P_R such that every component of V_1 is a Jordan domain containing exactly one point of P_R. By Böttcher's theorem, there exists a holomorphic function $\xi : V_1 \to \mathbb{C}$ with simple zeros at all points of P_R such that $\xi \circ R(z) = \xi(z)^{\nu(z)}$, where ν is a locally constant function on V_1 taking its values in $\mathbb{N}$. We can also assume that ξ is a holomorphic coordinate on every component of V_1. Note that R multiplies the metric $|d\xi|/|\xi|$ on V_1 by ν. Set

$$g_1(z) = \lambda(z)\frac{|d\xi(z)|}{|\xi(z)|},$$

where λ is a locally constant function on V_1, which we define below. It suffices to define λ on P_R. Set

$$E_1(z) = \lim_{n\to\infty} \left(\prod_{i=0}^{n-1} \nu(R^{\circ i}(z)) \right)^{1/n}.$$

This number is equal to the geometric mean of ν over the cycle, to which z eventually maps. In particular $E_1(z) > 1$. The function λ on P_R is now defined by the property

$$\lambda(R(z)) = \frac{E_1(z)\lambda(z)}{\nu(z)}.$$

If we fix an arbitrary positive value of λ at an arbitrarily chosen point of each periodic cycle in P_R, then this condition defines λ uniquely. The metric g_1 on $V_1 - P_R$ thus defined gets multiplied by $E_1(z)$ under the map R. Define the number $E_1 > 1$ as the minimum of $E_1(z)$ over all points in P_R.

We now want to combine the two metrics g_0 and g_1. We can assume that $V_0 \cup V_1 = \mathbb{C}P^1 - P_R$ and that both V_0 and V_1 are bounded by smooth curves. We can also assume that there is no point $z \in V_0 - V_1$ such that $R(z) \in V_1 - V_0$ (so that every R-orbit that visits both V_0 and V_1 must enter the "buffer zone" $V_0 \cap V_1$). Set $g = \varepsilon g_0$ on $V_0 - V_1$, $g = g_1$ on $V_1 - V_0$, and $g = \varepsilon g_0 + g_1$ on $V_0 \cap V_1$. As we will show, the map R is uniformly expanding with respect to g provided that the number $\varepsilon > 0$ is small enough so that e.g. $\varepsilon g_0 \leq (\sqrt{E_1} - 1)g_1$ everywhere on $V_0 \cap V_1$. Indeed, if $z \in V_0 - V_1$ and $R(z) \in V_0 \cap V_1$, then

$$g(R(z)) = \varepsilon g_0(R(z)) + g_1(R(z)) \geq E_0 \varepsilon g_0(z) = E_0 g(z).$$

If $z \in V_0 \cap V_1$ and $R(z) \in V_0 \cap V_1$, then

$$g(R(z)) = \varepsilon g_0(R(z)) + g_1(R(z)) \geq E_0 \varepsilon g_0(z) + E_1 g_1(z) \geq \tilde{E} g(z).$$

where $\tilde{E} = \min(E_0, E_1)$. Finally, if $z \in V_0 \cap V_1$ and $R(z) \in V_1 - V_0$, then

$$g(R(z)) = g_1(R(z)) \geq E_1 g_1(z) = (E_1 - \sqrt{E_1})g_1(z) + \sqrt{E_1} g_1(z) \geq$$

$$\sqrt{E_1}\varepsilon g_0(z) + \sqrt{E_1} g_1(z) = \sqrt{E_1} g(z).$$

□

In the sequel, we will write Y for the space $\mathbb{C}P^1 - P_R$ equipped with the metric g from Lemma 4.1. Note that the metric g is *proper*: every closed bounded set is compact. It follows that g is complete and locally compact. Let $E > 1$ be the expansion factor of R with respect to the metric g. In the notation of Lemma 4.1, we can set $E = \min(E_0, \sqrt{E_1})$.

We will use notation of Section 3. Note that there is an open neighborhood O of the set P_f in X, on which the map π is one-to-one and such that $f(O) \subset O$. We will assume that O is sufficiently small. The map π defines a Riemann surface structure on O. The maps ϕ_t are holomorphic on the set O equipped with this structure. Let $\mathcal{C}(O)$ denote the space of continuous maps $\chi : X \to \mathbb{C}P^1$ with the following properties:

1. $\chi = \iota$ on $f^{-1}(P_f)$;
2. $\chi^{-1}(P_R) \subseteq P_f$;
3. $\chi^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f)$;
4. the restriction of χ to O is holomorphic, and no point of P_f is a critical point of χ.

We will consider the following metric on $\mathcal{C}(O)$: the distance between maps χ and $\chi^* \in \mathcal{C}(O)$ is the uniform distance between the restrictions $\chi : X - P_f \to Y$ and $\chi^* : X - P_f \to Y$ measured with respect to the metric g on Y. We need to prove that the distance between any two elements χ and χ^* of $\mathcal{C}(O)$ is finite. It suffices to make a local estimate near each point $x \in P_f$. Let W_x be a small neighborhood of $\iota(x)$, and ξ a holomorphic coordinate on W_x such that $\xi(\iota(x)) = 0$, and $\xi(W_x)$ is a round disk centered at 0. Let O_x be a small neighborhood of x contained in O such that $\chi(O_x) \subset W_x$ and $\chi^*(O_x) \subset W_x$. Since both holomorphic functions $\xi \circ \chi$ and $\xi \circ \chi^*$ have simple zeros at x, their ratio extends to a holomorphic function on O_x taking a nonzero value at x. Note that the uniform distance between the maps $\chi : O_x - \{x\} \to Y$ and $\chi^* : O_x - \{x\} \to Y$ in the metric g is

$$const \cdot \sup_{x' \in O_x - \{x\}} \left| \log \left(\frac{\xi \circ \chi(x')}{\xi \circ \chi^*(x')} \right) \right|$$

for some local branch of the logarithm. Indeed, the metric g is equal to $const \cdot |d \log \xi|$ on W_x. We see that the distance between χ and χ^* is finite. A similar argument shows that the topology on $\mathcal{C}(O)$ coincides with the topology induced from the uniform metric on $C(X, \mathbb{C}P^1)$. Define $\mathcal{C}$ as the union of $\mathcal{C}(O)$ over all sufficiently small neighborhoods O of P_f such that $f(O) \subset O$. As a metric space, $\mathcal{C}$ is the inductive limit of the spaces $\mathcal{C}(O)$.

We will write $\tilde{Y}$ for $Y - R^{-1}(P_R)$. Then $R : \tilde{Y} \to Y$ is a proper expansion with expansion factor E. Being a proper map and a local homeomorphism, this map enjoys the unique path lifting property. This is a key to the following

Lemma 4.2. *If $\gamma : [0,1] \to \mathcal{C}(O)$ is a continuous path and $\tilde{\chi}_0 \in \mathcal{C}(O)$ is a map such that $R \circ \tilde{\chi}_0 = \gamma(0) \circ f$, then there is a unique continuous path $\tilde{\gamma} : [0,1] \to \mathcal{C}(O)$ with the properties $\tilde{\gamma}(0) = \tilde{\chi}_0$ and $R \circ \tilde{\gamma}(t) = \gamma(t) \circ f$ for all $t \in [0,1]$.*

We will call the path $\tilde{\gamma}$ a *lift* of the path γ.

Proof. For every t, consider the restriction of $\gamma(t)$ to $X - P_f$. We obtain a path $\gamma_* : [0,1] \to C(X - P_f, Y)$.

Consider the map $\mathcal{G} : C(X - f^{-1}(P_f), \tilde{Y}) \to C(X - f^{-1}(P_f), Y)$ given by the formula $\mathcal{G}(\chi) = R \circ \chi$. The map $R : \tilde{Y} \to Y$ is a proper expansion, and $X - f^{-1}(P_f)$ is a locally compact space (as a complement to finitely many points in a compact Hausdorff space). It is a standard fact from topology (see e.g. Spanier [S]) that in this case the map $\mathcal{G}$ has the path lifting property: given a path $\alpha : [0,1] \to C(X - f^{-1}(P_f), Y)$ and an element $\tilde{\chi}_0 \in C(X - f^{-1}(P_f), \tilde{Y})$ such that $R \circ \tilde{\chi}_0 = \alpha(0)$, there exists a unique path $\tilde{\alpha} : [0,1] \to C(X - f^{-1}(P_f), \tilde{Y})$ such that $R \circ \tilde{\alpha}(t) = \alpha(t)$ for all $t \in [0,1]$ and $\tilde{\alpha}(0) = \tilde{\chi}_0$. We take $\alpha(t) = \gamma_*(t) \circ f$ and consider the corresponding lift $\tilde{\alpha}$ (with $\tilde{\chi}_0$ as in the statement of the lemma).

It is obvious that every map $\tilde{\alpha}(t)$ extends to a continuous map $\tilde{\gamma}(t)$ from X to $\mathbb{C}P^1$ holomorphic on O. It remains to show that the maps $\tilde{\gamma}(t)$ belong to $\mathcal{C}(O)$. The following two properties imply this:

(1) $\tilde{\gamma}(t) = \gamma(t)$ on $f^{-1}(P_f)$;
(2) $\tilde{\gamma}(t)^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f)$.

Note that both $\gamma(t)$ and $\tilde{\gamma}(t)$ restricted to $f^{-1}(P_f)$ take values in $R^{-1}(P_R)$. For $\tilde{\gamma}(t)$, this follows from the defining identity $R \circ \tilde{\gamma}(t) = \gamma(t) \circ f$. Now property (1) holds by continuity (the two maps coincide for $t = 0$ and take values in finite sets). To prove property (2), take any $x \in X$ such that $\tilde{\gamma}(t)(x) \in R^{-1}(P_R)$. Then

$$\gamma(t)(f(x)) = R \circ \tilde{\gamma}(t)(x) \in P_R,$$

hence $f(x) \in P_f$, hence $x \in f^{-1}(P_f)$. □

Remark 4.3. Note that any lift of a rectifiable path in $\mathcal{C}$ is at least E times shorter than the path itself. This follows from the fact that the map $\mathcal{G}$ is a local expansion with expansion factor E.

Proposition 4.4. *There exists a continuous family of homeomorphisms $\psi_t : \mathbb{C}P^1 \to \mathbb{C}P^1$ defined for sufficiently large t with the following properties:*

- *$\psi_t \circ \phi_t \in \mathcal{C}$;*
- *$\psi_t \to id$ uniformly as $t \to \infty$;*
- *there is a neighborhood W of P_R such that the restrictions of ψ_t to W are holomorphic;*

Note that a priori we cannot fix a neighborhood O of P_f such that $\psi_t \circ \phi_t \in \mathcal{C}(O)$ for all t. However, as can be seen from the proof, such neighborhood exists for every bounded interval of values of t.

Proof. Suppose that t is sufficiently large so that $\phi_t(x) \neq \phi_t(x')$ for x, $x' \in f^{-1}(P_f)$ unless $\iota(x) = \iota(x')$. Recall that $P_t = \phi_t(P_f)$. For $z \in R_{t-1}^{-1}(P_{t-1})$, we set $\psi_t(z) = \iota(x)$, where $x \in f^{-1}(P_f)$ is any point such that $\phi_t(x) = z$. By our assumption, the point $\psi_t(z)$ thus defined does not depend on the choice of x. We have defined the map ψ_t on $R_{t-1}^{-1}(P_{t-1})$. It is clear that the map ψ_t is injective on $R_{t-1}^{-1}(P_{t-1})$ (different points of $R_{t-1}^{-1}(P_{t-1})$ do not merge in the limit). Since the sets $R_{t-1}^{-1}(P_{t-1})$ and $R^{-1}(P_R)$ have the same cardinality, this map is actually a bijection between these two sets.

The uniform distance between the map ψ_t on $R_{t-1}^{-1}(P_{t-1})$ and the identity is bounded above by the uniform distance between ϕ_t on $f^{-1}(P_f)$ and ι. Hence this distance tends to 0 as $t \to \infty$. It follows that we can choose a continuous family ψ_t so that $\psi_t \to id$ as $t \to \infty$ and that ψ_t are holomorphic on some neighborhood of P_R for all sufficiently large t.

It suffices to prove that $\psi_t \circ \phi_t$ belongs to $\mathcal{C}$. By definition $\psi_t \circ \phi_t = \iota$ on $f^{-1}(P_f)$. We need to check that:

(1) $(\psi_t \circ \phi_t)^{-1}(P_R) \subseteq P_f$;
(2) $(\psi_t \circ \phi_t)^{-1}(R^{-1}(P_R)) \subseteq f^{-1}(P_f)$.

To prove (1), suppose that $w = \psi_t \circ \phi_t(x) \in P_R$ for some $x \in X$. Since ψ_t is a homeomorphism taking P_t to P_R, and P_R has the same cardinality as P_t, we have $\psi_t^{-1}(w) \in P_t$. Then $x \in \phi_t^{-1}(P_t) = P_f$. Property (2) can be proved by the same argument. ☐

Theorem 3.5 (and hence the Main Theorem) reduces to the following:

Theorem 4.5. *The sequence $\chi_n = \psi_n \circ \phi_n$ converges in $\mathcal{C}$ as $n \to \infty$ through positive integers, hence in $C(X - P_f, Y)$ and in $C(X, \mathbb{C}P^1)$.*

Indeed, since ψ_n converge to the identity, we conclude from Theorem 4.5 that ϕ_n converge uniformly, q.e.d. The convergence in $\mathcal{C}$ is perhaps a little surprising because the space $\mathcal{C}$ is not complete.

5. Contracting lifting

In this section, we prove Theorem 4.5, hence also the Main Theorem. We will repeatedly use contraction properties of the lifting as defined in Lemma 4.2. Note that the map $\chi_t = \psi_t \circ \phi_t \in \mathcal{C}$ depends continuously on t with respect to the uniform metric in $C(X, \mathbb{C}P^1)$. It can also be arranged that, for every finite interval $[t_0, t_1]$, there exists a neighborhood O of P_f such that $\chi_t \in \mathcal{C}(O)$ for all $t \in [t_0, t_1]$. Hence χ_t form also a continuous family in $\mathcal{C}$.

Recall that we have the following commutative diagram:

$$\begin{tikzcd}
(X, f^{-1}(P_f)) \arrow[r, "f"] \arrow[d, "\phi_{t+1}"'] & (X, P_f) \arrow[d, "\phi_t"] \\
(\mathbb{C}P^1, R_t^{-1}(P_t)) \arrow[r, "R_t"] \arrow[d, "\psi_{t+1}"'] & (\mathbb{C}P^1, P_t) \arrow[d, "\psi_t"] \\
(\mathbb{C}P^1, R^{-1}(P_R)) & (\mathbb{C}P^1, P_R)
\end{tikzcd}$$

The family ψ_t can be thought of as a continuous path in $C(\mathbb{C}P^1, \mathbb{C}P^1)$ defined on the compact interval $[0, \infty]$: it suffices to set $\psi_\infty = id$. There exists a unique continuous path $t \mapsto \tilde{\psi}_t$, $t \in [1, \infty]$ such that $\tilde{\psi}_\infty = id$ and $R \circ \tilde{\psi}_t = \psi_{t-1} \circ R_{t-1}$ for all $t \geq 1$. We have $\tilde{\psi}_t = \psi_t$ on $R_{t-1}^{-1}(P_{t-1})$ by continuity, since both maps are equal to the identity for $t = \infty$, and both take values in $R^{-1}(P_R)$. Since $\psi_t \circ \phi_t \in \mathcal{C}$ and $\tilde{\psi}_t = \psi_t$ on $R_{t-1}^{-1}(P_{t-1})$, we also have $\tilde{\chi}_t = \tilde{\psi}_t \circ \phi_t \in \mathcal{C}$. Note that the family $\tilde{\chi}_t$ satisfies the following identity:

$$R \circ \tilde{\chi}_{t+1} = \chi_t \circ f.$$

Indeed, we have

$$R \circ \tilde{\psi}_{t+1} \circ \phi_t = \psi_t \circ R_t \circ \phi_{t+1} = \psi_t \circ \phi_t \circ f.$$

Note that both ψ_t and $\tilde{\psi}_t$ converge to the identity as $t \to \infty$ uniformly with respect to the spherical metric. Therefore, the distance between χ_t and $\tilde{\chi}_t$ in $C(X, \mathbb{C}P^1)$ tends to 0 as $t \to \infty$.

Proposition 5.1. *There is a continuous map* $\Gamma : [t_0, \infty) \times [0, 1] \to \mathcal{C}$, *where* t_0 *is a sufficiently large real number, such that*

$$\Gamma(t, 0) = \chi_t, \quad \Gamma(t, 1) = \tilde{\chi}_t,$$

and the length of the path $s \mapsto \Gamma(t, s)$, $s \in [0, 1]$ *in* $\mathcal{C}$ *tends to* 0 *as* $t \to \infty$. *Moreover, for every* $t \in [t_0, \infty)$, *there exists a neighborhood* O_t *of* P_f *such that* $\Gamma(t, s) \in \mathcal{C}(O_t)$ *for all* $s \in [0, 1]$. *For every finite interval* $[t_1, t_2]$, *there exists a neighborhood* O *of* P_f *that is contained in* O_t *for all* $t \in [t_1, t_2]$.

Proof. It suffices to define a continuous map $\Psi : [t_0, \infty) \times [0, 1] \to C(\mathbb{C}P^1, \mathbb{C}P^1)$ such that $\Psi(t, 0) = \psi_t$, $\Psi(t, 1) = \tilde{\psi}_t$, and $s \mapsto \Psi(t, s)$ is a rectifiable path in $C(\mathbb{C}P^1 - P_t, Y)$, whose length tends to 0 as $t \to \infty$, and such that all $\Psi(t, s)$ are holomorphic on some fixed neighborhood $\tilde{W}$ of P_R. Then we set $\Gamma(t, s) = \Psi(t, s) \circ \phi_t$.

Fix $x \in P_f$, and set $z_t = \phi_t(x)$. Let W_x be the component of W containing $\iota(x)$, and ξ a holomorphic coordinate on W_x such that

$\xi(\iota(x)) = 0$. We have $z_t \in W_x$ for all sufficiently large t, and $\xi(z_t) \to 0$ as $t \to \infty$. Since both $\xi \circ \psi_t(z)$ and $\xi \circ \tilde{\psi}_t(z)$ have simple zeros at z_t, the ratio $\xi \circ \tilde{\psi}_t / \xi \circ \psi_t$ extends holomorphically to W_x and converges to 1 on W_x as $t \to \infty$. It follows that the distance between ψ_t and $\tilde{\psi}_t$ in $C(W_x - \{z_t\}, Y)$ tends to zero as $t \to \infty$. We set

$$\xi \circ \Psi(t,s) = (\xi \circ \psi_t) \cdot \exp\left(s \log \frac{\xi \circ \tilde{\psi}_t}{\xi \circ \psi_t} \right),$$

on a neighborhood $\tilde{W}_x$ of $\iota(x)$ such that the right-hand side lies in $\xi(W_x)$ (we can choose one neighborhood that will work for all sufficiently large t). The branch of the logarithm is chosen to be the closest to 0. This defines the maps $\Psi(t,s)$ on $\tilde{W}_x$. Note that the path $s \mapsto \Psi(t,s)$ is rectifiable in $C(\tilde{W}_x - \{z_t\}, Y)$, and the length of the path is equal to the distance between ψ_t and $\tilde{\psi}_t$ in $C(\tilde{W}_x - \{z_t\}, Y)$. The same formula defines the maps $\Psi(t,s)$ on a neighborhood of any point from P_t. Clearly, we can extend $\Psi(t,s)$ to $\mathbb{C}P^1$ with desired properties. $\square$

In the proof of Theorem 4.5, we will need two more lemmas.

Lemma 5.2. *Consider a rectifiable path $\delta : [0,1] \to \mathcal{C}(O)$. Then there is a continuous extension $\delta : [0,\infty] \to \mathcal{C}(O)$ such that $R \circ \delta(t+1) = \delta(t) \circ f$ for all $t \in [0,\infty)$. Moreover, the length of the extended path δ is at most $E/(E-1)$ times the length of the original δ, and we have*

$$R \circ \delta(\infty) = \delta(\infty) \circ f.$$

Proof. Consider the lift α of the path δ as in Lemma 4.2. By Remark 4.3, the length of α is at most E^{-1} times the length of δ. Now we set $\delta(t) = \alpha(t-1)$ for $t \in [1,2]$. As we keep doing this extension process, we obtain more and more segments of δ, each segment being shorter than the preceding one by at least the factor E^{-1}. It follows that $\delta(t)$ converges in $C(X - P_f, Y)$ as $t \to \infty$. Denote the limit by $\delta(\infty)$. Thus we obtain the extended path $\delta : [0,\infty] \to C(X - P_f, Y)$. The length of this extended path can be estimated by a geometric series with the common ratio E^{-1}: it does not exceed $E/(E-1)$.

It remains to prove that $\delta(\infty) \in \mathcal{C}(O)$. The map $\delta(\infty)$ is holomorphic on O as a uniform limit of holomorphic maps. The only non-obvious property is that the preimage of $R^{-1}(P_R)$ under $\delta(\infty)$ is contained in $f^{-1}(P_f)$, or, equivalently, the image of $X - f^{-1}(P_f)$ under $\delta(\infty)$ is contained in $\tilde{Y}$. Indeed, the lift $\tilde{\delta}$ of the path $t \mapsto \delta(t)$, $t \in [0,\infty]$, such that $\tilde{\delta}(0) = \delta(1)$ is unique. Therefore, we must have $\tilde{\delta}(t) = \delta(t+1)$ on $X - f^{-1}(P_f)$ for all $t \in [0,\infty]$, in particular, $\tilde{\delta}(\infty) = \delta(\infty)$. On the other hand, by construction, the map $\tilde{\delta}(\infty)$ takes the set $X - f^{-1}(P_f)$ to $\tilde{Y}$.

Therefore, we have $\delta(\infty)(X - f^{-1}(P_f)) \subseteq \tilde{Y}$, as desired. The equality $R \circ \delta(\infty) = \delta(\infty) \circ f$ follows from the equality $R \circ \tilde{\delta}(\infty) = \delta(\infty) \circ f$ on $X - f^{-1}(P_f)$. □

Lemma 5.3. *There exists a real number $\varepsilon > 0$ such that the distance between two maps χ^*, $\chi^{**} \in \mathcal{C}$ is bigger than ε provided that*

$$R \circ \chi^* = \chi^* \circ f, \quad R \circ \chi^{**} = \chi^{**} \circ f.$$

Proof. There exists a neighborhood O of P_f such that every map $\chi \in \mathcal{C}$ such that $R \circ \chi = \chi \circ f$ is holomorphic on O. Indeed, χ is holomorphic on at least some small neighborhood of P_f but the formula $R \circ \chi = \chi \circ f$ says that χ is also holomorphic on iterated pullbacks of this neighborhood under f as long as f itself remains holomorphic.

Let ξ be an extended Böttcher coordinate on a neighborhood W of P_R so that $\xi \circ R(z) = \xi(z)^{\nu(z)}$ for some locally constant function ν taking positive integer values. Define the Green function G_R of R on W as $-\log|\xi|$. The function $G_R \circ \chi$ restricted to some neighborhood of P_f is the same for all maps $\chi \in \mathcal{C}$ with the property $R \circ \chi = \chi \circ f$ (by the uniqueness of Böttcher's coordinate). Denote this function by G_f and call it the Green function of f. The Green function G_R can be extended to $\mathbb{C}P^1$ by setting

$$G_R(z) = \frac{1}{d^n} G_R(R^{\circ n}(z))$$

if $R^{\circ n}(z) \in W$ for some $n \geq 0$, and $G_R(z) = 0$ otherwise. Here $d = \deg(R)$. Similarly, G_f extends to a function on X.

Take a sufficiently small number $\varepsilon_0 > 0$. The set $G_f \leq \varepsilon_0$ is mapped to the set $G_R \leq \varepsilon_0$ by both χ^* and χ^{**}. Note that $\xi \circ \chi^*$ and $\xi \circ \chi^{**}$ can only differ on the set $G_f \leq \varepsilon_0$ by a locally constant factor that is a root of unity of bounded degree. Therefore, the distance between restrictions of $\xi \circ \chi^*$ and $\xi \circ \chi^{**}$ to the set $G_f \leq \varepsilon_0$ cannot take arbitrarily small nonzero values. It follows that if χ^* and χ^{**} are sufficiently close, then their restrictions to the set $G_f \leq \varepsilon_0$ must coincide.

Since all critical values of R are poles of the Green function G_R, the distance between two different R-preimages of any point z with $G_R(z) \geq \varepsilon_0/d$ is bounded below by some positive number uniform with respect to z. It follows that χ^* and χ^{**} must also coincide on the set $\varepsilon_0 \leq G_f \leq d\varepsilon_0$ provided that χ^* and χ^{**} are sufficiently close. By induction, the two maps coincide on the set $G_f > 0$. This set is dense in X (because it is dense in $\mathbb{C}P^1 - Z$), hence $\chi^* = \chi^{**}$. □

Proof of Theorem 4.5. Throughout the proof, the parameters t and s will run through the interval $[0, 1]$. By a homotopy, we will always mean a homotopy between two paths in the metric space $\mathcal{C}$ with fixed endpoints.

Consider any rectifiable path $\gamma : [0,1] \to \mathcal{C}$ connecting χ_{n-1} with χ_n and homotopic to the path $t \mapsto \chi_{n-1+t}$. Consider the lift $\tilde{\gamma} : [0,1] \to \mathcal{C}$ of γ (as in Lemma 4.2) such that $\tilde{\gamma}(0) = \tilde{\chi}_n$. The path $\tilde{\gamma}$ is at least E times shorter than γ. We claim that $\tilde{\gamma}(1) = \tilde{\chi}_{n+1}$. Indeed, since the path γ is homotopic to the path $t \mapsto \chi_{n-1+t}$, the path $\tilde{\gamma}$ is homotopic to the path $t \mapsto \tilde{\chi}_{n+t}$ (the lifts of two homotopic paths are homotopic).

Set L_n to be the infimum of the lengths of rectifiable paths in $\mathcal{C}$ lying in $\mathcal{C}(O)$ for some neighborhood O of P_f, connecting χ_n to χ_{n+1} and homotopic to the path $t \mapsto \chi_{n+t}$ in $\mathcal{C}$. If ε_n denotes the maximum of the lengths of the paths $\Gamma_n : s \mapsto \Gamma(n,s)$ and $\Gamma_{n+1} : s \mapsto \Gamma(n+1,s)$, then we have

$$L_n \le E^{-1} \cdot L_{n-1} + 2\varepsilon_n$$

Indeed, if γ is a rectifiable path in $\mathcal{C}$ that connects χ_{n-1} with χ_n, then L_n is at most the length of the composition of the following paths:

- the path Γ_n from χ_n to $\tilde{\chi}_n$;
- the path $\tilde{\gamma}$ from $\tilde{\chi}_n$ to $\tilde{\chi}_{n+1}$;
- the reversed path Γ_{n+1} from $\tilde{\chi}_{n+1}$ to χ_{n+1}.

The length of this composition can be made smaller than any fixed number exceeding $E^{-1}L_{n-1} + 2\varepsilon_n$ by choosing the path γ in $\mathcal{C}(O)$ for some O to have length close to L_{n-1}. Note also that the composition is homotopic to $t \mapsto \chi_{n+t}$ provided that γ is homotopic to $t \mapsto \chi_{n-1+t}$. The corresponding homotopy can be easily constructed using the homotopy Γ.

Take any n_0, then, applying the previous inequality several times, we obtain

$$L_n \le E^{n_0-n} L_{n_0} + 2\tilde{\varepsilon}_{n_0}(1 + E^{-1} + \cdots + E^{n_0-n}),$$

where $\tilde{\varepsilon}_{n_0}$ is the supremum of $\varepsilon_{n_0+1}, \ldots$. The second term in the right-hand side can be made arbitrarily small (uniformly with n) by choosing n_0 large enough. After n_0 has been chosen, we can choose sufficiently large n to make the first term as small as we wish. It follows that $L_n \to 0$ (in particular, the distance between χ_n and χ_{n+1} tends to 0 in $\mathcal{C}$).

Consider the composition δ_n of some path γ of length at most $2L_n$ homotopic to $t \mapsto \chi_{n+t}$ and the path Γ_{n+1}. Reparameterize δ_n so that the parameter runs from 0 to 1. We can arrange that $\delta_n(t) \in \mathcal{C}(O)$ for some open neighborhood O of P_f and all $t \in [0,1]$. We have $R \circ \delta_n(1) = \delta_n(0) \circ f$ because $\delta_n(0) = \chi_n$ and $\delta_n(1) = \tilde{\chi}_{n+1}$.

Consider the extended path $\delta_n : [0,\infty] \to \mathcal{C}(O)$ as in Lemma 5.2. Then we have

$$R \circ \delta_n(\infty) = \delta_n(\infty) \circ f.$$

The distance between $\delta_n(\infty)$ and $\delta_m(\infty)$ tends to 0 as n and $m \to \infty$. By Lemma 5.3, the sequence $\delta_n(\infty)$ stabilizes, i.e. $\delta_n(\infty)$ is the same map χ_∞ for all sufficiently large n. We know that the distance between χ_n and χ_∞ in $\mathcal{C}$ tends to 0. Therefore, χ_n converge to χ_∞. ☐

References

[CT] Cui Guizhen, Tan Lei, "Hyperbolic-parabolic deformations of rational maps", preprint

[DH84] A. Douady, J.H. Hubbard, "Étude dynamique des polynômes complexes" I& II, Publ. Math. Orsay, (1984–85)

[DH93] A. Douady, J.H. Hubbard, "A proof of Thurston's topological characterization of rational functions", Acta Math., **171** (1993), 263–297

[E] A. Epstein, "Infinitesimal Thurston rigidity and the Fatou-Shishikura inequality", Stony Brook IMS preprint 1999/1

[M] J. Milnor, "Geometry and Dynamics of Quadratic Rational Maps" Experimental Math. **2** (1993) 37–83

[R] M. Rees, "A partial description of the parameter space of rational maps of degree two: Part 1", Acta Math. **168** (1992), 11–87

[S] E.H. Spanier, "Algebraic topology", Springer 1994

[T] V. Timorin, "Topological regluing of rational functions", Inventiones Math., **179** (2005), Issue 3, 461–506

FACULTY OF MATHEMATICS, HIGHER SCHOOL OF ECONOMICS, 7 VAVILOVA ST. 112312, MOSCOW, RUSSIA

E-mail address: `vtimorin@hse.ru`