# RANK AND CRANK MOMENTS FOR OVERPARTITIONS

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Abstract. We study two types of crank moments and two types of rank moments for overpartitions. We show that the crank moments and their derivatives, along with certain linear combinations of the rank moments and their derivatives, can be written in terms of quasimodular forms. We use this fact to prove exact relations involving the moments as well as congruence properties modulo 3, 5, and 7 for some combinatorial functions which may be expressed in terms of the second moments. Finally, we establish a congruence modulo 3 involving one such combinatorial function and the Hurwitz class number $H(n)$.

## 1. Introduction

Dyson's rank of a partition is the largest part minus the number of parts [13]. The Andrews-Garvan crank is either the largest part, if 1 does not occur, or the difference between the number of parts larger than the number of 1's and the number of 1's, if 1 does occur [1]. Let $N(m,n)$ denote the number of partitions of $n$ whose rank is $m$. For $n \neq 1$ let $M(m,n)$ denote the number of partitions of $n$ whose crank is $m$. Even though there is only one partition of one, for technical reasons we set $M(0,1) = -1$, $M(-1,1) = M(1,1) = 1$, and $M(m,1) = 0$ otherwise. Then the *kth rank moment* $N_k(n)$ and the *kth crank moment* $M_k(n)$ are given by

$$N_k(n) := \sum_{m \in \mathbb{Z}} m^k N(m,n), \tag{1.1}$$

and

$$M_k(n) := \sum_{m \in \mathbb{Z}} m^k M(m,n). \tag{1.2}$$

Since their introduction by Atkin and Garvan [4], the rank and crank moments and their linear combinations have been the subject of a number of works [2, 3, 5, 6, 15, 16]. A key role in several of these studies is played by the fact that the crank moments and their derivatives, along with a specific linear combination of the rank moments and their derivatives, can be expressed in terms of quasimodular forms [4]. Here we shall see that this holds in the case of overpartitions as well.

Recall that an overpartition [12] is a partition in which the first occurrence of each distinct number may be overlined. For example, the 14 overpartitions of 4 are

$$\begin{gathered} 4, \overline{4}, 3+1, \overline{3}+1, 3+\overline{1}, \overline{3}+\overline{1}, 2+2, \overline{2}+2, 2+1+1, \overline{2}+1+1, 2+\overline{1}+1, \\ \overline{2}+\overline{1}+1, 1+1+1+1, \overline{1}+1+1+1. \end{gathered} \tag{1.3}$$

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We denote by $\overline{P}$ the generating function for overpartitions [12],

$$\overline{P} = \prod_{n\geq 1} \frac{(1+q^n)}{(1-q^n)}.$$

The case of overpartitions is somewhat different from that of partitions. First, there are two distinct ranks of interest: Dyson's rank and the $M2$-rank [19]. The $M2$-rank is a bit more complicated than Dyson's rank. We use the notation $\ell(\cdot)$ to denote the largest part of an object, $n(\cdot)$ to denote the number of parts, and $\lambda_o$ for the subpartition of an overpartition consisting of the odd non-overlined parts. Then the $M2$-rank of an overpartition $\lambda$ is

$$M2\text{-rank}(\lambda) := \left\lceil \frac{\ell(\lambda)}{2} \right\rceil - n(\lambda) + n(\lambda_o) - \chi(\lambda),$$

where $\chi(\lambda) = 1$ if the largest part of $\lambda$ is odd and non-overlined and $\chi(\lambda) = 0$ otherwise.

Let $\overline{N}(m,n)$ (resp. $\overline{N2}(m,n)$) denote the number of overpartitions of $n$ whose rank (resp. $M2$-rank) is $m$. We define the rank moments $\overline{N}_k(n)$ and $\overline{N2}_k(n)$, along with their generating functions $\overline{R}_k$ and $\overline{R2}_k$, by

$$\overline{R}_k := \sum_{n\geq 0} \overline{N}_k(n) q^n := \sum_{n\geq 0} \left( \sum_{m\in\mathbb{Z}} m^k \overline{N}(m,n) \right) q^n, \tag{1.4}$$

and

$$\overline{R2}_k := \sum_{n\geq 0} \overline{N2}_k(n) q^n := \sum_{n\geq 0} \left( \sum_{m\in\mathbb{Z}} m^k \overline{N2}(m,n) \right) q^n. \tag{1.5}$$

We note that in light of the symmetries $\overline{N}(m,n) = \overline{N}(-m,n)$ [18] and $\overline{N2}(m,n) = \overline{N2}(m,n)$ [19], we have $\overline{R}_k = \overline{R2}_k = 0$ when $k$ is odd.

The second difference between partitions and overpartitions is that in the latter case no notion of crank has been defined. Indeed, the crank for partitions arose because of its relation to Ramanujan's congruences, and Choi has shown that no such congruences exist for overpartitions [10]. What will be required are two "residual cranks". The first residual crank of an overpartition is obtained by taking the crank of the subpartition consisting of the non-overlined parts. The second residual crank is obtained by taking the crank of the subpartition consisting of all of the even non-overlined parts divided by two.

Let $\overline{M}(m,n)$ (resp. $\overline{M2}(m,n)$) denote the number of overpartitions of $n$ with first (resp. second) residual crank equal to $m$. Here we make the appropriate modifications based on the fact that for partitions we have $M(0,1) = -1$ and $M(-1,1) = M(1,1) = 1$. For example, the overpartition $\overline{7} + \overline{5} + \overline{2} + 1$ contributes a $-1$ to the count of $\overline{M}(0,15)$ and a $+1$ to $\overline{M}(-1,15)$ and $\overline{M}(1,15)$. Define the crank moments $\overline{M}_k(n)$ and $\overline{M2}_k(n)$, along with their generating functions $\overline{C}_k$ and $\overline{C2}_k$, by

$$\overline{C}_k := \sum_{n\geq 0} \overline{M}_k(n) q^n := \sum_{n\geq 0} \left( \sum_{m\in\mathbb{Z}} m^k \overline{M}(m,n) \right) q^n, \tag{1.6}$$

and

$$\overline{C2}_k := \sum_{n\geq 0} \overline{M2}_k(n) q^n := \sum_{n\geq 0} \left( \sum_{m\in\mathbb{Z}} m^k \overline{M2}(m,n) \right) q^n. \tag{1.7}$$

As with the rank moments, the crank moments turn out to be 0 for $k$ odd (see (2.3) and (2.4)).

We are now ready to state the quasimodularity properties of the rank and crank moments for overpartitions.

**Theorem 1.1.** *For $k \geq 1$ let $\overline{\mathcal{W}}_k$ denote the space of quasimodular forms on $\Gamma_0(2)$ of weight at most $2k$ having no constant term. The following functions are in $\overline{P} \cdot \overline{\mathcal{W}}_k$:*

(*i*) *For $m \geq 0$, the functions in*

$$\overline{\mathcal{C}}_k := \{\delta_q^m(\overline{C}_{2j}) : 1 \leq j \leq k, j+m \leq k\},$$

(*ii*) *For $m \geq 0$, the functions in*

$$\overline{\mathcal{C}2}_k := \{\delta_q^m(\overline{C2}_{2j}) : 1 \leq j \leq k.j+m \leq k\},$$

(*iii*) *For $a = 2k$,*

$$(a^2 - 3a + 2)\overline{R}_a + 2\sum_{i=1}^{a/2-1} \binom{a}{2i}(3^{2i} - 2^{2i} - 1)\delta_q \overline{R}_{a-2i}$$

$$+ \sum_{i=1}^{a/2-1} \left(\binom{a}{2i}(2^{2i}+1) + 2\binom{a}{2i+1}(1 - 2^{2i+1}) + \frac{1}{2}\binom{a}{2i+2}(3^{2i+2} - 2^{2i+2} - 1)\right)\overline{R}_{a-2i},$$

(*iv*) *For $a = 2k$,*

$$(a^2 - 3a + 2)\overline{R2}_a + \frac{1}{2}\sum_{i=1}^{a/2-1} \binom{a}{2i}(3^{2i} - 2^{2i} - 1)\delta_q \overline{R2}_{a-2i}$$

$$+ \sum_{i=1}^{a/2-1} \left(\binom{a}{2i}(2^{2i}+1) + 2\binom{a}{2i+1}(1 - 2^{2i+1}) + \frac{1}{2}\binom{a}{2i+2}(3^{2i+2} - 2^{2i+2} - 1)\right)\overline{R2}_{a-2i}.$$

It turns out that for $k = 2, 3$ and $4$ the number of functions above exceeds the dimension of $\overline{\mathcal{W}}_k$, which implies relations among these functions. In Corollaries 3.1–3.3, we compute several such relations. This is the same approach taken by Atkin and Garvan in their study of rank and crank moments of partitions.

Then we show how Theorem 1.1 can be used to deduce congruence properties for two groups of combinatorial functions which can be expressed in terms of second rank and crank moments. For the first group, let $nov(n)$ (resp. $ov(n)$) denote the sum, over all overpartitions of $n$, of the non-overlined (resp. overlined) parts. For example, (1.3) shows that $ov(4) = 21$ and $nov(4) = 35$.

**Theorem 1.2.** *We have*

$$nov(3n) \equiv ov(3n) \pmod 3, \tag{1.8}$$

$$(n+2)nov(n) \equiv (n^2 + 4n + 3)ov(n) \pmod 5, \tag{1.9}$$

*and*

$$(n^2+1)nov(n) \equiv (4n^3 - n^2 - 1)ov(n) \pmod 7. \tag{1.10}$$

Notice that congruences like (1.9) and (1.10) imply congruences in arithmetic progressions for $ov(n)$ and $nov(n)$ modulo 5 and 7.

For the second group, let $\overline{spt1}(n)$ (resp. $\overline{spt2}(n)$) denote the sum, over all overpartitions $\lambda$ of $n$, of the number of occurrences of the smallest part of $\lambda$, provided this smallest part is odd (resp. even). Let $\overline{spt}(n)$ be the sum of these two functions. For example, using (1.3) we have $\overline{spt1}(4) = 20$, $\overline{spt2}(4) = 6$, and $\overline{spt}(4) = 26$. When the overpartition has no overlined parts, $\overline{spt}(n)$ reduces to Andrews' smallest parts function $spt(n)$ [3, 15, 16].

**Theorem 1.3.** *We have*

$$\overline{spt2}(3n) \equiv \overline{spt2}(3n+1) \equiv 0 \pmod 3, \tag{1.11}$$

$$\overline{spt}(3n) \equiv 0 \pmod 3, \tag{1.12}$$

$$\overline{spt2}(5n+3) \equiv 0 \pmod 5, \tag{1.13}$$

*and*

$$\overline{spt1}(5n) \equiv 0 \pmod 5. \tag{1.14}$$

For our last result we give a congruence modulo 3 between $\overline{spt1}(n)$ and the Hurwitz class number $H(n)$ of binary quadratic forms of discriminant $-n$ which depends not on Theorem 1.1, but on the fact that the generating function for $\overline{spt}1(n)$ is a quasimock theta function. To state this congruence, let $r(n)$ be defined by

$$r(n) := \begin{cases} 12H(4n) & \text{if } n \equiv 1,2 \pmod 4, \\ 24H(n) & \text{if } n \equiv 3 \pmod 8, \\ r(n/4) & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \equiv 7 \pmod 8, \end{cases} \tag{1.15}$$

and let $\overline{\alpha}(n)$ be defined by

$$(-1)^n \overline{\alpha}(n) := \begin{cases} -4H(4n) & \text{if } n \equiv 1,2 \pmod 4, \\ -24H(n) & \text{if } n \equiv 3 \pmod 8, \\ -16H(n) & \text{if } n \equiv 7 \pmod 8, \\ -16H(n) - \frac{1}{3}r(n/4) & \text{if } 4 \mid n. \end{cases} \tag{1.16}$$

**Theorem 1.4.** *We have*

$$\overline{spt1}(n) \equiv \left(\frac{-n}{3}\right) \overline{\alpha}(n) \pmod 3.$$

As a corollary, class number relations imply the following multiplicative formula:

**Corollary 1.5.** *Let $\ell \neq 2,3$ be a prime. Then we have*

$$\overline{spt1}\left(\ell^2 n\right) + \left(\frac{-n}{\ell}\right) \overline{spt1}(n) + \ell\, \overline{spt1}\left(\frac{n}{\ell^2}\right) \equiv (\ell+1)\overline{spt1}(n) \pmod 3.$$

The paper is organized as follows. In Section 2, we recall some facts about quasimodular forms and prove Theorem 1.1. In Section 3, we compute some exact relations involving rank and crank moments. In Section 4, we write the combinatorial functions in Theorems 1.2 and 1.3 in terms of the rank and crank moments and prove these theorems. In Section 5, we recall the notion of quasimock theta function along with some results from [8], and prove Theorem 1.4 and Corollary 1.5.

## 2. Proof of Theorem 1.1

Before proving Theorem 1.1, we recall a few facts about quasimodular forms [17]. First, quasimodular forms on $\Gamma_0(N)$ may be regarded as polynomials in the Eisenstein series $E_2$ whose coefficients are modular forms (of non-negative weight) on $\Gamma_0(N)$. The reader unfamiliar with the theory of modular forms may consult [20]. Here we have

$$E_2 := 1 - 24 \sum_{n \geq 1} \frac{nq^n}{(1-q^n)}. \tag{2.1}$$

Second, the space of quasimodular forms on $\Gamma_0(N)$ is preserved by the differential operator $\delta_q := q\frac{d}{dq}$. More specifically, this operator sends a quasimodular form of weight $2k$ to a quasimodular form of weight $2k+2$. Finally, replacing $q$ by $q^2$ sends a quasimodular form of weight $2k$ on $\Gamma_0(N)$ to a quasimodular form of weight $2k$ on $\Gamma_0(2N)$.

*Proof of Theorem 1.1.* We now prove parts $(i)$ and $(ii)$ of Theorem 1.1. Let $C(z,q)$ denote the two-variable generating function for the crank of a partition,

$$C(z,q) := \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} M(m,n) z^m q^n. \tag{2.2}$$

By definition, the residual cranks have two-variable generating functions

$$\overline{C}(z,q) := \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} \overline{M}(m,n) z^m q^n = (-q;q)_\infty C(z,q), \tag{2.3}$$

and

$$\overline{C2}(z,q) := \sum_{\substack{m \in \mathbb{Z} \\ n \geq 0}} \overline{M2}(m,n) z^m q^n = \frac{(-q;q)_\infty}{(q;q^2)_\infty} C(z,q^2). \tag{2.4}$$

Here we employ the standard $q$-series notation,

$$(a;q)_\infty := \prod_{k \geq 0} (1 - aq^k). \tag{2.5}$$

Now using the differential operator $\delta_z := z\frac{d}{dz}$ we have

$$\delta_z^j(\overline{C}(z,q))\Big|_{z=1} = \begin{cases} \overline{C}_j & j \text{ even}, \\ 0 & j \text{ odd}, \end{cases} \tag{2.6}$$

and

$$\delta_z^j(\overline{C2}(z,q))\Big|_{z=1} = \begin{cases} \overline{C2}_j & j \text{ even}, \\ 0 & j \text{ odd}. \end{cases} \tag{2.7}$$

But $\delta_z^j(\overline{C}(z,q)) = (-q;q)_\infty \delta_z^j(C(z,q))$ and Atkin and Garvan [4, Section 4] have already shown that if $j \geq 1$, then $\delta_z^j(C(z,q))|_{z=1}$ is in the space $P \cdot \mathcal{W}_j$, where $P = 1/(q;q)_\infty$ is the generating function for partitions and $\mathcal{W}_j$ is the space of quasimodular forms of weight at most $2j$ on $\Gamma_0(1)$ having no constant term. Since $\overline{P} = (-q;q)_\infty P$, we have that $\overline{C}_{2j}$ is in $\overline{P} \cdot \overline{\mathcal{W}}_j$. In a similar way we see that $\overline{C2}_{2j}$ is in $\overline{P} \cdot \overline{\mathcal{W}}_j$.

To finish we may calculate that

$$\delta_q(\overline{P}) = \overline{P}\left(\sum_{n\geq 1}\frac{2nq^n}{(1-q^n)} - \sum_{n\geq 1}\frac{2nq^{2n}}{(1-q^{2n})}\right), \tag{2.8}$$

and hence $\delta_q(\overline{P}) \in \overline{P}\cdot\overline{\mathcal{W}}_1$. By the Leibniz rule and the fact that $\delta_q$ maps the space $\overline{\mathcal{W}}_k$ into $\overline{\mathcal{W}}_{k+1}$, we have that $\delta_q f \in \overline{P}\cdot\overline{\mathcal{W}}_{k+1}$ if $f \in \overline{P}\cdot\overline{\mathcal{W}}_k$. This completes the proof of parts $(i)$ and $(ii)$.

For parts $(iii)$ and $(iv)$, we use partial differential equations established in [8]. Let $\overline{R}(z,q)$ denote the two-variable generating function for $\overline{N}(m,n)$,

$$\overline{R}(z,q) := \sum_{\substack{m\in\mathbb{Z}\\ n\geq 0}} \overline{N}(m,n)z^m q^n. \tag{2.9}$$

Thus we have

$$\delta_z^j(\overline{R}(z,q))\Big|_{z=1} = \begin{cases} \overline{R}_j & j \text{ even},\\ 0 & j \text{ odd}.\end{cases} \tag{2.10}$$

We have the following partial differential equation which relates $C(z,q)$ and $\overline{R}(z,q)$ [8]:

$$\begin{aligned} &z(1+z)\frac{(q)_\infty^2}{(-q)_\infty}[C(z,q)]^3(-zq;q)_\infty(-q/z;q)_\infty \\ &\quad = \left(2(1-z)^2(1+z)\delta_q + z(1+z) + 2z(1-z)\delta_z + \frac{1}{2}(1+z)(1-z)^2\delta_z^2\right)\overline{R}(z,q). \end{aligned} \tag{2.11}$$

Let $a$ be even and positive. After applying $\delta_z^a$ to both sides of (2.11) and setting $z=1$ we get

$$\begin{aligned} &\frac{1}{P\overline{P}}\sum_{j=0}^{a}\binom{a}{j}\delta_z^j\{(z^2+z)C(z,q)^3\}\delta_z^{a-j}\{(-zq;q)_\infty(-q/z;q)_\infty\}|_{z=1} - (2^a+1)\overline{P} - 2(3^a-2^a-1)\delta_q(\overline{P}) = \\ &\qquad (a^2-3a+2)\overline{R}_a + 2\sum_{i=1}^{a/2-1}\binom{a}{2i}(3^{2i}-2^{2i}-1)\delta_q\overline{R}_{a-2i} \\ &\quad + \sum_{i=1}^{a/2-1}\left(\binom{a}{2i}(2^{2i}+1) + 2\binom{a}{2i+1}(1-2^{2i+1}) + \frac{1}{2}\binom{a}{2i+2}(3^{2i+2}-2^{2i+2}-1)\right)\overline{R}_{a-2i}. \end{aligned} \tag{2.12}$$

We claim that the left hand side of (2.12) is in $\overline{P}\cdot\overline{\mathcal{W}}_k$ where $2k=a$. This is clearly true for the final term. For the first term, we have already noted that for $j\geq 1$ we have $\delta_z^j C(z,q)|_{z=1} \in P\cdot\overline{\mathcal{W}}_j$. As for $(-zq;q)_\infty(-q/z;q)_\infty$, we may compute that

$$\begin{aligned} \delta_z\Big((-zq;q)_\infty(-q/z;q)_\infty\Big) &= \left(z\sum_{m=1}^{\infty}\frac{q^m}{1+zq^m} - z^{-1}\sum_{m=1}^{\infty}\frac{q^m}{1+z^{-1}q^m}\right)(-zq;q)_\infty(-q/z;q)_\infty \\ &= \left(\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(-1)^s q^{ms}(z^{-s}-z^s)\right)(-zq;q)_\infty(-q/z;q)_\infty, \end{aligned}$$

and for $j \geq 1$,

$$\delta_z^j\Big(\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(-1)^s q^{ms}(z^{-s}-z^s)\Big)\Big|_{z=1} = \begin{cases} 0 & j \text{ even}, \\ -2\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(-1)^s s^j q^{ms} & j \text{ odd}. \end{cases}$$

Then one can check that

$$-2\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}(-1)^s s^j q^{ms} = -2^{j+2}\sum_{n\geq 1}\frac{n^j q^{2n}}{(1-q^{2n})} + 2\sum_{n\geq 1}\frac{n^j q^n}{(1-q^n)}.$$

Thus for $j \geq 1$ we have $\delta_z^j\{(-zq;q)_\infty(-q/z;q)_\infty\}|_{z=1} \in (\overline{P}^2/P^2)\cdot\overline{\mathcal{W}}_j$. Putting everything together we see that the only contribution from the first term on the left hand side which is not in $\overline{P}\cdot\overline{\mathcal{W}}_k$ is

$$\frac{1}{P\overline{P}}\delta_z^a\{(z^2+z)\}C(z,q)^3(-zq;q)_\infty(-q/z;q)_\infty|_{z=1}.$$

But this cancels with the second term on the left hand side. This establishes part $(iii)$.

The proof of part $(iv)$ is the same, except that we use the partial differential equation [8]

$$\begin{aligned} &2z(1+z)\left(q^2;q^2\right)_\infty^2\left[C(z,q^2)\right]^3(-zq;q)_\infty(-q/z;q)_\infty \\ &= \Big((1+z)(1-z)^2\delta_q + 2z(1+z) + 4z(1-z)\delta_z + (1+z)(1-z)^2\delta_z^2\Big)\overline{R2}(z,q). \end{aligned} \tag{2.13}$$

Here $\overline{R2}(z,q)$ is the two-variable generating function for $\overline{N2}(m,n)$, so that

$$\delta_z^j(\overline{R2}(z,q))\Big|_{z=1} = \begin{cases} \overline{R2}_j & j \text{ even}, \\ 0 & j \text{ odd}. \end{cases} \tag{2.14}$$

□

## 3. Exact relations

From part $(b)$ of Proposition 1 in [17] and known formulas for the dimensions of spaces of modular forms on $\Gamma_0(2)$ (see [20]), we have that the sequence $\{\dim(\overline{\mathcal{W}_k})\}_{k=1}^{\infty}$ begins $\{2,6,12,21,33,49,\dots\}$. Suppose first that $k=2$. Then there are 6 functions in parts $(i)$ and $(ii)$ of Theorem 1.1. Computation (with MAPLE, for example) shows that they are linearly independent. Hence, each function in parts $(iii)$ and $(iv)$ may be written as a linear combination of these 6 functions, and we compute the following:

**Corollary 3.1.** *We have*

$$\overline{N}_4(n) = (-8n-1)\overline{N}_2(n) + \left(\frac{-216+24n}{77}\right)\overline{M}_2(n) + \frac{192}{77}\overline{M}_4(n) + \left(\frac{260+184n}{77}\right)\overline{M2}_2(n) - \frac{40}{11}\overline{M2}_4(n) \tag{3.1}$$

*and*

$$\overline{N2}_4(n) = (-2n-1)\overline{N2}_2(n) + \left(\frac{-27+3n}{77}\right)\overline{M}_2(n) + \frac{24}{77}\overline{M}_4(n) + \left(\frac{71-131n}{77}\right)\overline{M2}_2(n) - \frac{16}{11}\overline{M2}_4(n). \tag{3.2}$$

Now let $k = 3$. Again we find that the 12 functions in parts $(i)$ and $(ii)$ of Theorem 1.1 are linearly independent, and so the functions in parts $(iii)$ and $(iv)$ may be written in terms of them. Following the lead of Atkin and Garvan, we use (3.1) and (3.2) to eliminate $\overline{N}_4(n)$ and $\overline{N2}_4(n)$, thus expressing $\overline{N}_6(n)$ (resp. $\overline{N2}_6(n)$) in terms of $\overline{N}_2(n)$ (resp. $\overline{N2}_2(n)$) and the crank moments.

**Corollary 3.2.** *We have*

$$\begin{aligned}\overline{N}_6(n) &= (3+20n+48n^2)\overline{N}_2(n) + \left(\frac{2192796}{274505} + \frac{123276n}{7595} + \frac{-5185344n^2}{1921535}\right)\overline{M}_2(n) \\ &+ \left(\frac{-445728}{54901} + \frac{-5730048n}{384307}\right)\overline{M}_4(n) + \left(\frac{5376}{3565}\right)\overline{M}_6(n) \\ &+ \left(\frac{-386988}{39215} + \frac{-54556468n}{1921535} + \frac{-30679392n^2}{1921535}\right)\overline{M2}_2(n) \\ &+ \left(\frac{96204}{7843} + \frac{1412352n}{54901}\right)\overline{M2}_4(n) + \left(\frac{-9056}{3565}\right)\overline{M2}_6(n)\end{aligned} \tag{3.3}$$

*and*

$$\begin{aligned}\overline{N2}_6(n) &= (3+5n+3n^2)\overline{N2}_2(n) + \left(\frac{249003}{274505} + \frac{36273n}{83545} + \frac{-162042n^2}{1921535}\right)\overline{M}_2(n) \\ &+ \left(\frac{-46014}{54901} + \frac{-179064n}{384307}\right)\overline{M}_4(n) + \left(\frac{168}{3565}\right)\overline{M}_6(n) \\ &+ \left(\frac{-765123}{274505} + \frac{6826601n}{1921535} + \frac{4805874n^2}{1921535}\right)\overline{M2}_2(n) \\ &+ \left(\frac{39102}{7843} + \frac{44136n}{54901}\right)\overline{M2}_4(n) + \left(\frac{-3848}{3565}\right)\overline{M2}_6(n).\end{aligned} \tag{3.4}$$

Now for $k = 4$, there are 22 functions in Theorem 1.1 and the dimension of $\overline{P} \cdot \overline{\mathcal{W}}_k$ is 21. We compute the implied relation, again using results above to eliminate the 4th and 6th rank moments in favor of the 2nd.

**Corollary 3.3.**

$$\begin{aligned}\overline{N}_8(n) &= (1071+1680n+672n^2)\overline{N}_2(n) + 128\overline{N2}_8(n) + +128\overline{M2}_8(n) \\ &+ \left(\frac{23636592}{39215} + \frac{2061072n}{11935} + \frac{-10370688n^2}{274505}\right)\overline{M}_2(n) \\ &+ \left(\frac{-4313856}{7843} + \frac{-11460096n}{54901}\right)\overline{M}_4(n) + \left(\frac{75264}{3565}\right)\overline{M}_6(n) \\ &+ \left(\frac{-162548772}{39215} + \frac{-223905936n}{274505} + \frac{-61358784n^2}{274505}\right)\overline{M2}_2(n) \\ &+ \left(\frac{28298256}{7843} + \frac{2824704n}{7843}\right)\overline{M2}_4(n) + \left(\frac{-3321024}{3565}\right)\overline{M2}_6(n).\end{aligned} \tag{3.5}$$

If we would like relations where only one type of rank moment occurs then we may combine the function $F := q(q;q)_\infty^6(q^2;q^2)_\infty^9$ with the functions in $\overline{\mathcal{C}}_4$ and $\overline{\mathcal{C}2}_4$ to get a basis for $\overline{P} \cdot \overline{\mathcal{W}}_4$. (The fact that $F$ is in this space follows from the fact that $q(q;q)_\infty^8(q^2;q^2)_\infty^8$ is a cusp form of weight 8 on $\Gamma_0(2)$.) Then each of the functions in (*iii*) and (*iv*) of Theorem 1.1 may expressed in terms of this basis.

When $k \geq 5$, the number of functions in Theorem 1.1 is smaller than the dimension of $\overline{P} \cdot \overline{\mathcal{W}}_k$. Presumably this could be remedied by adding functions of the form $\overline{P}f$, where $f$ is a cusp form, along with their $\delta_q$- derivatives. We shall not pursue this here.

## 4. Proof of Theorems 1.2 and 1.3

We begin this section by expressing our combinatorial functions in terms of the second moments $\overline{N}_2(n)$, $\overline{N2}_2(n)$, $\overline{M}_2(n)$, and $\overline{M2}_2(n)$.

**Proposition 4.1.** *We have* $nov(n) = \frac{1}{2}\overline{M}_2(n)$ *and* $ov(n) = \frac{1}{2}\overline{M}_2(n) - \overline{M2}_2(n)$.

*Proof.* Dyson [14] has shown that $M_2(n) = 2np(n)$, where $p(n)$ is the number of partitions of $n$. Since

$$\sum_{n\geq 0} M_2(n)q^n = \delta_z^2 C(z,q)|_{z=1},$$

we have that

$$\begin{aligned}\sum_{n\geq 0} \overline{M}_2(n)q^n &= (-q;q)_\infty \delta_z^2 C(z,q)|_{z=1} \\ &= (-q;q)_\infty \sum_{n\geq 0} 2np(n)q^n \\ &= 2\sum_{n\geq 0} nov(n)q^n.\end{aligned}$$

Similarly, we find that $\overline{M2}_2(n)$ may be interpreted as $enov(n)$, where $enov(n)$ denotes the sum, over all overpartitions of $n$, of the even non-overlined parts. Using Euler's identity between the number of partitions of $n$ into odd parts and the number of partitions of $n$ into distinct parts, we see that $nov(n) - enov(n) = ov(n)$. □

We now turn to the smallest parts functions.

**Proposition 4.2.** *We have* $\overline{spt}(n) = \overline{M}_2(n) - \overline{N}_2(n)$ *and* $\overline{spt2}(n) = \overline{M2}_2(n) - \overline{N2}_2(n)$.

*Proof.* By the work in [8], we find that

$$\overline{spt}(n) = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{2nq^n}{(1-q^n)} - \overline{N}_2(n), \tag{4.1}$$

and

$$\overline{spt2}(n) = \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{2nq^{2n}}{(1-q^{2n})} - \overline{N2}_2(n). \tag{4.2}$$

By applying $\delta_q$ to $P$, we see that

$$\frac{1}{(q;q)_\infty} \sum_{n\geq 0} \frac{nq^n}{(1-q^n)} = \sum_{n\geq 0} np(n)q^n.$$

Combining this with (4.1), (4.2), and Proposition 4.1 finishes the proof. □

We now prove the congruences in Theorems 1.2 and 1.3.

*Proof of Theorem 1.2.* To prove (1.8), we remove $\delta_q^2\overline{C}_2$ from the set $\overline{\mathcal{C}}_3$ and replace it with $\overline{C}_2\overline{C}_4/\overline{P}$. We compute that together with the functions in $\overline{\mathcal{C}2}_3$ we still have a basis for $\overline{P}\cdot\overline{\mathcal{W}}_3$. Computing $\delta_q^2\overline{C}_2$ in terms of this basis and reducing modulo 3 gives

$$n^2\overline{M}_2(n) \equiv \overline{M}_6(n) + 2\overline{M2}_4(n) + n^2\overline{M2}_2(n) \pmod 3.$$

Since $m^6 \equiv m^4 \equiv m^2 \pmod 3$, by definition of the rank moments we have

$$(n^2-1)\overline{M}_2(n) \equiv (n^2-1)\overline{M2}_2(n) \pmod 3. \tag{4.3}$$

This is equivalent to (1.8).

For (1.9), we simply multiply (3.3) by 5 and reduce modulo 5. The result is

$$(2n^2+n+2)\overline{M}_2(n) + (n^2+4n+3)\overline{M2}_2(n) \equiv 0 \pmod 5,$$

which implies the desired congruence.

For (1.10), we first multiply (3.1) by 7 and reduce modulo 7. The result is

$$(2+6n)\overline{M}_2(n) + 6\overline{M}_4(n) + (2+4n)\overline{M2}_2(n) \equiv 0 \pmod 7. \tag{4.4}$$

Next we take the set $\overline{\mathcal{C}}_4 \cup \overline{\mathcal{C}2}_4 \cup \{F\}$ and replace $\delta_q\overline{C}_4$ by $\overline{C}_2\overline{C}_4/\overline{P}$ and $\delta_q^2\overline{C}_4$ by $\overline{C}_2\overline{C}_6/\overline{P}$. This turns out to be a basis for $\overline{P}\cdot\overline{\mathcal{W}}_4$. Expressing the function in part *(iii)* of Theorem 1.1 in terms of this basis, multiplying by 7 and reducing modulo 7 gives

$$(4+6n+2n^2+3n^3)\overline{M}_2(n) + 6\overline{M}_4(n) + (4n+5n^2+n^3)\overline{M2}_2(n) \equiv 0 \pmod 7.$$

Using (4.4) to substitute for $\overline{M}_4(n)$ gives

$$(2n^3+3n^2+3)\overline{M2}_2(n) \equiv (n^3+3n^2+3)\overline{M}_2(n),$$

and the congruence (1.10) follows. □

We now treat Theorem 1.3.

*Proof of Theorem 1.3.* First reduce (3.2) modulo 3 to obtain

$$(2n+2)\overline{N2}_2(n) \equiv (2n+2)\overline{M2}_2(n) \pmod 3.$$

Since $\overline{spt2}(n) = \overline{M2}_2(n) - \overline{N2}_2(n)$, we have (1.11).

Reducing (3.1) modulo 3 we obtain

$$(2n+2)\overline{N}_2(n) \equiv (2n+2)\overline{M2}_2(n) \pmod 3.$$

Combined with (4.3) when $n$ is replaced by $3n$ and the fact that $\overline{spt}(n) = \overline{M}_2(n) - \overline{N}_2(n)$ gives (1.12).

Next we perform the same computation used to obtain (3.4), except that we replace $\delta_q^2\overline{C2}_2$ by $\overline{C}_2\overline{C}_4/\overline{P}$. Reducing the result modulo 5 gives

$$(1-n^2)\overline{N2}_2(n) \equiv (2n^2+3)\overline{M}_2(n) \pmod 5. \tag{4.5}$$

Combining this with (4.3) when $n$ is replaced by $5n+3$ gives (1.13).

Finally we perform the same calculation used to obtain (3.3), again replacing $\delta_q^2\overline{C2}_2$ by $\overline{C}_2\overline{C}_4/\overline{P}$. Reducing the result modulo 5 gives

$$(3+2n^2)\overline{N}_2(n) \equiv (n+4n^2)\overline{M}_2(n) + (4+4n)\overline{M2}_2(n).$$

Combining this with (4.5) and (4.3) when $n$ is replaced by $5n$, together with the fact that $\overline{spt1}(n) = \overline{M}_2(n) - \overline{N}_2(n) - \overline{M2}_2(n) + \overline{N2}_2(n)$, gives (1.14).

□

## 5. Proof of Theorem 1.4 and Corollary 1.5

*Proof of Theorem 1.4.* Let $\overline{Spt1}(q)$ denote the generating function for $\overline{spt1}(n)$. Define the integral

$$\overline{NH}(z) := \frac{1}{2\sqrt{2}\pi i}\int_{-\bar{z}}^{i\infty} \frac{\eta^2(\tau)}{\eta(2\tau)(-i(\tau+z))^{\frac{3}{2}}}\, d\tau$$

where $\eta$ is Dedekind's eta function. From the work in [8], we may conclude that

$$\overline{\mathcal{M}}_3(z) := \overline{Spt1}(q) + \frac{1}{12}\frac{\eta(2z)}{\eta^2(z)}E_2\left(z\right) - \frac{1}{3}\frac{\eta(2z)}{\eta^2(z)}E_2\left(2z\right) + \overline{NH}(z)$$

is a weight $\frac{3}{2}$ weak Maass form on $\Gamma_0(16)$.

Next let $\overline{f}(q)$ denote the generating function for $\overline{\alpha}(n)$. From [7] we have that

$$\overline{\mathcal{M}}(z) := \overline{f}(q) - 4\overline{NH}(z)$$

is a weak Maass form of weight $\frac{3}{2}$ on $\Gamma_0(16)$. Since by (1.12) we have

$$\overline{spt1}(3n) \equiv 0 \pmod 3,$$

to prove Theorem 1.4 it is enough to show that

$$G(q) := \left(\frac{\bullet}{3}\right) \otimes \left(8\overline{Spt1}(q) + \left(\frac{\bullet}{3}\right) \otimes \overline{f}(q)\right) \equiv 0 \pmod 3,$$

where for a character $\chi$ and a $q$-series $g$, $\chi \otimes g$ denotes the twist of $g$ by $\chi$, i.e., the $n$th Fourier coefficient of $g$ is multiplied by $\chi(n)$. Firstly it is not hard to check that

$$G(q) \equiv \left(\frac{\bullet}{3}\right) \otimes \left(8\left(\overline{Spt}_3(q) + \frac{1}{24}\frac{\eta(2z)}{\eta^2(z)}E_2\left(2z\right) + \frac{1}{24}\frac{\eta(2z)}{\eta^2(z)}E_2\left(z\right)\right)\right.$$
$$\left. + \frac{\eta(2z)}{30\eta^2(z)}\left(4E_4\left(2z\right) - E_4(z)\right) + \left(\frac{\bullet}{3}\right) \otimes \overline{f}(q)\right) \pmod 3.$$

We call the right hand side $H(q)$.

Next one can show that the non-holomorphic parts of $\overline{\mathcal{M}}_3(q)$ and $\overline{\mathcal{M}(q)}$ are supported on negative squares. This easily yields that $H(q)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ on $\Gamma_0(144)$. Now $\overline{f}(q)$, $E_4(2z)$ and $E_4(z)$ have no poles. Moreover $M_3(q)$ and $\frac{\eta(2z)}{\eta^2(z)}$ have poles only in $\frac{a}{c}$ with $c$ odd. Using this one can compute, using properties of twists, that $H(q)$ has poles at most in $\frac{a}{c}$ if $c$ is odd. If $9|c$, its order can bound by $-\frac{1}{16}$, if $3 \parallel c$ by $-\frac{9}{16}$ and if $3 \nmid c$ by $-\frac{1}{144}$. This now easily yields that $\frac{\eta^{18}(z)}{\eta^9(2z)}G(q)$ is the sum of two holomorphic modular forms of weight $\frac{3}{2}$ and $\frac{7}{3}$, respectively. Using that $\frac{\eta^6(z)}{\eta^2(3z)}$ is a holomorphic weight 2 modular form on $\Gamma_0(72)$ satisfying

$$\frac{\eta^6(z)}{\eta^2(3z)} \equiv 1 \pmod 3,$$

we can show that $G(q)$ is congruent modulo 3 to a holomorphic modular form of weight 8. Sturm's Theorem now gives that this form is congruent to 0 if the first

$$\left[\frac{8}{12}\left[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(144)\right]\right] + 1 = 193$$

coefficients are congruent 0 modulo 3. This can be done by MAPLE. (to be checked) □

Corollary 1.5 now follows easily from Theorem 1.4 and the following

**Proposition 5.1.** *We have for a prime $\ell \neq 2, 3$*

$$\overline{\alpha}\left(\ell^2 n\right) + \left(\frac{-n}{\ell}\right)\overline{\alpha}(n) + \ell\,\overline{\alpha}\left(\frac{n}{\ell^2}\right) = (\ell+1)\overline{\alpha}(n). \tag{5.1}$$

*Proof.* To prove (5.1), we have to show that

$$g_\ell(\tau) := \overline{f}|T_{\ell^2}(q) - (\ell+1)\overline{f}(q) = 0,$$

where $T_\ell$ is the usual half-integral weight Hecke-operator. Using that $\frac{\eta^2(z)}{\eta(2z)}$ is a Hecke eigenform with eigenvalue $1+\frac{1}{\ell}$, one obtains from [9] that $g_\ell(z)$ is a weakly holomorphic modular form of weight $\frac{3}{2}$ on $\Gamma_0(16)$. Since the coefficients of $\bar{f}$ have only polynomial growth it is a holomorphic forms. The valence formula now gives that $g_\ell = 0$ if its first 4 Fourier coefficients equal 0. Thus to finish the proof, we have to show that (5.1) is true for $0 \leq n \leq 3$. For $n = 0$ this claim is trivial. For the other cases recall from [7] it is known that

$$(-1)^n\,\overline{\alpha}(n) = \begin{cases} -4H(4n) & \text{if } n \equiv 1, 2 \pmod 4, \\ -24H(n) & \text{if } n \equiv 3 \pmod 8, \\ -16H(n) & \text{if } n \equiv 7 \pmod 8, \\ -16H(n) - \frac{1}{3}r(n/4) & \text{if } 4 \mid n, \end{cases} \tag{5.2}$$

where

$$r(n) = \begin{cases} 12H(4n) & \text{if } n \equiv 1, 2 \pmod 4, \\ 24H(n) & \text{if } n \equiv 3 \pmod 8, \\ r(n/4) & \text{if } n \equiv 0 \pmod 4, \\ 0 & \text{if } n \equiv 7 \pmod 8. \end{cases} \tag{5.3}$$

Moreover we need the fact [11] that if $-n = Df^2$ where $D$ is a fundamental discriminant, then

$$H(n) = \frac{h(D)}{w(D)} \sum_{d|f} \mu(d)\left(\frac{D}{d}\right)\sigma_1(f/d). \tag{5.4}$$

Here $h(D)$ is the class number of $\mathbb{Q}(\sqrt{D})$, $w(D)$ is half the number of units in the ring of integers of $\mathbb{Q}(\sqrt{D})$, $\sigma_1(n)$ is the sum of the divisors of $n$, and $\mu(n)$ is the Möbius function. We only show (5.1) for $n = 1$. The other cases follow similarly. In this case we have to show that

$$\overline{\alpha}\left(\ell^2\right) = 2\left(\ell + 1 - \left(\frac{-1}{\ell}\right)\right).$$

Firstly we have from (5.2) that

$$\overline{\alpha}\left(\ell^2\right) = 4H\left(-4\ell^2\right).$$

Since $H(-4) = 1$ and $\omega(-4) = 2$, (5.4) yields

$$\overline{\alpha}\left(\ell^2\right) = 2\left(\sigma_1(\ell) - \left(\frac{-1}{\ell}\right)\right) = 2\left(\ell + 1 - \left(\frac{-1}{\ell}\right)\right),$$

as claimed.

□

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