# Birational automorphisms of algebraic varieties with a pencil of double quadrics

A.V.Pukhlikov

Institute for Systems Analysis
Prospekt 60-letya Oktyabrya 9
117312 Moscow
RUSSIA
and
Max-Planck-Insitut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY
e-mail: *dost@dost.mccme.rssi.ru*

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**Abstract**

We prove that higher-dimensional algebraic varieties with a pencil of double quadrics of index 1 are birationlly rigid, provided they are sufficiently twisted over the base.

# Introduction

**0.1.** Let $\pi: V \to S$ be a Fano fibration. Here $V$ is a smooth projective variety of dimension $\geq 4$, satisfying the condition $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \pi^* \operatorname{Pic} S$, $S$ is a smooth projective variety of dimension $\geq 1$, and the anticanonical divisor $(-K_V)$ is ample on the fibers of $\pi$. The generic fiber $F_\eta$ (that is, the very variety $V$, considered over the non-closed field $\mathbf{C}(S)$) is a Fano variety. The subgroup $\operatorname{Bir} F_\eta \subset \operatorname{Bir} V$ of birational self-maps, preserving the fibers of $\pi$, is called the group of *proper* birational automorphisms of the Fano fibration $V/S$.

Recall [P2-P5], that $V/S$ is said to be *birationally rigid*, if for each birational map

$$\chi: V - - \to V'$$

onto a uniruled variety $V'$ of the same dimension the monotonicity condition

$$c(D^*, V) \leq c(D', V')$$

holds. Here $D' \in \Sigma'$, $\Sigma'$ is a linear system of divisors on $V'$ with no fixed components, $D^* \in \Sigma^*$, $\Sigma^*$ is the proper inverse image of $\Sigma'$ on $V$ with respect to the composition

$$\chi^* \circ \chi \colon V - - \to V',$$

where $\chi^* \in \mathrm{Bir}\, F_\eta$ is a certain proper birational self-map of $V/S$ (depending on $\Sigma'$), and

$$c(Y, X) = \sup\{\frac{b}{a} | b \geq 1, a \geq 1, |aY + bK_X| \neq \emptyset\}$$

is the *threshold of canonical adjunction.* If for each $V'$, $\chi$, $\Sigma'$ we can take $\chi^* = \mathrm{id}$, that is,

$$c(D, V) \leq c(D', V'),$$

then the Fano fibration $V/S$ is said to be *birationally superrigid.*

**0.2.** The aim of the present paper is to prove birational superrigidity of higher-dimensional algebraic varieties with a pencil of double quadrics. Take $S = \mathbf{P}^1$ and let $\pi \colon V \to \mathbf{P}^1$ be a smooth variety of dimension $M + 1 \geq 5$, each fiber of which is a double quadric of index 1. More precisely, there is a commutative diagram

$$\begin{array}{ccccccc} & V & \stackrel{\sigma}{\longrightarrow} & X & \subset & \mathbf{P}(\mathcal{E}^*) & \\ \pi & \downarrow & & & & \downarrow & \pi \\ & \mathbf{P}^1 & \longleftrightarrow & & & \mathbf{P}^1, & \end{array}$$

where $\sigma$ is a double cover, $\mathbf{P}(\mathcal{E}^*)$ is a locally trivial $\mathbf{P} = \mathbf{P}^{M+1}$-fibration over $\mathbf{P}^1$, the fiber $X_t = \pi^{-1}(t) \subset \mathbf{P}$ is an $M$-dimensional quadric and the branch divisor of $\sigma$ is cut out on $X_t$ by a hypersurface of degree $2(M-1)$ in $\mathbf{P}$ (for the details of construction of a pencil of double quadrics, see Sec. 1). The main result of the present paper is given by the following

**Theorem 1.** *Let $V$ be a general variety, satisfying the $K^2$-condition: the numerical class $NK_V^2 - f$ is not effective for any $N \in \mathbf{Z}$, where $f$ is the class of an effective divisor in a fiber of the pencil $\pi$. Then the Fano fibration $\pi \colon V \to \mathbf{P}^1$ is birationally superrigid.*

For a Fano fibration $\pi \colon V \to \mathbf{P}^1$, satisfying the $K^2$-condition, superrigidity implies a number of standard properties:

(i) the only non-trivial structure of a Fano fibration on $V$ is the very pencil of fibers of the morphism $\pi$; in other words, each birational map

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & V' & \\ \pi & \downarrow & & \downarrow & \pi' \\ & \mathbf{P}^1 & & S' & \end{array}$$

onto an arbitrary Fano fibration of the same dimension is fiber-wise, that is, there is a map $\alpha \colon \mathbf{P}^1 \to S'$ such that $\pi' \circ \chi = \alpha \circ \pi$;

(ii) any birational self-map of $V$ is fiber-wise; for a general $V$ we have $\mathrm{Bir}\, V = \mathrm{Aut}\, V = \mathbf{Z}/2\mathbf{Z}$;

(iii) the variety $V$ is not rational.

For those Fano fibrations $V \to \mathbf{P}^1$, which cover $X = Q \times \mathbf{P}^1$, where $Q \subset \mathbf{P}$ is a smooth quadric hypersurface, our theorem was proved in [P3]. All the arguments of the paper [P3] work for the general case of an arbitrary $X$, too, but they do not exclude the most difficult case of a maximal singularity, lying over the vertex of the cone $Q_t \subset \mathbf{P}$. It is exactly to avoid the singular fibers on $X$, that in [P3] the class of varieties under consideration was restricted by the direct product. However, today the strengthened techniques of hypertangent systems makes it possible to study the case of singular fibers exhaustively, too.The proof of the theorem is given in Sec. 2.

The results of the papers [P1,P3,P4] and Theorem 1 suggest that the implication ($K^2$-condition $\Rightarrow$ birational rigidity) is true for all varieties with a pencil of Fano double hypersurfaces.

**Conjecture.** *Let $\pi: V \to \mathbf{P}^1$ be a Fano fibration with $V$ smooth, where each fiber $\pi^{-1}(t)$ is the double cover of a hypersurface $Q_m \subset \mathbf{P}$ of degree $m$, branched over $W^*_{2l} \cap Q_m$, $l + m = M + 1$. Assume that $V$ satisfies the $K^2$-condition. Then $\pi: V \to \mathbf{P}^1$ is birationally superrigid.*

The technique of hypertangent systems makes it possible to realize some of the principal steps in the proof of Conjecture (following the traditional scheme of the method of maximal singularities). More precisely, one can exclude all the possibilities for a maximal singularity but one exceptional case, when its centre on $V$ is a singular point of a fiber of $\pi$, which comes from a singularity on the branch divisor (if the singularity comes from a double point on the hypersurface $Q_t \subset \mathbf{P}^M$, then the arguments of [P5] successfully exclude the case). To complete the proof, we need to strengthen the technique.

**0.3.** The present paper is a direct continuation of [P3,P4]. The double spaces and double quadrics of index 1 have small degree and thus are most suitable for experiments in studying birational geometry by means of the method of maximal singularities. It was these varieties that made the first examples of birationally (super)rigid Fano varieties of arbitrary dimension [P1].

The new technique of studying Fano fibrations, which was developed in [P4], made it possible at once to prove superrigidity of varieties with a pencil of double quadrics ([P4], Sec. 8). However, the degree 4 turned out to be too high to obtain automatically a similar result for the double quadrics. It was obvious, though, that such a result should be true. The methods of the paper [P4] were strong enough for the double covers of the direct products only [P3], but they failed to exclude a maximal singularity over the vertex of a cone.

The paper [P3] was actually written in 1997 during my stay at the Max-Planck-Institut für Mathematik in Bonn. The technique of hypertangent systems, a modification of which yielded the main theorem of the present paper, was also developed there. I want to use this opportunity to thank the Institute for the marvellous conditions of work.

# 1 Varieties with a pencil of double quadrics

## 1.1 Construction

We study smooth projective $(M+1)$-dimensional varieties $V$ over the field $\mathbf{C}$ of complex numbers, $M \geq 4$, which permit a morphism $\pi\colon V \to \mathbf{P}^1$, the fibers of which are double quadrics of index 1. More precisely, let

$$\mathcal{E} = \bigoplus_{i=0}^{M+1} \mathcal{O}_{\mathbf{P}^1}(a_i)$$

be a locally free sheaf of rank $M+2$ on $\mathbf{P}^1$, where for convenience we assume that $0 = a_0 \leq a_1 \leq \ldots \leq a_{M+1}$. Its projectivization $X = \mathbf{Proj}(\mathcal{E}) = \mathbf{P}(\mathcal{E}^*)$ is a $\mathbf{P}^{M+1}$-bundle over the line. The projection onto the base is denoted by the same symbol $\pi$.

Let $G$ be a fiber of the morphism $\pi$, then $\operatorname{Pic} X = \mathbf{Z}L \oplus \mathbf{Z}G$, where $L$ is the class of the tautological sheaf $\mathcal{O}_{X/\mathbf{P}^1}(1)$, i.e. $\pi_*\mathcal{O}_X(L) = \mathcal{E}$. By our assumption the sheaf $\mathcal{O}_X(L)$ is generated by global sections, i.e. the linear system $|L|$ is free. The canonical class of the variety $X$ is equal to

$$K_X = -(M+2)L + \left(\sum_{i=1}^{M+1} a_i - 2\right) G.$$

Consider a smooth divisor $Q \in |2L + \mu G|$ on $X$, $\mu \in \mathbf{Z}$. Its fibers $Q_t = Q \cap \pi^{-1}(t)$ over the points $t \in \mathbf{P}^1$ are quadric hypersurfaces. The canonical class of the variety $Q$ is equal to

$$K_Q = \left.\left(-ML + \left(\sum_{i=1}^{M+1} a_i + \mu - 2\right) G\right)\right|_Q.$$

Now take a prime Weil divisor $W \in |2(M-1)L + 2\lambda G|$ on $X$ such that $W \cap Q$ is a smooth divisor on $Q$.

Let $V$ be the (smooth) double cover $\sigma\colon V \to Q$, branched over $W \cap Q$. The just constructed family of smooth varieties $V$ is denoted by $\mathcal{V}(a_*, \mu, \lambda)$. The projection of $V$ onto $\mathbf{P}^1$ we denote by the symbol $\pi$, and its fiber — by the symbol $F_t$, $t \in \mathbf{P}^1$. Obviously, $\operatorname{Pic} V = \mathbf{Z}K_V \oplus \mathbf{Z}F$ and the morphism $\pi\colon V \to \mathbf{P}^1$ fibers $V$ into Fano varieties of index 1 and degree 4. The canonical class of the variety $V$ takes the form

$$K_V = -\sigma^*(L|_Q) + \left(\sum_{i=1}^{M+1} a_i + \mu + \lambda - 2\right) F.$$

## 1.2 The condition of general position

We will assume that the following three conditions of general position hold.

(G1) If $Q_t \subset \mathbf{P}^{M+1}$ is a singular quadric, then $Q_t$ has exactly one singular point, that is, $Q_t$ is a cone over a non-singular quadric in $\mathbf{P}^M$.

(G2) If the variety $W \cap Q_t$ is singular (this may happen for a finite number of points $t \in \mathbf{P}^1$ only), then $Q_t$ is a smooth quadric, the set $\operatorname{Sing} W \cap Q_t$ is finite and each singularity $x \in W \cap Q_t$ is a non-degenerate quadratic singularity. Moreover, the quadric tangent cones

$$T_x(T_xQ_t \cap Q_t) \quad \text{and} \quad T_x(W \cap Q_t)$$

are in general position, that is, the intersection of their projectivizations in the projective space $\mathbf{P}(T_xQ_t) \cong \mathbf{P}^{M-1}$ is a smooth variety of dimension $M-3$ and degree 4.

(G3) If $Q_t \subset \mathbf{P}^{M+1}$ is a quadric cone with the vertex $x$, then consider a system of linear coordinates $(z_1, \ldots, z_{M+1})$ on $\mathbf{C}^{M+1} \subset \mathbf{P}^{M+1}$ with the origin at $x$. Let $q = 0$ and $g = w_0 + w_1 + \ldots + w_{2(M-1)}$ be the equations of the quadric $Q_t$ and the hypersurface $W_t = W \cap G_t$, respectively. First of all, $w_0 \neq 0$, i.e. $x \notin W$. From this moment on we assume that $w_0 = 1$.

Consider the power series $(1+a)^{1/2} = 1 + \sum \gamma_i a^i = 1 + \frac{1}{2}a + \ldots$, where $a$ is nothing but a formal variable. Set

$$\sqrt{g} = (1 + w_1 + \ldots + w_{2(M-1)})^{1/2} = 1 + \sum_{i=1}^{\infty} \gamma_i (w_1 + \ldots + w_{2(M-1)})^i =$$
$$= 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \ldots, w_{2(M-1)}),$$

where $\Phi_i(w_1(z_*), \ldots, w_{2(M-1)}(z_*))$ are homogeneous of degree $i$ in $z_*$. Obviously, $\Phi_i(w_*) = w_i + \ldots$ for $i \leq 2(M-1)$, where the dots stand for a polynomial in $w_1, \ldots, w_{i-1}$. For $k \geq 1$ set

$$[\sqrt{g}]_k = 1 + \sum_{i=1}^{k} \Phi_i(w_*), \quad g^{(k)} = g - [\sqrt{g}]_k^2.$$

It is easy to see that the first non-zero homogeneous component in $g^{(k)}$ is of degree $k+1$ in $z_*$. This component for $k+1 \leq 2(M-1)$ is of the form $w_{k+1} + \ldots$, where the dots stand for a polynomial in $w_1, \ldots, w_k$.

Now our condition of general position (G3) is formulated as follows. For a certain system of coordinates $z_*$ the set of polynomials

$$q, g^{(M-1)}, \ldots, g^{(2M-3)}$$

makes a regular sequence in $\mathcal{O}_{x,\mathbf{P}^{M+1}} = \mathcal{O}_{x,\mathbf{C}^{M+1}}$. Moreover, in a neighborhood of the point $x$ the system of equations

$$q = g^{(M-1)} = \ldots = g^{(2M-4)} = 0$$

defines a surface, no one component of which lies in a hyperplane, containing the point $x$. In other words, if $l(z)$ is an arbitrary non-zero linear form, then the sequence

$$l, q, g^{(M-1)}, \ldots, g^{(2M-4)}$$

remains regular in $\mathcal{O}_{x,\mathbf{P}^{M+1}}$. For $M = 4$ we assume in addition that for each linear form $l(z)$ the scheme $l = q = g^{(3)} = g = 0$ is reduced at the generic point.

It is easy to see that for a general variety $V \in \mathcal{V}(a_*, \mu, \lambda)$ the conditions (G1-G3) are true: each of them deals with a finite set of points.

### 1.3 The main result

Now let us give a precise formulation of our main result.

**Theorem 1$^*$.** *Let $V \in \mathcal{V}(a_*, \mu, \lambda)$ be a variety, satisfying the conditions (G1-G3) and the $K^2$-condition. Then the Fano fibration $\pi: V \to \mathbf{P}^1$ is birationally superrigid.*

Note that since the linear system $|L|$ is free, the $K^2$-condition is guaranteed by the inequality

$$\left(K_V^2 \cdot \sigma^*(L|_Q)^{M-1}\right) \leq 0.$$

Explicitly, this inequality takes the form

$$3\mu + 4\lambda + 2\sum_{i=1}^{M+1} a_i \geq 8.$$

For instance, if $X = \mathbf{P}^1 \times \mathbf{P}^{M+1}$, i.e. $a_1 = \ldots = a_{M+1} = 0$, then for $\mu \geq 3$ we have no restrictions for the hypersurface $W$ except for the conditions of general position. If $\mu = 0$, i.e. $Q = \mathbf{P}^1 \times Q_t$, then we get the case, which was studied in [P3]. In any case, however, we get one more confirmation of the general principle: "twistedness" over the base implies rigidity.

# 2 Proof of Theorem 1$^*$

## 2.1 The method of maximal singularities

The principal concepts, constructions and theorems of the method have been published many times, starting from the fundamental paper of V.A.Iskovskikh and Yu.I.Manin on the quartic [IM], see [IP,P2,P4]. Therefore there is no need to reproduce the details here. For Fano fibrations over $\mathbf{P}^1$ the technique of the method of maximal singularities was developed in [P4] and later used in [P3,P5]. In the present paper we formulate a sufficient condition of superrigidity in terms of degrees and multiplicities of algebraic cycles and verify it for our Fano fibrations. The details of the proof of this sufficient condition the reader can find in [P3-P5].

Let $\pi: Y \to \mathbf{P}^1$ be a smooth Fano fibration with $\operatorname{Pic} Y = \mathbf{Z}K_Y \oplus \pi^* \operatorname{Pic} \mathbf{P}^1$, satisfying the $K^2$-condition, i.e. the numerical class $NK_Y^2 - f$, where $f \in A^2 Y$ is the class of a hyperplane section of a fiber $F_t = \pi^{-1}(t)$, does not represent an effective cycle of codimension 2 for any $N \in \mathbf{Z}$. Following [P3,P4], we call an irreducible cycle $B \subset Y$ *horizontal*, if it covers the base, $\pi(B) = \mathbf{P}^1$, and *vertical*, if it lies in a fiber of $\pi$. The *degree* of a vertical cycle $B$ is defined to be the integer

$$\deg B = (B \cdot (-K_Y)^{\dim B}) > 0.$$

The *degree* of a horizontal cycle $B$ is defined to be the integer $\deg B = \deg(B \cap F_t)$, where $F_t$ is a fiber of general position.

**Proposition 2.1. (a sufficient condition of birational superrigidity)** *Assume that for each moving linear system $\Sigma \subset |-nK_Y + lF|$, each irreducible vertical cycle $A \subset F_t$ of codimension 2 (with respect to $Y$), each irreducible horizontal* *cycle $B \subset Y$ of codimension 2, each irreducible vertical cycle $C \subset F_t$ of codimension 3 (with respect to $Y$) and each point $x \in F_t$ the following conditions are satisfied:*

(i) $\mathrm{mult}_A \Sigma \leq n$, $\mathrm{mult}_B \Sigma \leq n$;

(ii) $\mathrm{mult}_x C \leq (4/\deg Y)\, \mathrm{mult}_x F_t \deg C$;

(iii) $\mathrm{mult}_x A \leq (2\, \mathrm{mult}_x F_t/\deg Y) \deg A$, $\mathrm{mult}_y \widetilde{A} \leq (2/\deg Y) \deg A$, *where $y \in E_x \subset \widetilde{F}_t$ is an arbitrary infinitely near point over $x \in F_t$ and $\widetilde{A}$ is the proper inverse image of $A$ with respect to the blow-up of the point $x$, $\varphi\colon \widetilde{F}_t \to F_t$, $E_x = \varphi^{-1}(x)$.*

*Then the Fano fibration $\pi\colon Y \to \mathbf{P}^1$ is birationally superrigid.*

Here $\deg Y = (F \cdot (-K_Y)^{\dim Y - 1})$, in accordance with the definition of degree, given above.

For the **proof**, see [P3-P5]. The scheme of arguments is as follows. If $Y \to \mathbf{P}^1$ is not birationally superrigid, then there is a moving linear system $\Sigma \subset |-nK_Y + lF|$ on $Y$ with a maximal singularity $\nu$. If the centre of the maximal singularity is of codimension 2, we get a contradiction with the condition (i). Let $x \in F_t$ be a point of the centre. Consider the effective cycle $Z = (D_1 \bullet D_2)$, $D_i \in \Sigma$, and decompose it into the vertical and horizontal components

$$Z = Z^h + Z^v = Z^h + \sum_{s \in \mathbf{P}^1} Z_s^v,$$

where the support of the cycle $Z_s^v$ is contained in the fiber $F_s$. Note that by the condition (ii)

$$\mathrm{mult}_x Z^h \leq 4n^2. \tag{1}$$

Indeed, take the effective cycle $C = (Z^h \bullet F_t)$. Obviously, $\mathrm{mult}_x C \geq \mathrm{mult}_x Z^h\, \mathrm{mult}_x F_t$, whereas $\deg Z^h = \deg C = n^2 \deg Y$. Applying (ii), we get the estimate (1). Furthermore, by the condition (iii) we get

$$\mathrm{mult}_x Z_t^v \leq 4nl. \tag{2}$$

We may assume that the maximal singularity $\nu$ is *supermaximal* [P3,P4]. Now arguing as in [P3,P4], we get a contradiction.

Thus to prove Theorem 1* it is sufficient to check that the conditions (i)-(iii) are satisfied for our Fano fibration $\pi\colon V \to \mathbf{P}^1$.

## 2.2 Maximal cycles

The condition (i) has been actually verified in [P3]. Since a linear system on a smooth double quadric of dimension $\geq 4$ can not have a maximal singularity [P1], a moving linear system $\Sigma$ on $V$ can not have maximal singularities covering the base, i.e. $\mathrm{mult}_B \Sigma \leq n$ for each horizontal cycle $B \subset V$ of codimension 2. Furthermore, for each Weil divisor $A \subset F_t$ we have: $\sigma(A) \subset Q_t$ is a prime Weil divisor on the quadric $Q_t$. Divisors of the family $\sigma(\Sigma|_{F_t})$ are cut out on $Q_t$ by hypersurfaces of degree $2n$, so that if $\mathrm{mult}_A \Sigma > n$, then $\sigma(A) = H \cap Q_t$ is a hyperplane section and, moreover, the morphism $\sigma\colon A \to \sigma(A)$ is birational. Therefore the scheme-theoretic intersection $H \cap Q_t \cap W$ is non-reduced at every point. But this is impossible: by Lemma 3.4.2 from [IP] $\dim \mathrm{Sing}\, H \cap Q_t \cap W \leq 1$. Contradiction.

Q.E.D. for the condition (i).

## 2.3 Proof of condition (ii)

Obviously, $\deg V = 4$, so that (ii) takes the form of the inequality

$$\mathrm{mult}_x\, C \leq \mathrm{mult}_x\, F_t \deg C,$$

which is certainly true, since a stronger inequality is true: $\mathrm{mult}_x\, C \leq \deg C$.

## 2.4 Proof of condition (iii)

If $z = \sigma(x)$ is a smooth point of the quadric $Q_t$, then we prove (iii), repeating the arguments of the paper [P3] word for word. The scheme of the proof is as follows. There are two possibilities: $\sigma(A) = T_z Q_t \cap Q_t$ and $\sigma(A) \neq T_z Q_t \cap Q_t$. In the first case (iii) can be verified directly, taking into account (G1), (G2) and the above-mentioned Lemma 3.4.2 from [IP]. In the second case we obtain (iii), applying the estimate (ii), which has already been proved, to the effective cycle $(A \bullet \sigma^{-1}(T_z Q_y \cap Q_t))$ of codimension 2.

Assume that $z = \sigma(x)$ is the vertex of the cone $Q_t$. First of all, note that of the two inequalities (iii) the first one in this case is trivial: it asserts that $\mathrm{mult}_x\, A \leq \deg A$. Assume that for a prime Weil divisor $A \subset F_t$ the second inequality is not true:

$$\mathrm{mult}_y\, \widetilde{A} > \frac{1}{2} \deg A,$$

where $y \in E_x = \varphi^{-1}(x) \subset \widetilde{F}_t$ is an infinitely near point of the first order over $x$, $\varphi\colon \widetilde{F}_t \to F_t$ is the blowing up of $x$. Let us show that this assumption leads to a contradiction. We do it in three steps.

**Step 1.** The point $z = \sigma(x)$ does not lie on the branch divisor, so that $\sigma(y)$ is a well-defined infinitely near point of the first order over $z$, defining a line $(z, \sigma(y)) \subset Q_t$. Denote by $T_y Q_t$ the hyperplane in $\mathbf{P}$, which is tangent to $Q_t$ along this line. Set $T = \sigma^{-1}(T_y Q_t \cap Q_t) \subset F_t$. Obviously, $\mathrm{mult}_y\, \widetilde{T} = 2$, $\deg T = 4$, so that the condition (iii) holds for $T$. In particular, $A \neq T$.

**Step 2.** Set $B = E_x \cap \widetilde{T}$ (if $T_y Q_t \cap Q_t$ is looked at as an affine cone with the vertex $x$, then $B$ is its projectivization).

**Lemma 2.1.** *The following estimate holds:*

$$\mathrm{mult}_B\, \widetilde{A} \geq \frac{1}{4}(2\, \mathrm{mult}_y\, \widetilde{A} - \mathrm{mult}_x\, A).$$

**Proof:** see [P5], Sec. 3.6.

Now consider an algebraic cycle $(A \bullet T) = C$. It is easy to see that $C$ satisfies the inequality

$$\mathrm{mult}_x\, C > \frac{3}{4} \deg C.$$

**Step 3.** Let us show that this is impossible. Let us assume that $M \geq 5$. Obviously, the cycle $C$ can be assumed to be irreducible. Let $H = T_y Q_t \cong \mathbf{P}^M$ be the tangent hyperplane, $\sigma\colon R \to H$ the double cover, branched over the hypersurface

$W_H = W \cap H$. The variety $R$ is irreducible and possibly singular. The cycle $T$ naturally identifies with a divisor in $R$. Thus $C$ can be identified with a codimension 2 cycle in $R$, too. Meaning this identification, we write $x \in R$ etc.

Let $P \in \operatorname{Pic} R$ be the class of a hyperplane section, i.e. $\mathcal{O}_R(P) \cong \sigma^* \mathcal{O}_H(1)$, $H \hookrightarrow \mathbf{P}$. We define the *k-th hypertangent system* at the point $x \in R$ by the condition

$$\Lambda_k = \{ \ D \in |kP| \ \mid \ \operatorname{mult}_x D \geq k+1 \}.$$

**Lemma 2.2.** *The hypertangent linear systems $\Lambda_k$ are non-empty for $k = M-1, \ldots, 2M-3$. Moreover,*

$$\operatorname{codim}_T \operatorname{Bs}(\Lambda_{M-2+i}|_T) \geq i$$

*for $i = 1, \ldots, M-2$.*

**Proof.** It is immediately implied by the condition (G3), where we take $H$ as that hyperplane, containing the point $p = \sigma(x) \in Q_t \cap H$. Indeed, let $z_1, \ldots, z_{M+1}$ be a system of linear coordinates with the origin at $p$, where $\mathbf{P} \supset H = \{z_{M+1} = 0\}$. Take $U = \mathbf{C}^M_{(z_1,\ldots,z_M)}$. Over the set $U$ the variety $R$ is defined by the equation $y^2 = g(z_1, \ldots, z_M, 0)$ in $\mathbf{C}_{(y)} \times U$. The coordinate $y$ defines a section of the sheaf $\mathcal{O}_R((M-1)P)$. Therefore, the functions

$$f_k = y - [\sqrt{g}]_k,$$

regular on $\mathbf{C} \times U$, for $k \geq M-1$ define sections of the sheaves $\mathcal{O}_R(kP)$, respectively. Moreover, for $k \geq M-1$ set

$$\Delta_k = \overline{\{f_i = 0\}} \subset R.$$

It is easy to see that $\Delta_k \in \Lambda_k$. The image of the divisor $\Delta_k$ on $U = \mathbf{C}^M$ is given by the equation $g^{(k)}(z_1, \ldots, z_M, 0) = 0$. Now applying (G3), we get the assertion of our lemma.

Let $D_i \in \Lambda_{M-2+i}$, $i = 1, \ldots, M-1$, be general divisors. Let us construct a sequence of irreducible cycles $C_2, C_3, \ldots, C_{M-1} \subset R$, $\operatorname{codim} C_j = j$, in the following way. Set $C_2 = C$. Assume that $C_j$, $j = 2, \ldots, l$, have already been constructed and $l \leq M-2$. By Lemma 2.2 a general divisor $D_l$ does not contain $C_l$. By the definition of hypertangent systems there exists an irreducible component $Z$ of the cycle $(C_l \bullet D_l)$, satisfying the inequality

$$\frac{\operatorname{mult}_x}{\deg} Z \geq \frac{\operatorname{mult}_x}{\deg} C_l \cdot \frac{M-1+l}{M-2+l};$$

set $C_{l+1} = Z$. The cycle $C_{M-1} \subset R$ is an irreducible curve. We get the estimate

$$\frac{\operatorname{mult}_x}{\deg} C_{M-1} \ > \ \frac{3}{4} \cdot \frac{M+1}{M} \cdot \underbrace{\frac{M+2}{M+1} \cdot \ldots \cdot \frac{2M-3}{2M-4}}_{} = \frac{3(2M-3)}{4M},$$

and it is easy to see that $3(2M-3)/4M \geq 1$, which is impossible.

Now assume that $M=4$. Here the linear system $\Lambda_3$ contains the single effective divisor $D_1$. There are two possible cases: either $C \subset D_1$, or $C \not\subset D_1$. In the first case (iii) is implied by the condition of general position, since $C = T \cap D_1$. In the second case the cycle $C_3 = (C \bullet D_1)$ is effective. However, it satisfies the inequality $(\mathrm{mult}_x / \deg) C_3 > 1$, which is impossible. Q.E.D. for the property (iii) and thus for Theorem 1*.

# 3 Three-dimensional case

Most of the arguments above work for $M=3$ without any modifications. There are, however, two points of possible complications. First of all, the dimension no more allows to use Lemma 3.4.2 from [IP]. However, imposing additional conditions of general position on $V$, we can avoid using this lemma. The reader will easily reconstruct the necessary details.

The second complication is more essential: in dimension three maximal cycles can appear on $V$! The reason is that on a smooth three-dimensional double quadric of index 1 each line outside the branch divisor can be a maximal singularity, see [IP]. Accordingly, the group of birational self-maps of this variety is rather big (it was described by V.A.Iskovskikh in the mid-70s).

Let $\mathcal{L}(V)$ be the (quasi-projective) variety of lines in the fibers of the morphism $\pi: V \to \mathbf{P}^1$, which lie outside the branch divisor, i.e.

$$\mathcal{L}(V) = \{L \subset F_t \quad | \quad (L \cdot K_V) = -1, \sigma(L) \not\subset W\}.$$

The natural projection $\mathcal{L}(V) \to \mathbf{P}^1$ will be denoted by the same symbol $\pi$. Each section (defined, generally speaking, outside a finite set of points) $s: \mathbf{P}^1 \to \mathcal{L}(V)$, $\pi \circ s = \mathrm{id}$, defines a fiber-wise birational self-map $\tau_s \in \mathrm{Bir}\, F_\eta \subset \mathrm{Bir}\, V$, where $F_\eta$ is the fiber over the generic point, that is, a Fano variety over the non-closed field $\mathbf{C}(\mathbf{P}^1)$. The involutions $\tau_s$ freely generate in $\mathrm{Bir}\, V$ a normal subgroup

$$B(V) = \mathop{*}_{\substack{s: \mathbf{P}^1 \to \mathcal{L}(V) \\ \pi \circ s = \mathrm{id}}} \langle \tau_s \rangle,$$

see [IP]. The following exact sequence

$$1 \to B(V) \to \mathrm{Bir}\, V \to \mathrm{Aut}\, V \to 1 \tag{3}$$

holds, where the group $\mathrm{Aut}\, V$ is finite, generically isomorphic to $\mathbf{Z}/2\mathbf{Z}$.

Repeating the arguments of Sec.2 word for word and taking into account these remarks, we get the following result.

**Theorem 2.** *A general four-dimensional Fano fibration $V$, satisfying the $K^2$-condition, is birationally rigid. Its group of birational self-maps is described by the exact sequence (3).*

One may ask: do the sections $s: \mathbf{P}^1 \to \mathcal{L}(V)$ exist for a general Fano fibration $V$? The author does not know the answer, but it seems to be negative. If it *is* negative, then for $M=3$ we also get birational superrigidity.

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