

Classifying Dihedral $O(2)$ -Equivariant Spectra

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Abstract

The category of rational $O(2)$ -equivariant spectra splits as a product of cyclic and dihedral parts. Using the classification of rational G -equivariant spectra for finite groups G , we classify the dihedral part of rational $O(2)$ -equivariant spectra in terms of an algebraic model.

1 Main Results

The category of rational $O(2)$ -spectra, $O(2)\mathcal{M}_{\mathbb{Q}}$, is the category of $O(2)$ -equivariant EKMM S -modules ([MM02]) with weak equivalences the rational π_* -isomorphisms. There is a strong symmetric monoidal Quillen equivalence

$$\Delta : O(2)\mathcal{M}_{\mathbb{Q}} \xrightarrow{\simeq} L_{EW_+}O(2)\mathcal{M}_{\mathbb{Q}} \times L_{S^{\infty\delta}}O(2)\mathcal{M}_{\mathbb{Q}} : \prod .$$

Where δ is the determinant representation of $O(2)$ on \mathbb{R} and W is the group of order two. We call $L_{EW_+}O(2)\mathcal{M}_{\mathbb{Q}}$ the category of cyclic $O(2)$ -spectra and $\mathcal{DM}_{\mathbb{Q}} := L_{S^{\infty\delta}}O(2)\mathcal{M}_{\mathbb{Q}}$ the category of dihedral spectra. This splitting result was proven at the homotopy level in [Gre98] and the model category statement above is [Bar08, Theorem 6.1.3].

An object V of the category $\mathcal{A}(\mathcal{D})$ consists of a differential graded rational vector space V_{∞} and $dgQW$ -modules V_k for each $k \geq 1$, with a map of $dgQW$ -modules $\sigma_V : V_{\infty} \rightarrow \text{colim}_n \prod_{k \geq n} V_k$. A map $f : V \rightarrow V'$ in this category consists of a map $f_{\infty} : V_{\infty} \rightarrow V'_{\infty}$ in dgQ -mod and $dgQW$ -module maps $f_k : V_k \rightarrow V'_k$ such that $\sigma_{V'} \circ f_{\infty} = \text{colim}_n \prod_{k \geq n} f_k \circ \sigma_V$.

To define $\mathcal{A}(\mathcal{D})$, we have simply taken the algebraic model for the homotopy category of dihedral spectra in [Gre98] (a graded category) and added the requirement that each piece have a differential. A map f in $\mathcal{A}(\mathcal{D})$ is called a weak equivalence or fibration if f_{∞} and each f_k is (in dgQ -mod and $dgQW$ -mod respectively). This defines a monoidal model structure on the category $\mathcal{A}(\mathcal{D})$.

Theorem 1.1 *There is a zig-zag of monoidal Quillen equivalences between the model category of dihedral spectra $\mathcal{DM}_{\mathbb{Q}}$ and the algebraic model for dihedral spectra $\mathcal{A}(\mathcal{D})$.*

Proof We first use Lemma 3.1 to replace $\mathcal{DM}_{\mathbb{Q}}$ by $S_{\mathcal{D}}\text{-mod}$, a category with every object fibrant. Theorem 3.4 moves us to a category of right modules over a ringoid \mathcal{E}_{top} (enriched over Sp_+^{Σ}). We then alter this ringoid to obtain \mathcal{E}_t , a ringoid over $dg\mathbb{Q}\text{-mod}$, see Theorem 3.5. On the algebraic side we have $\mathcal{A}(\mathcal{D})$ which is equivalent to $\text{mod-}\mathcal{E}_a$ by Theorem 4.12. Theorem 5.5 proves that the $dg\mathbb{Q}$ -ringoids \mathcal{E}_t and \mathcal{E}_a are quasi-isomorphic, completing the argument. \blacksquare

This paper exists to take the homotopy level classification of dihedral spectra in [Gre98] and lift it to the level of model categories. This strengthens the result and allows detailed consideration of monoidal structures. The proof of the theorem above is an adaptation of the results of [Bar08, Chapter 4], which itself follows the basic plan of [GS]. The new points are the comparisons of the ringoids and the construction of a model structure on $\mathcal{A}(\mathcal{D})$. This classification is possible since the homotopy of the endomorphism ringoid \mathcal{E}_{top} is concentrated in degree zero. Consequently, one could have followed the method of [SS03, Example 5.1.2] and replaced \mathcal{E}_{top} by an Eilenberg-MacLane ringoid $\underline{H}\mathcal{E}_a$ and then the results of that paper would prove that $\text{mod-}\mathcal{E}_{top}$, $\text{mod-}\underline{H}\mathcal{E}_a$ and $\text{mod-}\mathcal{E}_a$ are Quillen equivalent. This alternative method would not be compatible with the monoidal structures.

Corollary 1.2 *The above zig-zag induces a zig-zag of Quillen equivalences between the category of algebras in $S_{\mathcal{D}}\text{-mod}$ and the category of algebras in $\mathcal{A}(\mathcal{D})$.*

Let $i: C_0\mathcal{E}_t \rightarrow \mathcal{E}_t$ and $p: C_0\mathcal{E}_t \rightarrow H_*\mathcal{E}_t$ be the maps constructed in Theorem 5.5. Recall from [GS, Theorem 4.1] and [Shi07, Corollary 2.16] the composites $\text{Hom}(\mathcal{G}_t, -) \circ D \circ \phi^*N \circ \tilde{\mathbb{Q}}$ and $U \circ L'\hat{c} \circ R \circ (\hat{c}-) \otimes_{\mathcal{E}_t} \mathcal{G}_t$. These are the derived composite functors of Theorem 3.5. Cofibrant replacements (\hat{c}) are not needed in the first of these composites as we are working rationally. The functor L' is the alteration of L from [Shi07] to modules over a ringoid ([SS03, Theorem 6.5]). We will then L' so that it acts on categories of algebras, we call this L'' .

Corollary 1.3 *Let Θ be the derived functor*

$$(-) \otimes_{\mathcal{E}_a} \mathcal{G}_a \circ (\hat{c}-) \square_{C_0\mathcal{E}_t} \mathcal{E}_a \circ i^* \circ \text{Hom}(\mathcal{G}_t, -) \circ D \circ \phi^*N \circ \tilde{\mathbb{Q}} \circ \text{Hom}(\mathcal{G}_{top}, -)$$

from $\mathcal{DM}_{\mathbb{Q}}$ to $\mathcal{A}(\mathcal{D})$. Then for each $S_{\mathcal{D}}$ -algebra A there is a zig-zag of Quillen equivalences between $A\text{-mod}$ and $\Theta A\text{-mod}$. Let \mathbb{H} be the derived functor

$$(\hat{c}-) \wedge_{\mathcal{E}_{top}} \mathcal{G}_{top} \circ U \circ L''\hat{c} \circ R \circ (-) \otimes_{\mathcal{E}_t} \mathcal{G}_t \circ (\hat{c}-) \square_{C_0\mathcal{E}_t} \mathcal{E}_t \circ p^* \circ \text{Hom}(\mathcal{G}_a, -)$$

from $\mathcal{A}(\mathcal{D})$ to $\mathcal{DM}_{\mathbb{Q}}$. Then for each algebra object B in $\mathcal{A}(\mathcal{D})$ there is a zig-zag of Quillen equivalences between $B\text{-mod}$ and $\mathbb{H}B\text{-mod}$.

2 The group $O(2)$

We will use the notation D_{2n}^h to represent the dihedral subgroup of order $2n$ containing h , an element of $O(2) \setminus SO(2)$. The closed subgroups of $O(2)$ are $O(2)$, $SO(2)$, the finite dihedral groups D_{2n}^h for each h and the cyclic groups C_n ($n \geq 1$). We will

always have $W = O(2)/SO(2)$. Let $H \leq O(2)$, $N_{O(2)}(H)$ is the normaliser in $O(2)$ of H , it is the largest subgroup of $O(2)$ in which H is normal. The Weyl-group of H in $O(2)$ is $W_{O(2)}(H) := N_{O(2)}(H)/H$. The normaliser of D_{2n}^h in $O(2)$ is D_{4n}^h , thus the Weyl group of D_{2n}^h is W . The cyclic groups are normal, hence the Weyl group of C_n is $O(2)/C_n \cong O(2)$. The Weyl group of $SO(2)$ is again W .

Recall the following material from [LMSM86, Chapter V, Section 2]. Define $\mathcal{FO}(2)$ to be the set of those subgroups of $O(2)$ with finite index in their normaliser (or equally, with finite Weyl-group), equipped with the Hausdorff topology. This is an $O(2)$ -space via the conjugation action of $O(2)$ on its subgroups. This space is of interest due to tom Dieck's ring isomorphism:

$$A(O(2)) \otimes \mathbb{Q} := [S^0, S^0]^{O(2)} \otimes \mathbb{Q} \xrightarrow{\cong} C(\mathcal{FO}(2)/O(2), \mathbb{Q})$$

where $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ is the ring of continuous maps from $\mathcal{FO}(2)$ to \mathbb{Q} , considered as a discrete space. We draw $\mathcal{FO}(2)/O(2)$ below as Figure 1. We will sometimes write D_{2n} for (D_{2n}^h) , the conjugacy class of D_{2n}^h . The point $O(2)$ is a limit point of this space.

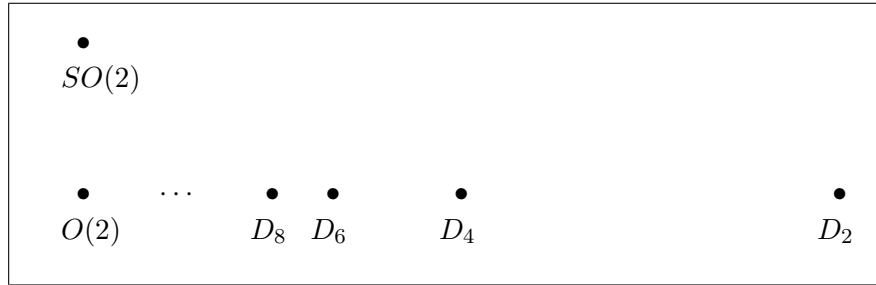


Figure 1: $\mathcal{FO}(2)/O(2)$.

Definition 2.1 Define \mathcal{C} to be the set consisting of the cyclic groups and $SO(2)$, this is a family of subgroups (that is, \mathcal{C} is closed under conjugation and subgroups). Let \mathcal{D} be the complement of \mathcal{C} in the set of all (closed) subgroups of $O(2)$.

We define idempotents of $C(\mathcal{FO}(2)/O(2), \mathbb{Q})$ as follows: $e_{\mathcal{C}}$ is the characteristic function of $SO(2)$, $e_{\mathcal{D}} = e_{\mathcal{C}} - 1$ and e_n is the characteristic function of D_n for each $n \geq 1$, note that $e_{\mathcal{D}} * e_n = e_n$.

Lemma 2.2 The rational Burnside ring of $O(2)$ is $\mathbb{Q}e_{\mathcal{C}} \oplus \mathbb{Q}e_{\mathcal{D}} \oplus \bigoplus_{n \geq 1} \mathbb{Q}e_n$, with multiplication defined by the multiplication of the idempotents.

3 Dihedral Spectra

Following the work of [Bar08, Chapter 3] we have the following constructions. The category $O(2)\mathcal{M}_{\mathbb{Q}}$ is the category of $O(2)$ -equivariant S -modules ([MM02]) localised

at the spectrum $S_{\mathcal{M}}^0\mathbb{Q}$. This spectrum is the cofibre of a map $g: \bigvee_R \widehat{cS} \rightarrow \bigvee_F \widehat{cS}$ where $0 \rightarrow R \xrightarrow{f} F \rightarrow \mathbb{Q}$ is a free resolution of \mathbb{Q} over \mathbb{Z} and $F = \bigoplus_{q \in \mathbb{Q}} \mathbb{Z}$. The map g is a representative for the homotopy class $f \otimes \text{Id}: R \otimes \pi_*^{O(2)}(S) \rightarrow F \otimes \pi_*^{O(2)}(S)$. We have a map $\widehat{cS} \rightarrow S_{\mathcal{M}}^0\mathbb{Q}$ corresponding to the inclusion of \mathbb{Z} into the 1-factor of F . This map induces an isomorphism $\pi_*^H(\widehat{cS} \wedge X) \otimes \mathbb{Q} \rightarrow \pi_*^H(S_{\mathcal{M}}^0\mathbb{Q} \wedge X)$ for any spectrum X . The cofibrations of $O(2)\mathcal{M}_{\mathbb{Q}}$ are the same as for $O(2)\mathcal{M}$ and the weak equivalences $O(2)\mathcal{M}_{\mathbb{Q}}$ are those maps f such that $\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism of graded groups for all H . There is a strong symmetric monoidal Quillen equivalence

$$\Delta: O(2)\mathcal{M}_{\mathbb{Q}} \xrightarrow{\simeq} L_{EW_+} O(2)\mathcal{M}_{\mathbb{Q}} \times L_{S^{\infty\delta}} O(2)\mathcal{M}_{\mathbb{Q}}: \prod.$$

This result was proven at the homotopy level in [Gre98]. Note that EW_+ is rationally equivalent to $e_{\mathcal{E}}S$ and $S^{\infty\delta}$ is rationally equivalent to $e_{\mathcal{D}}S$. We write $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$ for $L_{S^{\infty\delta}} O(2)\mathcal{M}_{\mathbb{Q}}$.

We now examine the category $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$ in more detail. The weak equivalences are those maps f such that $f \wedge \text{Id}_{S^{\infty\delta}}$ induces an isomorphism of rational homotopy groups, or equally, such that $e_{\mathcal{D}}\pi_*^H(f) \otimes \mathbb{Q}$ is an isomorphism for all H . The cofibrations of this model category are the same as for $O(2)\mathcal{M}$. If X is fibrant in this model structure, then X is $S^{\infty\delta}$ -local and hence $\pi_*^H(X) = 0$ for all $H \in \mathcal{C}$. Furthermore all the homotopy groups of such an X are rational vector spaces.

Lemma 3.1 *There is a commutative S -algebra, $S_{\mathcal{D}}$, such that $S \rightarrow S_{\mathcal{D}}$ is a weak equivalence in $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$ and the adjunction below is a weak equivalence.*

$$- \wedge S_{\mathcal{D}}: \mathcal{D}\mathcal{M}_{\mathbb{Q}} \xrightarrow{\simeq} S_{\mathcal{D}}\text{-mod}: U$$

Proof The existence of $S_{\mathcal{D}}$ comes from the fact that one can localise S at a cell- S -module (in this case $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$) and obtain a commutative S -algebra, $S_{\mathcal{D}}$, such that the unit map is a $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ -equivalence (see [EKMM97, Chapter VIII, Theorem 2.2]).

We claim that $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ and $S_{\mathcal{D}}$ are π_* -isomorphic. The construction of $S_{\mathcal{D}}$ comes with a π_* -isomorphism $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta} \rightarrow S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta} \wedge S_{\mathcal{D}}$. This last term is π_* -isomorphic to $S_{\mathcal{M}}^0\mathbb{Q} \wedge e_{\mathcal{D}}S_{\mathcal{D}}$. The spectra $S_{\mathcal{M}}^0\mathbb{Q} \wedge S_{\mathcal{D}}$ and $(S_{\mathcal{M}}^0\mathbb{Q} \wedge e_{\mathcal{E}}S_{\mathcal{D}}) \vee (S_{\mathcal{M}}^0\mathbb{Q} \wedge e_{\mathcal{D}}S_{\mathcal{D}})$ are π_* -isomorphic. Since $S_{\mathcal{D}}$ is $e_{\mathcal{D}}S$ -local, it follows that $S_{\mathcal{M}}^0\mathbb{Q} \wedge e_{\mathcal{E}}S_{\mathcal{D}}$ is π_* -isomorphic to a point. Since $S_{\mathcal{D}}$ is $S_{\mathcal{M}}^0\mathbb{Q}$ -local, $S_{\mathcal{D}}$ and $S_{\mathcal{M}}^0\mathbb{Q} \wedge S_{\mathcal{D}}$ are π_* -isomorphic and we have proven our claim. It then follows by a standard argument that all $S_{\mathcal{D}}$ -modules are $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ -local.

Now we can prove that the above adjunction is a strong symmetric monoidal Quillen equivalence. Note that the weak equivalences and fibrations of $S_{\mathcal{D}}\text{-mod}$ are defined in terms of their underlying maps in $O(2)\mathcal{M}$. The left adjoint preserves cofibrations and it takes acyclic cofibrations to $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ -equivalences between $S_{\mathcal{D}}$ -modules. Since such a module is $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ -local, it follows that the left adjoint takes acyclic cofibrations to π_* -isomorphisms.

The right adjoint preserves and detects all weak-equivalences, so we must prove that for X , cofibrant in $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$, the map $S \rightarrow X \wedge S_{\mathcal{D}}$ is a weak-equivalence in $\mathcal{D}\mathcal{M}_{\mathbb{Q}}$. This

is immediate, since smashing with a cofibrant object will preserve the $S_{\mathcal{M}}^0\mathbb{Q} \wedge S^{\infty\delta}$ -equivalence $S \rightarrow S_{\mathcal{D}}$. \blacksquare

Note that an $S_{\mathcal{D}}$ -module X has rational homotopy groups and $\pi_*^H(X) = 0$ for any $H \in \mathcal{C}$.

Lemma 3.2 *The model category $S_{\mathcal{D}}\text{-mod}$ is generated by $S_{\mathcal{D}}$ and the countably infinite collection $\{S_{\mathcal{D}} \wedge \widehat{ce}_H O(2)/H_+ \mid H \in \mathcal{D} \setminus \{O(2)\}\}$. Furthermore, these objects are compact.*

Proof We must prove that if X is an object of $S_{\mathcal{D}}\text{-mod}$ such that $[\sigma, X]_*^{S_{\mathcal{D}}} = 0$ (maps in the homotopy category of $S_{\mathcal{D}}\text{-mod}$) for all σ as above, then $X \rightarrow *$ is a π_* -isomorphism. As mentioned above $\pi_*^H(X) = 0$ for any $H \in \mathcal{C}$ and since $[S_{\mathcal{D}}, X]_*^{S_{\mathcal{D}}} = 0$ we see that $\pi_*^{O(2)}(X) = 0$. So now we must consider a finite dihedral group H : by [Gre98, Example C(i)], $\pi_*^H(X)$ is given by $\bigoplus_{(K) \leq H} (e_K \pi_*^K(X))^{W_H K}$. We have assumed that $[S_{\mathcal{D}} \wedge \widehat{ce}_K O(2)/K_+, X]_*^{S_{\mathcal{D}}} = 0$, for each finite dihedral K , but this is precisely the condition that $e_K \pi_*^K(X) = 0$ for each K . Thus $\pi_*^H(X) = 0$ and our set generates the homotopy category. Compactness follows from the isomorphisms $[\sigma_H, X]_*^{S_{\mathcal{D}}} \cong e_H \pi_*^H(X)$. \blacksquare

Proposition 3.3 *The category of dihedral spectra is a spectral model category.*

Proof This takes a little work and we must use the category of $O(2)$ -equivariant orthogonal spectra. We begin with the composite functor $i_* \varepsilon_{O(2)}^* : \mathcal{I}\mathcal{S}_+ \rightarrow O(2)\mathcal{I}\mathcal{S}_+$ as defined in [MM02, Chapter V, Proposition 3.4], which states that this functor is part of a Quillen pair $(i_* \varepsilon_{O(2)}^*, (i^*(-))^{O(2)})$. This is a strong symmetric monoidal adjunction, as noted on [MM02, Chapter V, Page 80]. Now we use the following diagram:

$$Sp_+^{\Sigma} \xleftarrow[\text{Sing} \circ \mathbb{U}]{\mathbb{P} \circ |-|} \mathcal{I}\mathcal{S}_+ \xleftarrow[(i^*(-))^{O(2)}]{i_* \varepsilon_{O(2)}^*} O(2)\mathcal{I}\mathcal{S}_+ \xleftarrow[\mathbb{N}\#]{\mathbb{N}} O(2)\mathcal{M} \xleftarrow[U]{S_{\mathcal{D}} \wedge (-)} S_{\mathcal{D}}\text{-mod}$$

For a pair of $S_{\mathcal{D}}$ -modules X and Y , the Sp_+^{Σ} -function object $\underline{\text{Hom}}(X, Y) \in Sp_+^{\Sigma}$ is given by $\text{Sing} \mathbb{U}(i^* \mathbb{N}\# \mathbb{U} F_{S_{\mathcal{D}}}(X, Y))^{O(2)}$. \blacksquare

So now we have a spectral model category and a set of compact, fibrant generators such that every non-unit object is fibrant. Take the closure of this set under the smash product, call it \mathcal{G}_{top} . Take the subcategory of $S_{\mathcal{D}}\text{-mod}$ on object set \mathcal{G}_{top} considered as a category enriched over symmetric spectra, we denote this spectral category by \mathcal{E}_{top} . We will sometimes call an enriched category a ringoid. A right module over \mathcal{E}_{top} is a contravariant enriched functor $M : \mathcal{E}_{top} \rightarrow Sp_+^{\Sigma}$, the category of such functors is denoted $\text{mod-}\mathcal{E}_{top}$. Later we will use other categories in place of Sp_+^{Σ} , such as $dg\mathbb{Q}$ -modules. The category of right modules over \mathcal{E}_{top} has a model structure with weak equivalences and fibrations defined objectwise in Sp_+^{Σ} , see [SS03, Subsection 3.3] for more details.

Theorem 3.4 *The adjunction below is a Quillen equivalence with strong symmetric monoidal left adjoint and lax symmetric monoidal right adjoint.*

$$(-) \otimes_{\mathcal{E}_{top}} \mathcal{G}_{top} : \text{mod-}\mathcal{E}_{top} \xleftrightarrow{\quad} S_{\mathcal{D}}\text{-mod} : \underline{\text{Hom}}(\mathcal{G}_{top}, -)$$

Proof This follows from [SS03, Theorem 3.9.3] and [GS, Theorem 4.1], see also [Bar08, Theorem 9.1.2]. \blacksquare

Theorem 3.5 *There is a zig-zag of monoidal Quillen equivalences between $\text{mod-}\mathcal{E}_{top}$ (enriched over Sp_+^Σ) and a category $\text{mod-}\mathcal{E}_t$ (enriched over $dg\mathbb{Q}\text{-mod}$). This zig-zag induces an isomorphism of monoidal graded \mathbb{Q} -categories: $\pi_*(\mathcal{E}_{top}) \cong H_*\mathcal{E}_t$.*

Proof This is contained in the proof of [GS, Theorem 4.1] which is based on [Shi07, Corollary 2.16]. Further details can be found in [Bar08, Section 9.3], noting that \mathcal{E}_t is, in fact, a monoidal $dg\mathbb{Q}$ -category. We describe the adjoint pairs below. \blacksquare

The zig-zag consists of the following functors applied to modules over ringoids. The free simplicial \mathbb{Q} -module functor induces an adjunction: $\tilde{\mathbb{Q}} : Sp^\Sigma \rightleftarrows Sp^\Sigma(\mathbb{s}\mathbb{Q}\text{-mod}) : U$. Normalisation of simplicial \mathbb{Q} -modules gives an adjunction:

$$\phi^*N : Sp^\Sigma(\mathbb{s}\mathbb{Q}\text{-mod}) \rightleftarrows Sp^\Sigma(dg\mathbb{Q}\text{-mod}_+) : L.$$

Finally there is the adjoint pair $D : Sp^\Sigma(dg\mathbb{Q}\text{-mod}_+) \rightleftarrows dg\mathbb{Q}\text{-mod} : R$, where the functor R takes a chain complex Y to the symmetric spectrum with $RY_n = C_0(Y \otimes \mathbb{Q}[m])$.

These adjoint pairs induce a zig-zag of Quillen equivalences between $\text{mod-}\mathcal{E}_{top}$ and $\text{mod-}D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}$. One also needs to apply an adjoint pair similar to that of Theorem 3.4. Consider the set $\{S_{\mathcal{D}} \wedge \widehat{e}_H O(2)/H_+ \mid H \in \mathcal{D} \setminus \{O(2)\}\}$. Apply $\text{Hom}(\mathcal{G}_{top}, -)$ and the above functors to this set to obtain a collection of objects in $\text{mod-}D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}^H$. Take cofibrant replacements and consider all such products of these objects. This set is \mathcal{G}_t and gives the $dg\mathbb{Q}$ -ringoid \mathcal{E}_t . We then have a Quillen equivalence between $\text{mod-}D\phi^*N\tilde{\mathbb{Q}}\mathcal{E}_{top}^H$ and $\text{mod-}\mathcal{E}_t$. Note that the functor D , from [Shi07], is symmetric monoidal. There has been some confusion over this issue, which has now been resolved by a detailed note by Neil Strickland: [Str08]. Thus we have an algebraic model for dihedral spectra, albeit one which is not particularly explicit. We spend the rest of this paper constructing a better algebraic model and proving that it is monoidally Quillen equivalent to $\text{mod-}\mathcal{E}_t$.

4 The Algebraic Model

We take the work of [Gre98] and consider the algebraic model for the homotopy category of dihedral spectra. We give this category a monoidal model structure and then replace it by modules over a $dg\mathbb{Q}$ -ringoid.

Definition 4.1 *An object V of the algebraic model $\mathcal{A}(\mathcal{D})$ consists of a differential graded rational vector space V_∞ and $dg\mathbb{Q}W$ -modules V_k for each $k \geq 1$, with a map of $dg\mathbb{Q}W$ -modules $\sigma_V : V_\infty \rightarrow \text{colim}_n \prod_{k \geq n} V_k$. We will always consider a $dg\mathbb{Q}$ -module as a $dg\mathbb{Q}W$ -module with trivial W -action.*

A map $f : V \rightarrow V'$ in this category consists of a map $f_\infty : V_\infty \rightarrow V'_\infty$ in $dg\mathbb{Q}\text{-mod}$ and $dg\mathbb{Q}W$ -module maps $f_k : V_k \rightarrow V'_k$ making the square below commute. We often write

$V(\text{end})$ for $\text{colim}_n \prod_{k \geq n} V_k$.

$$\begin{array}{ccc} V_\infty & \xrightarrow{\sigma_V} & \text{colim}_n \prod_{k \geq n} V_k \\ f_\infty \downarrow & & \downarrow \text{colim}_n \prod_{k \geq n} f_k \\ V'_\infty & \xrightarrow{\sigma_{V'}} & \text{colim}_n \prod_{k \geq n} V'_k \end{array}$$

The following definition and theorem are taken from [Gre98]. Note that for any compact Lie group G and closed subgroup H , the action of $N_G H/H$ on G/H induces an action of $N_G H/H$ on $[G/H_+, X]_*^G \cong \pi_*^H(X)$.

Definition 4.2 For an $O(2)$ -spectrum X with rational homotopy groups, let $\pi_*(X)$ denote the following object of $\mathcal{A}(\mathcal{D})$ with trivial differential. Let $k \geq 1$ and take H a dihedral group with $|H| = 2k$, then $\pi_*(X)_k = e_H \pi_*^H(X) \otimes \mathbb{Q}$. Let $f_n = e_{\mathcal{D}} - \sum_{i=1}^n e_i$ and set $\pi_*(X)_\infty = \text{colim}_n (f_n \pi_*^{O(2)}(X) \otimes \mathbb{Q})$. The structure map is induced by the collection of maps

$$f_n \pi_*^{O(2)}(X) \otimes \mathbb{Q} \longrightarrow (\iota_H)_*(f_n) \pi_*^H(X) \otimes \mathbb{Q} \rightarrow e_H (\iota_H)_*(f_n) \pi_*^H(X) \otimes \mathbb{Q}$$

where the first arrow arises from the forgetful functor ι_H^* (ι_H is the inclusion $H \rightarrow O(2)$) and the second is applying e_H .

This construction defines a functor $\pi_*: \text{Ho } S_{\mathcal{D}}\text{-mod} \rightarrow \mathcal{A}(\mathcal{D})$, using the forgetful functor $\text{Ho } S_{\mathcal{D}}\text{-mod} \rightarrow \text{Ho } O(2)\mathcal{M}$. Thus one has a map of \mathbb{Q} -modules $e_{\mathcal{D}}[X, Y]^{O(2)} \rightarrow \text{Hom}_{\mathcal{A}(\mathcal{D})}(\pi_*(X), \pi_*(Y))$.

Theorem 4.3 For X and Y , $O(2)$ spectra with rational homotopy groups, there is a short exact sequence as below.

$$0 \rightarrow \text{Ext}_{\mathcal{A}(\mathcal{D})}(\pi_*(\Sigma X), \pi_*(Y)) \rightarrow e_{\mathcal{D}}[X, Y]^{O(2)} \rightarrow \text{Hom}_{\mathcal{A}(\mathcal{D})}(\pi_*(X), \pi_*(Y)) \rightarrow 0$$

The homotopy category of $\mathcal{A}(\mathcal{D})$ agrees with the algebraic model for dihedral spectra in [Gre98]. We now introduce a construction that we will make much use of, we will soon see that this is an explicit description of the ‘global sections’ of an object of $\mathcal{A}(\mathcal{D})$.

Definition 4.4 Let $n \geq 1$ and take $V \in \mathcal{A}(\mathcal{D})$. Then $\hat{\bigoplus}_N V$ is defined as the following pullback in the category of $dg\mathbb{Q}W$ -modules.

$$\begin{array}{ccc} \hat{\bigoplus}_N V & \longrightarrow & \prod_{k \geq N} V_k \\ \downarrow & & \downarrow \\ V_\infty & \longrightarrow & \text{colim}_n \prod_{k \geq n} V_k \end{array}$$

We also have $\hat{\bigoplus}_N^W V := (\hat{\bigoplus}_N V)^W$ the W -fixed points of $\hat{\bigoplus}_N V$.

Note that $V(\text{end})^W = \text{colim}_n \prod_{k \geq n} V_k^W$ and that the structure map of V induces a map $V_\infty \rightarrow V(\text{end})^W$. So we can construct $\widehat{\text{fr}}_N^W V$ in terms of a pullback of $dg\mathbb{Q}$ -modules. The notation $\widehat{\text{fr}}_N$ is to make the reader think of some combination of a direct product and a direct sum. Indeed if $V_\infty = 0$, then $\widehat{\text{fr}}_N V = \bigoplus_{k \geq n} V_k$.

Let \mathcal{P} be the space $\mathcal{F}O(2)/O(2) \setminus \{SO(2)\}$ (\mathcal{P} for points) and let \mathcal{O} be the constant sheaf of \mathbb{Q} , this is a sheaf of rings. To specify an \mathcal{O} -module M one only needs to give the stalks at the points k and ∞ and a \mathbb{Q} map $M_\infty \rightarrow M(\text{end})$. The global sections of M are then given by $\widehat{\text{fr}}_1 M$. One can then consider W -equivariant objects in $\mathcal{O}\text{-mod}$, we denote this category by $W\mathcal{O}\text{-mod}$. The category $\mathcal{A}(\mathcal{P})$ is a full subcategory of $W\mathcal{O}\text{-mod}$; we have an inclusion functor $\text{inc}: \mathcal{A}(\mathcal{P}) \rightarrow W\mathcal{O}\text{-mod}$ and this has a right adjoint: fix . On an W -equivariant \mathcal{O} -module V , $\text{fix}(V)_k = V_k$, $\text{fix}(V)_\infty = V_\infty^W$ and the structure map is $V_\infty^W \rightarrow V_\infty \rightarrow V(\text{end})$. Our definitions and constructions are, therefore, slight adjustments to the usual definitions of modules over a sheaf of rings.

Lemma 4.5 *The category $\mathcal{A}(\mathcal{P})$ contains all small limits and colimits.*

Proof Take some small diagram V^i of objects of $\mathcal{A}(\mathcal{P})$. Define $(\text{colim}_i V^i)_\infty = \text{colim}_i (V_\infty^i)$ and $(\text{colim}_i V^i)_k = \text{colim}_i (V_k^i)$. The map below induces (via the universal properties of colimits) a structure map for $\text{colim}_i V^i$.

$$V_\infty^i \longrightarrow \text{colim}_n \prod_{k \geq n} V_k^i \longrightarrow \text{colim}_n \prod_{k \geq n} \text{colim}_i V_k^i$$

Limits are harder to define because we are working with a stalk-based description of a ‘sheaf’. Let $(\lim_i V^i)_k = \lim_i (V_k^i)$ and $(\lim_i V^i)_\infty = \text{colim}_N \lim_i \widehat{\text{fr}}_N^W V^i$. The structure map is then as below.

$$(\lim_i V^i)_\infty = \text{colim}_N \lim_i \widehat{\text{fr}}_N^W V^i \longrightarrow \text{colim}_N \lim_i \prod_{k \geq N} V_k^i = (\lim_i V^i)(\text{end})$$

■

One could also describe the limit of some diagram V^i in $\mathcal{A}(\mathcal{P})$ as $\text{fix} \lim_i \text{inc} V^i$, where the limit on the right is taken in the category of $W\mathcal{O}$ -modules. Let M be a $dg\mathbb{Q}$ module, R a $dg\mathbb{Q}W$ -module and $V \in \mathcal{A}(\mathcal{P})$. Then we define $i_k R$ to be that object of $\mathcal{A}(\mathcal{P})$ with $(i_k R)_\infty = 0$, $(i_k R)_n = 0$ for $n \neq k$ and $(i_k R)_k = R$. Let $p_k V = V_k$, an object of $dg\mathbb{Q}W\text{-mod}$ and $p_\infty V = V_\infty$, a $dg\mathbb{Q}$ -module. The functor i_∞ takes M to that object of $\mathcal{A}(\mathcal{P})$ with $(i_\infty M)_\infty = M$ and $(i_\infty M)_k = 0$. We set cM to be the object of $\mathcal{A}(\mathcal{P})$ with $cM_k = M = cM_\infty$ and structure map induced by the diagonal map $M \rightarrow \prod_{k \geq 1} M$.

Lemma 4.6 *We have the following series of adjoint pairs.*

$$\begin{aligned} i_k : dg\mathbb{Q}W\text{-mod} &\rightleftarrows \mathcal{A}(\mathcal{P}) : p_k \\ p_k : \mathcal{A}(\mathcal{P}) &\rightleftarrows dg\mathbb{Q}W\text{-mod} : i_k \\ p_\infty : \mathcal{A}(\mathcal{P}) &\rightleftarrows dg\mathbb{Q}\text{-mod} : i_\infty \\ c : dg\mathbb{Q}\text{-mod} &\rightleftarrows \mathcal{A}(\mathcal{P}) : \widehat{\text{fr}}_1^W \end{aligned}$$

Proof This is a routine check, we just wish to comment that $(c, \widehat{\text{fr}}_1^W)$ is the constant sheaf and global sections adjunction. \blacksquare

Lemma 4.7 *The category $\mathcal{A}(\mathcal{D})$ is a closed symmetric monoidal category. Furthermore there is a symmetric monoidal adjunction as below.*

$$c : dg\mathbb{Q}\text{-mod} \rightleftarrows \mathcal{A}(\mathcal{D}) : \widehat{\text{fr}}_1^W$$

Proof The category of \mathcal{O} -modules is closed symmetric monoidal, hence so is the category $W\mathcal{O}\text{-mod}$. That is, for M and N in $W\mathcal{O}\text{-mod}$, W acts diagonally on $(M \otimes_{\mathcal{O}} N)(U)$ and by conjugation on $\text{Hom}_{\mathcal{O}}(M, N)(U)$ (for U an open subset of \mathcal{D}). We ‘restrict’ this structure to $\mathcal{A}(\mathcal{D})$. Take A, B and C in $\mathcal{A}(\mathcal{D})$, the tensor product $A \otimes_{\mathcal{O}} B$ is in $\mathcal{A}(\mathcal{D})$ and is given by: $(A \otimes_{\mathcal{O}} B)_k = A_k \otimes_{\mathbb{Q}} B_k$, $(A \otimes_{\mathcal{O}} B)_{\infty} = A_{\infty} \otimes_{\mathbb{Q}} B_{\infty}$ and the structure map is given by the composite of the three maps below.

$$\begin{array}{ccc} A_{\infty} \otimes B_{\infty} & \longrightarrow & \text{colim}_n(\prod_{k \geq n} A_k) \otimes \text{colim}_n(\prod_{k \geq n} B_k) \\ \text{colim}_n(\prod_{k \geq n} A_k \otimes \prod_{k \geq n} B_k) & \xrightarrow{\cong} & \text{colim}_n(\prod_{k \geq n} A_k) \otimes \text{colim}_n(\prod_{k \geq n} B_k) \\ \text{colim}_n(\prod_{k \geq n} A_k \otimes \prod_{k \geq n} B_k) & \longrightarrow & \text{colim}_n \prod_{k \geq n} (A_k \otimes B_k) \end{array}$$

We now have a series of natural isomorphisms, where we suppress notation for inc .

$$\begin{aligned} \mathcal{A}(\mathcal{D})(A \otimes_{\mathcal{O}} B, C) &= W\mathcal{O}\text{-mod}(A \otimes_{\mathcal{O}} B, C) \\ &\cong W\mathcal{O}\text{-mod}(A, \text{Hom}_{\mathcal{O}}(B, C)) \\ &\cong \mathcal{A}(\mathcal{D})(A, \text{fix Hom}_{\mathcal{O}}(B, C)) \end{aligned}$$

Thus we take the internal homomorphism object for $\mathcal{A}(\mathcal{D})$ to be $\text{fix Hom}_{\mathcal{O}}(B, C)$. It is routine to prove that $(c, \widehat{\text{fr}}_1^W)$ is a strong symmetric monoidal adjunction. \blacksquare

Our model structure is an alteration of the flat model structure from [Hov01], noting that every sheaf on \mathcal{D} is automatically flasque so the fibrations are precisely the stalk-wise surjections. Let $I_{\mathbb{Q}}$ and $J_{\mathbb{Q}}$ denote the sets of generating cofibrations and acyclic cofibrations for the projective model structure on $dg\mathbb{Q}\text{-mod}$, see [Hov99, Section 2.3]. Similarly we have $I_{\mathbb{Q}W}$ and $J_{\mathbb{Q}W}$ for $dg\mathbb{Q}W\text{-mod}$.

Theorem 4.8 *Define a map f in $\mathcal{A}(\mathcal{D})$ to be a weak equivalence or fibration if f_{∞} and each f_k is (in $dg\mathbb{Q}\text{-mod}$ and $dg\mathbb{Q}W\text{-mod}$ respectively). This defines a monoidal model structure on the category $\mathcal{A}(\mathcal{D})$. Furthermore this model structure is cofibrantly generated, proper and satisfies the monoid axiom. The generating cofibrations, I , are the collections $cI_{\mathbb{Q}}$ and $i_k I_{\mathbb{Q}W}$, for $k \geq 1$. The generating acyclic cofibrations, J , are $cJ_{\mathbb{Q}}$ and $i_k J_{\mathbb{Q}W}$, for $k \geq 1$.*

Proof We prove that the above sets of maps define a model structure using [Hov99, Theorem 2.1.19]. We prove in Lemma 4.9 below that $\widehat{\text{fr}}_1^W$ preserves filtered colimits. From this it follows that the domains of the generating cofibrations and acyclic cofibrations are small.

We identify the maps with the right lifting property with respect to I . Let $f: A \rightarrow B$ be such a map, using the adjunctions of Lemma 4.6 it follows that each $f_k: A_k \rightarrow B_k$ must be a surjection and a homology isomorphism, as must $\widehat{\text{fr}}_1^W f: \widehat{\text{fr}}_1^W A \rightarrow \widehat{\text{fr}}_1^W B$.

In turn, each f_k^W is a homology isomorphism and a surjection so $\hat{\mathfrak{F}}_n^W f$ is a surjection and a homology isomorphism for each $n \geq 1$. Taking colimits over n we see that f_∞ is a surjection and homology isomorphism.

Now take a map $f: A \rightarrow B$ that is stalk-wise a surjection and homology isomorphism. Since $\hat{\mathfrak{F}}_1^W A$ and $\hat{\mathfrak{F}}_1^W B$ are homotopy limits in $dg\mathbb{Q}\text{-mod}$ we see that $\hat{\mathfrak{F}}_1^W f$ is a homology isomorphism. It is also a surjection by similar arguments to that of Lemma 4.9. Hence f has the right lifting property with respect to I . Similarly the maps with the right lifting property with respect to J are precisely the stalk-wise surjections.

To complete the proof that we have a cofibrantly generated model category we must prove that transfinite compositions of pushouts of elements of J are weak equivalences that are I -cofibrations. Showing that such maps are I -cofibrations is routine. The maps in J are stalk-wise injective homology isomorphisms, such maps are preserved by pushouts and transfinite compositions and so the result follows.

Checking the pushout product axiom for the generators is routine. The monoid axiom holds because the functors p_k and p_∞ are strong monoidal left adjoints such that if each $p_k f$ and $p_\infty f$ are weak equivalences then f is a weak equivalence in $\mathcal{A}(D)$.

Left properness is immediate because colimits are defined stalk-wise. Right properness only requires work over ∞ , let P be the pull back of $X \rightarrow Z \leftarrow Y$, with $X \rightarrow Z$ a weak equivalence and $Y \rightarrow Z$ a fibration. Applying the right Quillen functor $\hat{\mathfrak{F}}_n^W$ gives a pullback diagram in $dg\mathbb{Q}\text{-mod}$, which is right proper. Hence $\hat{\mathfrak{F}}_n^W P \rightarrow \hat{\mathfrak{F}}_n^W X$ is a weak equivalence for each $n \geq 1$. Taking colimits over n shows that $P_\infty \rightarrow X_\infty$ is a homology isomorphism. \blacksquare

Lemma 4.9 *The functors $\hat{\mathfrak{F}}_n$ preserves filtered colimits for all $n \geq 1$.*

Proof One has a canonical map $\text{colim}_i \hat{\mathfrak{F}}_1 V^i \rightarrow \hat{\mathfrak{F}}_1 \text{colim}_i V^i$. For $X \in \mathcal{A}(\mathcal{D})$, we write elements of $\hat{\mathfrak{F}}_1 X$ as $(x_\infty, x_1, x_2, \dots)$, for $x_\infty \in X_\infty$ and $x_i \in X_i$, such that x_∞ and (x_1, x_2, \dots) agree in $X(\text{end})$. For each $n > 1$ there is an isomorphism of $dg\mathbb{Q}W$ -modules $\hat{\mathfrak{F}}_1 X \cong \hat{\mathfrak{F}}_n X \oplus \bigoplus_{1 \leq k < n} X_k$, which sends $(x_\infty, x_1, x_2, \dots)$ to the term $((x_\infty, x_n, x_{n+1}, \dots), x_1, x_2, \dots, x_{n-1})$. Hence, for each $n > 1$ we have specified an isomorphism between $\text{colim}_i \hat{\mathfrak{F}}_1 V^i$ and $\text{colim}_i \hat{\mathfrak{F}}_n V^i \oplus \bigoplus_{1 \leq k < n} \text{colim}_i V_k^i$.

It suffices to prove the result for $n = 1$, let $\{V^i\}_{i \in I}$ be a filtered diagram in $\mathcal{A}(\mathcal{D})$. Take some $(v_\infty, v_1, v_2, \dots)$ in V^j which maps to zero in $\hat{\mathfrak{F}}_1 \text{colim}_i V^i$. Thus there is some $m \in I$, with a map $j \rightarrow m$ in I such that v_∞ is sent to zero in V_∞^m . Consider $(0, w_1, w_2, \dots) \in \hat{\mathfrak{F}}_1 V^m$, the image of $(v_\infty, v_1, v_2, \dots)$. Since the term at infinity is zero, only finitely many w_k are non-zero, let N be such that for $k \geq N$, $w_k = 0$. Note that each w_k is zero in $\text{colim}_i V_k^i$. Write $\hat{\mathfrak{F}}_1 V^m$ as $\hat{\mathfrak{F}}_N V^m \oplus \bigoplus_{1 \leq k < N} V_k^m$. Thus $(0, w_1, w_2, \dots)$ is given by $((0, 0, \dots), w_1, w_2, \dots, w_{N-1})$ in $\hat{\mathfrak{F}}_N V^m \oplus \bigoplus_{1 \leq k < N} V_k^m$. The element $((0, 0, \dots), w_1, w_2, \dots, w_{N-1})$ is clearly zero in the colimit, hence so are $(0, w_1, w_2, \dots)$ and $(v_\infty, v_1, v_2, \dots)$.

Now we consider surjectivity, take $([z_\infty], [z_1], [z_2], \dots) \in \hat{\mathfrak{F}}_1 \text{colim}_i V^i$. So there is some $j \in I$ with $z_\infty \in V_\infty^j$. Pick a representative $(y_1, y_2, \dots) \in V^j(\text{end})$ for the image of z_∞ . Thus we have $(z_\infty, y_1, y_2, \dots) \in \hat{\mathfrak{F}}_1 V^j$. The difference between the image of

$(z_\infty, y_1, y_2, \dots)$ in $\widehat{\mathfrak{A}}_1 \operatorname{colim} V^i$ and $([z_\infty], [z_1], [z_2], \dots)$ has only finitely many non-zero terms. We now proceed as for injectivity. \blacksquare

Corollary 4.10 *The adjunctions of Lemma 4.6 are strong symmetric monoidal Quillen pairs.*

Lemma 4.11 *The collection $i_k \mathbb{Q}W$ for $k \geq 1$ and $c\mathbb{Q}$ are a set of compact, cofibrant and fibrant generators for this category.*

Proof That these are cofibrant follows immediately from the fact that \mathbb{Q} and $\mathbb{Q}W$ are cofibrant in $dg\mathbb{Q}\text{-mod}$ and $dg\mathbb{Q}W\text{-mod}$ respectively.

Take V in $\mathcal{A}(\mathcal{D})$, we must show that $M \rightarrow 0$ is a weak equivalence if and only if $[G, M]_*^{\mathcal{A}(\mathcal{D})} = 0$ for each object in our set. First we know (since every object is fibrant) that $[i_k \mathbb{Q}W, V]_*^{\mathcal{A}(\mathcal{D})} \cong [\mathbb{Q}W, V_k]_*^{\mathbb{Q}W}$, where the right hand side is maps in $\operatorname{Ho} dg\mathbb{Q}W\text{-mod}$. Secondly $[c\mathbb{Q}, V]_*^{\mathcal{A}(\mathcal{D})} \cong [\mathbb{Q}, \widehat{\mathfrak{A}}_1^W V]_*^{\mathbb{Q}}$, with the right hand side is maps in the homotopy category of $dg\mathbb{Q}\text{-mod}$. Since \mathbb{Q} generates $dg\mathbb{Q}\text{-mod}$ and $\mathbb{Q}W$ generates $dg\mathbb{Q}W\text{-mod}$, we know that V_k is acyclic for all k and that $\widehat{\mathfrak{A}}_1^W V$ is acyclic. The following isomorphism then tells us that $\widehat{\mathfrak{A}}_N^W V$ is acyclic for each N .

$$\widehat{\mathfrak{A}}_1^W V \cong \widehat{\mathfrak{A}}_N^W V \oplus \bigoplus_{1 \leq k < N} V_k^W$$

So now we have the following series of isomorphisms. $H_* V_\infty \cong H_*(\operatorname{colim}_N \widehat{\mathfrak{A}}_N^W V) \cong \operatorname{colim}_N H_*(\widehat{\mathfrak{A}}_N^W V) = 0$. \blacksquare

Let \mathcal{E}_a denote the full subcategory of $\mathcal{A}(\mathcal{D})$ on tensor products of the generators \mathcal{G}_a , considered as a category enriched over $dg\mathbb{Q}\text{-mod}$. This result below follows from [GS, Proposition 3.6].

Theorem 4.12 *The adjunction below is a Quillen equivalence with strong symmetric monoidal left adjoint and lax symmetric monoidal right adjoint.*

$$(-) \otimes_{\mathcal{E}_a} \mathcal{G}_a : \operatorname{mod}\text{-}\mathcal{E}_a \xleftrightarrow{\quad} \mathcal{A}(\mathcal{D}) : \underline{\operatorname{Hom}}(\mathcal{G}_a, -)$$

5 Comparison of Ringoids

Proposition 5.1 *For H a dihedral subgroup of $O(2)$ of order $2k$, and $i \geq 1$, let $\sigma_H^i = (S_{\mathcal{D}} \wedge \widehat{e}_H O(2)/H_+)^i$, with the smash product taken in the category of $S_{\mathcal{D}}$ -modules. Then $\pi_*(S_{\mathcal{D}}) = c\mathbb{Q}$ and $\pi_*(\sigma_H^i) = i_k \mathbb{Q}^{\otimes i} W$.*

Proof We already know the homotopy groups of $S_{\mathcal{D}}$: $\pi_*^K(S_{\mathcal{D}}) \cong \iota_K^*(e_{\mathcal{D}})\pi_*^K(S) \otimes \mathbb{Q}$. For $\pi_*^K(\sigma_H^i)$ we use the following facts: $\Phi^H X \simeq (e_H X)^H$, $\Phi^H(X \wedge Y) \simeq \Phi^H(X) \wedge \Phi^H(Y)$, $\Phi^H \Sigma^\infty A \simeq \Sigma^\infty A^H$ and $(O(2)/H)^H = W$, where X and Y are $O(2)$ -spectra and Y is an $O(2)$ -space. Thus $\pi_*^K(\sigma_H^i) \cong \iota_*(e_H)\pi_*^K(O(2)/H_+^{\wedge i}) \otimes \mathbb{Q}$. \blacksquare

Recall that $H_* \mathcal{E}_i \cong \pi_* \mathcal{E}_{top}$ as monoidal enriched categories, so we work with $\pi_* \mathcal{E}_{top}$. This is a category enriched over graded rational vector spaces and is isomorphic to

the full $g\mathbb{Q}$ -subcategory of $\text{Ho } S_{\mathcal{D}}\text{-mod}$ on object set \mathcal{G}_{top} . Furthermore this is an isomorphism of monoidal ringoids. Thus we calculate $[\sigma, \sigma']_*^{S_{\mathcal{D}}}$ for each pair of objects in \mathcal{G}_{top} and we have an isomorphism from this graded \mathbb{Q} -vector space to $H_* \mathcal{E}_t(\sigma, \sigma')$. The following proposition is immediate from the calculation above and Theorem 4.3.

Proposition 5.2 *Let $i, j, k, m \geq 1$ and let H and K be finite dihedral groups with $|H| = 2k$ and $|K| = 2m$. Then the functor π_* from Definition 4.2 gives isomorphisms as below.*

$$\begin{aligned} [S_{\mathcal{D}}, S_{\mathcal{D}}]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [c\mathbb{Q}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_H^i, S_{\mathcal{D}}]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_k \mathbb{Q}W^{\otimes i}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} \\ [S_{\mathcal{D}}, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [c\mathbb{Q}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_H^j, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_k \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \\ [\sigma_K^j, \sigma_H^i]_*^{S_{\mathcal{D}}} &\xrightarrow{\cong} [i_m \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} \end{aligned}$$

Proposition 5.3 *Let $i, j, k \geq 1$, then*

$$\begin{aligned} [c\mathbb{Q}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} &= e_{\mathcal{D}} A(O(2)) \otimes \mathbb{Q} \\ [i_k \mathbb{Q}W^{\otimes i}, c\mathbb{Q}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}W^{\otimes i}, \mathbb{Q})^W \\ [c\mathbb{Q}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}, \mathbb{Q}W^{\otimes i})^W \\ [i_k \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= \text{Hom}_{\mathbb{Q}}(\mathbb{Q}W^{\otimes i}, \mathbb{Q}W^{\otimes j})^W \\ [i_n \mathbb{Q}W^{\otimes j}, i_k \mathbb{Q}W^{\otimes i}]_*^{\mathcal{A}(\mathcal{D})} &= 0. \end{aligned}$$

Theorem 5.4 *There is an isomorphism of monoidal $dg\mathbb{Q}$ -categories $H_* \mathcal{E}_t \cong \mathcal{E}_a$.*

Proof As mentioned above it suffices to give an isomorphism of monoidal $g\mathbb{Q}$ -categories (and hence of $dg\mathbb{Q}$ -categories with trivial differentials) $\pi_* \mathcal{E}_{top} \rightarrow \mathcal{E}_a$. We already have a suitable enriched functor: π_* , our calculations above show that this gives an isomorphism of enriched categories. This functor respects the monoidal structures since everything is concentrated in degree zero. \blacksquare

Theorem 5.5 *Let C_0 denote the (-1) -connective cover functor on $dg\mathbb{Q}$ -modules. There is a zig-zag of quasi-isomorphisms of monoidal $dg\mathbb{Q}$ -categories.*

$$\mathcal{E}_t \xleftarrow{\sim} C_0 \mathcal{E}_t \xrightarrow{\sim} H_* \mathcal{E}_t \cong \mathcal{E}_a$$

hence there is a zig-zag of monoidal Quillen equivalences of $dg\mathbb{Q}\text{-mod-model}$ categories.

$$\text{mod-} \mathcal{E}_t \xleftarrow{\sim} \text{mod-} C_0 \mathcal{E}_t \xrightarrow{\sim} \text{mod-} H_* \mathcal{E}_t \cong \text{mod-} \mathcal{E}_a$$

Proof This result is [Bar08, Theorem 4.3.9 and Corollary 4.3.10]. The map $C_0 \mathcal{E}_t \rightarrow \mathcal{E}_t$ is the inclusion and $C_0 \mathcal{E}_t \rightarrow H_* \mathcal{E}_t$ is the projection. These are quasi-isomorphisms because the homology of \mathcal{E}_t is concentrated in degree zero. That quasi-isomorphisms induce Quillen equivalences of module categories is [SS03, Theorem A.1.1]. \blacksquare

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