MULTIPLE POLYLOGARITHMS, POLYGONS, TREES AND ALGEBRAIC CYCLES

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Abstract. We construct, for a field F and a natural number n, algebraic cycles in Bloch's cubical cycle group of codimension n cycles in $\big(\mathbb{P}^1_F\setminus\{1\}\big)^{2n-1}$, which correspond to weight n multiple polylogarithms with generic arguments if $F\subset\mathbb{C}$. Moreover, we construct out of them a Hopf subalgebra in the Bloch-Kriz cycle Hopf algebra χ_{cycle}. In the process, we are led to other Hopf algebras built from trees and polygons, which are mapped to χ_{cycle}. We relate the coproducts to the one for Goncharov's motivic multiple polylogarithms and to the Connes-Kreimer coproduct on plane trees and produce the associated Hodge realization for polygons.

Contents

1. Introduction 2
1.1. Multiple polylogarithms 2
1.2. From iterated integrals to Hopf algebras 2
1.3. Multiple polylogarithms and the Hopf algebra of algebraic cycles 3
1.4. Strategy 3
1.5. The structure of the paper 5
2. Background on algebraic cycles 6
3. The differential graded algebra of R-deco forests 8
3.1. The orientation torsor 8
3.2. The algebra of R-deco forests 8
3.3. The differential 10
4. Mapping forests to algebraic cycles 13
5. Tree sums with decomposable boundary and polygons 15
5.1. Tree sums with decomposable boundary 15
5.2. The algebra of R-deco polygons 16
5.3. Relating the polygon algebra to the tree algebra 19
5.4. Appendix: Relation with the dihedral Lie coalgebra from [12] 20
6. Cycles in the bar construction of the polygon algebra 22
6.1. Higher dissections of polygons 22
6.2. Proof of Proposition 5.15 23
6.3. The sign of a dissection 24
6.4. The cocycle attached to a polygon in the bar construction 25
7. Recognizing the coproduct for $\mathrm{B}(\pi,\partial)$ 29
7.1. Level structure on plane trees 29
7.2. Regrouping the terms in the coproduct $\Delta\big(\mathrm{B}(\pi)\big)$ 30
8. Comparing the coproduct for $\mathrm{B}(\pi)$ and the coproduct for the Hopf algebra of iterated integrals from [15] 32

8.1. The Hopf algebra $\mathcal{I}_\bullet(R)$ of iterated integrals. 32
8.2. The Lie coalgebra of iterated integrals and polygons. 33
8.3. Comparing the two coproducts. 34
8.4. Comparison with the coproduct from [6]. 37
9. Associating integrals to the multiple logarithm cycles 37
9.1. Brief review of the Hodge realization construction from [3], §8 37
9.2. Algebraic-topological chains 38
9.3. An example: the double logarithm case 39
9.4. Enhanced polygons as a comodule over the polygon algebra 40
9.5. The Hodge realization for polygons 45
10. Algebraic cycles for multiple polylogarithms 46
References 46

1. Introduction

1.1. **Multiple polylogarithms.** We start from the analytic background. The multiple polylogarithm functions were defined in [10] by the power series

$$Li_{n_1,\dots,n_m}(z_1,\dots,z_m)=\sum_{0<k_1<\dots<k_m}\frac{z_1^{k_1}}{k_1^{n_1}}\frac{z_2^{k_2}}{k_2^{n_2}}\cdots\frac{z_m^{k_m}}{k_m^{n_m}}\qquad(z_i\in\mathbb{C},|z_i|<1)\,.$$

They admit an analytic continuation to a Zariski open subset of $\mathbb{C}^m$. When $m=1$, we recover the classical polylogarithm function. Putting $n_1=...=n_m=1$, we get the *multiple logarithm* function.

Let x_i be complex numbers. Recall that an iterated integral is defined as

$$I(x_0;x_1,\dots,x_m;x_{m+1})=\int\limits_{\Delta_\gamma}\frac{dt_1}{t_1-x_1}\wedge\dots\wedge\frac{dt_m}{t_m-x_m}\,, \tag{1}$$

where γ is a path from x_0 to x_{m+1} in $\mathbb{C}-\{x_1,...,x_m\}$, and the cycle of integration Δ_γ consists of all m-tuples of points $(\gamma(t_1),...,\gamma(t_m))$ with $t_i\leqslant t_j$ for $i<j$.

Multiple polylogarithms can be written as iterated integrals (cf. loc.cit.). In particular, here is the iterated integral presentation of the multiple logarithm function:

$$Li_{1,\dots,1}(z_1,\dots,z_m)=(-1)^m I(0;x_1,\dots,x_m;1)\,, \tag{2}$$

where we set

$$x_1:=(z_1\cdots z_m)^{-1},\quad x_2:=(z_2\cdots z_m)^{-1},\quad\dots\quad,x_m:=z_m^{-1}\,. \tag{3}$$

Observe that in (3) the parameters $x_1,\dots,x_m$ are non-zero. Moreover, many properties of the iterated integrals will change if we put some of the x_i's equal to zero. As a result, many features of the theory of multiple polylogarithms are different from the ones of multiple logarithms. We will use the notation $I_{1,\dots,1}(x_1,\dots,x_m)$ for $I(0;x_1,\dots,x_m;1)$ when the x_i's are non-zero.

1.2. **From iterated integrals to Hopf algebras.** In [15] it was shown that any iterated integral (1) gives rise to an element

$$I^{\mathcal{H}}(x_0;x_1,\dots,x_m;x_{m+1})\in\mathcal{A}^{\mathcal{H}}_m,\qquad x_i\in\mathbb{C}, \tag{4}$$

of a certain commutative graded Hopf algebra $\mathcal{A}^{\mathcal{H}}_\bullet$, called the *fundamental Hodge-Tate Hopf algebra*. The category of graded finite-dimensional comodules over this

Hopf algebra is canonically equivalent to the category of mixed $\mathbb{Q}$-Hodge-Tate structures. The element (4) is called a *Hodge iterated integral*.

It was shown in loc. cit. that this construction has l-adic and motivic counterparts. In particular, given a number field F, there are the *motivic iterated integrals*

$$I^{\mathcal{M}}(x_0; x_1, \ldots, x_m; x_{m+1}) \in \mathcal{A}^{\mathcal{M}}_m, \qquad x_i \in F, \tag{5}$$

which live in the *fundamental motivic Tate Hopf algebra* $\mathcal{A}^{\mathcal{M}}_{\bullet}(F)$ of F.

The benefits gained by working with the Hodge/motivic iterated integrals are explained in the introduction to [15]. In particular the coproduct in the Hopf algebra is a new powerful structure, which is not visible on the level of numbers. It is conjectured that any relation of algebraic-geometric origin between the iterated integrals gives rise to a similar relation between the corresponding Hodge/motivic iterated integrals. Therefore upgrading iterated integrals to elements of a Hopf algebra we see new structures, but conjecturally do not lose any information.

1.3. **Multiple polylogarithms and the Hopf algebra of algebraic cycles.** Given any field F, Bloch and Kriz [3] constructed, using algebraic cycles, yet another graded Hopf algebra $\chi_{\mathrm{cycle}}(F)$, denoted $\chi_{\mathrm{Mot}}(F)$ in loc.cit., and conjectured that it is isomorphic to the fundamental motivic Tate Hopf algebra of F.

Thus it is natural to ask whether we can find elements

$$\mathbb{L}\mathrm{i}^{\mathrm{cycle}}_{n_1,\ldots,n_m}(z_1, \ldots, z_m) \in \chi_{\mathrm{cycle}}(F), \quad z_i \in F^{\times}\,, \tag{6}$$

corresponding to multiple polylogarithms in the *Hopf algebra of algebraic cycles* $\chi_{\mathrm{cycle}}(F)$ (defined in (8) below). The elements corresponding to the classical polylogarithms have been defined in [3] by using the Bloch-Totaro cycles [4]. The main result of our paper is a construction of elements (6) for generic $z_i \in F^{\times}$, where generic means that the products of the arguments $\prod_{j=i}^{k} z_j$ for any pair (i, k), where $1 \leqslant i \leqslant k \leqslant n$, are all different from 1. We show that the $\mathbb{Q}$-linear combinations of these elements form a Hopf subalgebra

$$\widetilde{\chi}^{\mathrm{MP}}_{\mathrm{cycle}}(F) \subset \chi_{\mathrm{cycle}}(F)\,. \tag{7}$$

Presumably one can define the elements (6) for all $z_i \in F^{\times}$. They should generate the "true" multiple polylogarithm Hopf subalgebra $\chi^{\mathrm{MP}}_{\mathrm{cycle}}(F) \subset \chi_{\mathrm{cycle}}(F)$.

It follows from Conjecture 17a) in [10] that, for any field F, the $\mathbb{Q}$-linear combinations of the elements (6) should span over $\mathbb{Q}$ the fundamental motivic Hopf algebra. In other words, one should have

$$\chi^{\mathrm{MP}}_{\mathrm{cycle}}(F) \stackrel{?}{=} \chi_{\mathrm{cycle}}(F)\,.$$

1.4. **Strategy.** Let $\mathcal{N}^{\bullet}(F, p)$ be Bloch's weight p cubical cycle complex [1]. The direct sum of these complexes

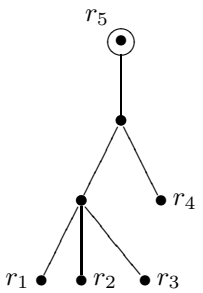
$$\mathcal{N}^{\bullet}(F, *) := \bigoplus_{p \geqslant 0} \mathcal{N}^{\bullet}(F, p)$$

has a structure of graded-commutative augmented differential graded algebra (DGA). Applying the bar construction functor B to it and then taking the zero$^{\mathrm{th}}$ cohomology of the resulting complex we get the Bloch-Kriz Hopf algebra:

$$\chi_{\mathrm{cycle}}(F) := \mathrm{H}^0\mathrm{B}(\mathcal{N}^{\bullet}(F, *))\,. \tag{8}$$

Our Hopf subalgebra $\widetilde{\chi}^{\mathrm{MP}}_{\mathrm{cycle}}(F)$ is also defined using the functor $\mathrm{H}^0\mathrm{B}$. Its definition is organized into three steps outlined below.

1. **Algebraic cycles from trees**. Let R be a set. An *R-deco tree* is a plane tree with additional data: a root vertex of valency 1 and a decoration of the external vertices by elements r_i of R, as in the picture (where the encircled vertex, decorated by r_5, is the root vertex).



We define the grading of a tree as the number of its edges. Consider the graded $\mathbb{Q}$-vector space generated by R-deco trees. Let $\mathcal{T}_\bullet(R)$ be the graded-commutative algebra generated by this vector space. A set of linear generators for $\mathcal{T}_\bullet(R)$ is given by monomials in R-deco trees, called *R-deco forests*. It has another grading, given by the number of leaves. So we get a bigraded algebra $\mathcal{T}^*_\bullet(R)$. There is a differential $\mathcal{T}^*_\bullet(R) \to \mathcal{T}^*_{\bullet-1}(R)$ defined by contracting individual edges of a forest. It equips $\mathcal{T}^*_\bullet(R)$ with a structure of graded-commutative DGA with an extra grading. It is a rooted version of the DGA defined by the second author in [11].

Now take $R = F^\times$, where F is a field. We consider the subclass of "generic" decorations (as in Definition 4.1) on plane rooted trees. The corresponding DGA is denoted by $\widetilde{\mathcal{T}}^*_\bullet(F^\times)$. We associate to such an $F^\times$-decorated tree with m edges an element of the cubical cycle group $\mathcal{N}^{2p-m}(F,p)$ given by the codimension p admissible algebraic cycles in the m-dimensional algebraic cube. We show that by extending this map to forests by multiplicativity we get a homomorphism of DGAs

$$\Phi : \widetilde{\mathcal{T}}^*_\bullet(F^\times) \longrightarrow \mathcal{N}^{2*-\bullet}(F,*)$$

which we call the *forest cycling map*.

2. **Trees from polygons.** In a second step, we single out distinguished combinations of the above (R-deco) trees encoded by (R-deco) polygons. Namely, given a polygon with additional data (a root side and a decoration), we consider all its triangulations. Each triangulation defines by duality a plane trivalent tree. Let Ψ be the map which assigns to the polygon the formal sum of these trees. This map is extended to the graded-commutative algebra generated by the R-deco polygons. The latter is equipped with a differential, which gives rise to a graded-commutative DGA $\mathcal{P}^*_\bullet(R)$. The DGA $\mathcal{P}^*_\bullet(R)$ is identified with the standard cochain complex of one of the Lie coalgebras of iterated integrals defined in [15].

Let us consider its "generic part", a sub-DGA $\widetilde{\mathcal{P}}^*_\bullet(R)$. The map Ψ provides a homomorphism of DGAs

$$\Psi : \widetilde{\mathcal{P}}^*_\bullet(R) \longrightarrow \widetilde{\mathcal{T}}^*_\bullet(R)\,.$$

The *crucial fact* is the following. The image of the composition

$$\widetilde{\mathcal{P}}^*_\bullet(F^\times) \stackrel{\Psi}{\longrightarrow} \widetilde{\mathcal{T}}^*_\bullet(F^\times) \stackrel{\Phi}{\longrightarrow} \mathcal{N}^{2*-\bullet}(F,*) \tag{9}$$

lies in the subspace of elements with "decomposable" differentials.

3. **The bar construction.** Applying the $\mathrm{H}^0\mathrm{B}$ functor to (9) we get the Hopf algebras

$$\chi_{\widetilde{\mathcal{P}}}(F^\times) := \mathrm{H}^0\mathrm{B}(\widetilde{\mathcal{P}}^*_\bullet(F^\times)), \quad \chi_{\widetilde{\mathcal{T}}}(F^\times) := \mathrm{H}^0\mathrm{B}(\widetilde{\mathcal{T}}^*_\bullet(F^\times)),$$

and homomorphisms between them

$$\chi_{\widetilde{\mathcal{P}}}(F^\times) \stackrel{\mathrm{H}^0\mathrm{B}(\Psi)}{\longrightarrow} \chi_{\widetilde{\mathcal{T}}}(F^\times) \stackrel{\mathrm{H}^0\mathrm{B}(\Phi)}{\longrightarrow} \chi_{\mathrm{cycle}}(F)\,. \tag{10}$$

We prove that any $F^\times$-decorated polygon with the generic decoration gives rise to an element of $\chi_{\mathrm{cycle}}(F)$, called the *multiple logarithm cycle*. Its components in the bar complex are parametrized by dissections of the original polygon. By a slight generalization of this construction, we obtain our *multiple polylogarithm cycle*.

4. **Comparison theorems.** The very name of the cycles suggests that they correspond to the motivic multiple polylogarithms from [15]. We give two pieces of evidence for this:

(i) The coproduct for the multiple polylogarithm cycles agrees with the one for the motivic multiple polylogarithms computed in [15].

(ii) Using the Hodge realization of the cycle Hopf algebra ([3]), we recover from our algebraic cycles the corresponding analytic functions.

1.5. **The structure of the paper.** In Section 2 we recall a few facts about Bloch's DGA of cubical algebraic cycles $\mathcal{N}^\bullet(F,p)$, for a field F.

In Section 3 we define, for a set R, another DGA $\mathcal{T}^*_\bullet(R)$, built from R-decorated rooted forests. A very similar differential graded algebra, for non-rooted forests, was introduced in [11].

In Section 4 we relate the forest DGA to the cycle DGA in the case when $R = F^\times$. More precisely, we define a subalgebra $\widetilde{\mathcal{T}}^*_\bullet(F^\times)$ of $\mathcal{T}^*_\bullet(F^\times)$ by imposing some explicit genericity condition on the decoration. Then we construct a map of graded DGA's

$$\Phi : \widetilde{\mathcal{T}}^*_\bullet(F^\times) \longrightarrow \mathcal{N}^{2*-\bullet}(F,*)\,.$$

In Section 5 we introduce the second important ingredient of our construction: given an ordered collection of elements $x_1, \dots, x_m \in F^\times$, we define an element

$$\tau(x_1, \dots, x_m) \in \mathcal{T}^*_\bullet(F^\times)$$

with decomposable differential. Under the genericity conditions on the x_i's, it belongs to $\widetilde{\mathcal{T}}^*_\bullet(F^\times)$, and the algebraic cycle $\Phi\tau(x_1, \dots, x_m)$ corresponds to the multiple logarithm (cf. 1.1). Then we introduce R-deco polygons as more convenient combinatorial objects, and define a DGA of R-deco polygons, which maps to the tree DGA from Section 4.

We show in Section 8.2 that the DGA of R-deco polygons is identified with the standard cochain complex of a Lie coalgebra of (formal) iterated integrals defined in [15]. There is a yet another differential $\overline{\partial}$ on the graded algebra of R-deco polygons, which plays only a technical role in the paper. In Section 5.4 the corresponding

DGA is related to the rooted version of one of the graded Lie coalgebras introduced in [12] under the name of the dihedral Lie coalgebras.

In Section 6, we perform the bar construction on polygons. It gives rise to a Hopf algebra with an induced map to χ_{cycle}.

In Section 7, an explicit formula for the coproduct in this Hopf algebra of polygons is given.

In Section 8, we prove a coproduct comparison theorem: the coproduct of our multiple polylogarithm cycles agrees with the coproduct of the corresponding motivic iterated integrals obtained in [15]. This ensures that our multiple polylogarithm cycles agree with the corresponding motivic iterated integrals. We relate the formula for the coproduct in the Hopf algebra of polygons to the one from [6].

In Section 9, we show how to get the original multivalued analytic functions (2) from the constructed cycles.

In Section 10, we indicate how to modify our construction in order to obtain cycles corresponding to multiple *poly*logarithms.

See Sections 7-9 of [13] for connections between the sums of plane decorated trivalent trees and multiple polylogarithms closely related to the one discussed in our paper.

Our paper [9] can serve as an introduction to the present paper.

During the preparation of the final version of this paper, we learned that H. Furusho and A. Jafari obtained similar results, see [7].

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2. Background on algebraic cycles

We review some of the main properties of cubical algebraic cycles. Our main references for the general set-up are the papers by Bloch [4], [5], §2.4.1, and by Bloch and Kriz [3], §5. The reader can find more examples in [8].

Let F be a field. Following [5], we define the algebraic 1-cube $\square_F$ as a pair

$$\square_F = \Big(\mathbb{P}^1_F \setminus \{1\} \simeq \mathbb{A}^1_F\,, (0) - (\infty)\Big).$$

Here we consider the standard coordinate z on the projective line $\mathbb{P}^1_F$ and remove from it the point 1. Furthermore, $(0)-(\infty)$ denotes the divisor defined by the two points 0 and ∞. The algebraic n-cube is defined by setting $\square^n_F = (\square_F)^n$.

Bloch defined (we use the conventions from [3], §5), for $p, n \in \mathbb{N}$, the cycle groups

$$\mathrm{Cycle}^p\langle F, n\rangle = \mathbb{Z}\big[\{\text{admissible closed irreducible subvarieties over } F, \text{ of codimension } p \text{ in } \square^n_F\}\big]\,.$$

Here a cycle is called **admissible** if it intersects all the faces (of any codimension) of $\square^n_F$ properly, i.e., in codimension p or not at all.

Let $G(k)$ be the semidirect product of the symmetric group S_k and the group $(\mathbb{Z}/2\mathbb{Z})^k$, acting by permuting and inverting the coordinates in $\square^k_F$. Let ε_k be the sign representation of this group. We set in a slight variation to [3], indicated by a subscript c for "coinvariants",

$$\mathcal{N}^n_{\mathrm{c}}(F,p) = \big(\mathrm{Cycle}^p\langle F, 2p-n\rangle \otimes \varepsilon_{2p-n}\big)_{G(2p-n)}\,.$$

Remark: The cycle groups were defined in [4] using invariants instead of coinvariants, and therefore the groups above agree with those rationally. We will refer to them as the *cubical algebraic cycle groups*.

Putting the group $\mathcal{N}^n_{\mathrm{c}}(F,p)$ in degree n, we get, for a fixed p, a complex

$$\cdots \to \mathcal{N}^n_{\mathrm{c}}(F,p) \xrightarrow{\partial} \mathcal{N}^{n+1}_{\mathrm{c}}(F,p) \to \dots$$

where the differential ∂ is given by

$$\partial = \sum_{i=1}^{n} (-1)^{i-1}(\partial^i_0 - \partial^i_\infty)$$

and ∂^i_ε denotes the operator given by the intersection with the coordinate hyperplane $\{z_i = \varepsilon\}$, $\varepsilon \in \{0,\infty\}$.

The concatenation of coordinates, followed by taking the corresponding coinvariants, provides a product on algebraic cycles, and together with the above one gets ([4], Lemma 1.1, and [3], Prop. 5.1):

Proposition 2.1. *Given a field F, the algebraic cycle groups $\mathcal{N}^n_{\mathrm{c}}(F,p)$ give rise to a differential graded algebra $\mathcal{N}^\bullet_{\mathrm{c}} = \bigoplus_{p\geqslant 0}\mathcal{N}^\bullet_{\mathrm{c}}(F,p)$. Its cohomology groups are the higher Chow groups of F.*

The higher Chow groups of a field F are known, by Bloch ([1] and [2]) and also by Levine [16], to be rationally isomorphic to its algebraic K-groups.

Example 2.2:

a) In $\square^3_F$, a curve intersecting the line $z_1 = z_2 = 0$ is not admissible, since an admissible curve has to meet faces of dimension 1 in codimension 2, i.e. not at all.

b) For any element a in F, one can associate a cubical algebraic cycle which corresponds to the dilogarithm $Li_2(a)$ if $F \hookrightarrow \mathbb{C}$. This cycle has been given by Totaro as the image of the map

$$\begin{aligned} \varphi_a : \mathbb{P}^1_F &\to \big(\mathbb{P}^1_F\big)^3\,, \\ t &\mapsto (t, 1-t, 1-\frac{a}{t})\,, \end{aligned}$$

restricted to the algebraic cube $\square^3_F$: we write

$$C_a := \Big[t, 1-t, 1-\frac{a}{t}\Big] := \varphi_a\big(\mathbb{P}^1_F\big) \cap \square^3_F\,.$$

The cycle C_a belongs to the group $\mathcal{N}^1_{\mathrm{c}}(F,2)$. One has

$$\partial C_a = [a, 1-a] \in \square^2_F$$

(only ∂^3_0 gives a non-empty contribution). The same computation shows that C_a is in fact admissible. Observe the apparent similarity with the formula $d\,Li_2(a) = -\log(1-a)\,d\log(a)$ for the differential of the dilogarithm.

c) Recall that the double logarithm can be defined by an iterated integral

$$I_{1,1}(x_1,x_2)=\int_{\Delta_\gamma}\frac{dt_1}{t_1-x_1}\wedge\frac{dt_2}{t_2-x_2}\,,$$

in the notation of (1) in the Introduction. Its differential can be easily computed as

$$d\,I_{1,1}(x_1,x_2)=I_1(x_1)\,d\,I_1(x_2)-I_1(x_1)\,d\,I_1\Big(\frac{x_2}{x_1}\Big)+I_1(x_2)\,d\,I_1\Big(\frac{x_1}{x_2}\Big)\,. \tag{11}$$

The cycle

$$Z_{x_1,x_2}:=\Big[1-\frac{1}{t},1-\frac{t}{x_1},1-\frac{t}{x_2}\Big]\,, \tag{12}$$

will play the role of the double logarithm among the algebraic cycles. Its boundary is readily evaluated as

$$\partial Z_{x_1,x_2}=\underbrace{\Big[1-\frac{1}{x_1},1-\frac{1}{x_2}\Big]}_{\text{from } z_1=0}-\underbrace{\Big[1-\frac{1}{x_1},1-\frac{x_1}{x_2}\Big]}_{\text{from } z_2=0}+\underbrace{\Big[1-\frac{1}{x_2},1-\frac{x_2}{x_1}\Big]}_{\text{from } z_3=0}\in\mathcal{N}^2_{\rm c}(F,2) \tag{13}$$

whose individual terms are already very reminiscent of the three terms in (11).

The cycle associated to the triple logarithm $I_{1,1,1}(x_1,x_2,x_3)$ is given in Example 5.2 below.

3. The differential graded algebra of R-deco forests

In this paper, a *plane tree* is a finite tree whose internal vertices are of valency $\geqslant 3$, and where at each vertex a cyclic ordering of the incident edges is given. We assume that all the other vertices are of valency 1, and call them *external* vertices. A plane tree is *planted* if it has a distinguished external vertex of valency 1, called its *root*; in particular, a planted tree has at least one edge. A *forest* is a disjoint union of trees.

Below we work with algebras over $\mathbb{Q}$, although we may replace it by any field.

3.1. The orientation torsor.

Recall that a *torsor* under a group G is a set on which G acts freely transitively.

Let S be a finite (non-empty) set. We impose on the set of orderings of S an equivalence relation, given by even permutations of the elements. The equivalence classes form a 2-element set Or_S of *orientation classes*, for short also *orientations*. It has an obvious $\mathbb{Z}/2\mathbb{Z}$-torsor structure and is called the *orientation torsor of* S.

Definition 3.1. *The* **orientation torsor** *of a plane forest is the orientation torsor of the set of its edges.*

3.2. The algebra of R-deco forests.

Definition 3.2. *Let R be a (non-empty) set. An R-***deco tree** *is a planted plane tree with a map, called R-***decoration***, from its external vertices to R. An R-***deco forest** *is a disjoint union of R-deco trees.*

Remark 3.3: (1) There is a canonical direction for each edge in an R-deco tree, away from the root.

(2) An edge ordering in a tree τ provides an orientation on τ. For a planted plane tree there is a canonical linear order of the edges, starting from the root edge, which is induced by the cyclic order of edges at internal vertices.

We draw the trees so that the cyclic order of edges around the vertices is displayed in counterclockwise direction, and the root vertex is at the top.

Example 3.4: We draw an R-deco tree τ with root vertex decorated by $x_4 \in R$; its other external vertices are decorated by x_1, x_2, $x_3 \in R$. The above-mentioned ordering of the edges e_i coincides with the natural ordering of their indices, while the direction of the edges (away from the root) is indicated by small arrows along the edges. The root vertex is marked by a circle around it.

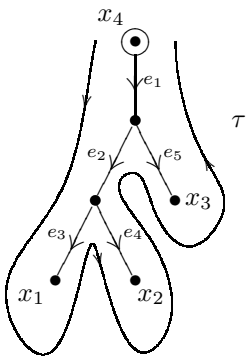


Figure 1: An R-deco tree τ with root vertex decorated by x_4.

We define the **grading** of an R-deco tree τ by

$$e(\tau) = \#\{\text{edges of } \tau\}$$

and extend it to forests by linearity: $e(\phi_1 \sqcup \phi_2) := e(\phi_1) + e(\phi_2)$. (Here $\sqcup$ denotes the disjoint union.)

Let $V_\bullet^{\mathrm{tr}}(R)$ be the graded vector space where $V_n^{\mathrm{tr}}(R)$ for $n > 0$ has a basis given by R-deco trees of grading n and where $V_0^{\mathrm{tr}}(R) := \mathbb{Q} \cdot \mathbb{1}$ for an extra generator $\mathbb{1}$.

Definition 3.5. *The algebra $\mathcal{T}_\bullet(R)$ is the free graded-commutative algebra with unit $\mathbb{1}$, generated by the graded vector space $V_\bullet^{\mathrm{tr}}(R)$.*

Here is a definition which does not use a choice of orientation.

Definition 3.6. *For a set R, $\mathcal{T}_\bullet^{\mathrm{Or}}(R)$ is generated as a vector space by pairs (ϕ, ω), where ϕ is an R-deco forest and ω an orientation on it, subject to the relation $(\phi, -\omega) = -(\phi, \omega)$.*

We define an algebra structure $\star$ on $\mathcal{T}_\bullet^{\mathrm{Or}}(R)$ by letting $\mathbb{1}$ be the neutral element with respect to $\star$ and by setting ($\sqcup$ denotes the disjoint union)

$$(\phi_1, \omega_1) \star (\phi_2, \omega_2) := (\phi_1 \sqcup \phi_2, \omega_1 \otimes \omega_2)\,.$$

It makes $\mathcal{T}_\bullet^{\mathrm{Or}}(R)$ into a graded-commutative algebra, so the basis elements of $V_\bullet^{\mathrm{tr}}(R)$ commute in the R-deco tree algebra via the rule

$$(\tau_1, \omega_1) \star (\tau_2, \omega_2) = (-1)^{e(\tau_1)e(\tau_2)} (\tau_2, \omega_2) \star (\tau_1, \omega_1)\,.$$

Since each R-deco tree comes equipped with a canonical orientation (cf. Remark 3.3), we can—and in the following will—identify $\mathcal{T}_\bullet(R)$ with $\mathcal{T}_\bullet^{\mathrm{Or}}(R)$. This identification will be useful in describing the differential, since the orientation torsor takes care of the signs which are more complicated to describe for $\mathcal{T}_\bullet(R)$. We will refer to either algebra as the **R-deco tree algebra**.

3.3. **The differential.** A differential on $\mathcal{T}_\bullet(R) \cong \mathcal{T}_\bullet^{\mathrm{Or}}(R)$ is a map

$$d : \mathcal{T}_\bullet(R) \longrightarrow \mathcal{T}_{\bullet-1}(R)$$

satisfying $d^2 = 0$ and the Leibniz rule

$$d\big((\tau_1,\omega_1)\star(\tau_2,\omega_2)\big) = d\big((\tau_1,\omega_1)\big)\star(\tau_2,\omega_2) + (-1)^{e(\tau_1)}(\tau_1,\omega_1)\star d\big((\tau_2,\omega_2)\big)\,.$$

Since $\mathcal{T}_\bullet(R)$ is a free graded-commutative algebra, it is sufficient to define the differential on the algebra generators, that is on the elements (τ,ω), where τ is an R-deco tree and ω is an orientation of τ.

The terms in the differential of a tree τ arise by contracting an edge of τ—they fall into two types, according to whether the edge is internal or external. We will need the notion of a splitting.

Definition 3.7. *A* **splitting** *of a tree τ at an internal vertex v is the disjoint union of the trees which arise as $\tau_i \cup v$ where the τ_i are the connected components of $\tau \setminus v$.*

The following further structures on τ are inherited for each $\tau_i \cup v$: a decoration at v, planarity of τ and an ordering of its edges. Also, if τ has a root r, then v plays the role of the root for all $\tau_i \cup v$ which do not contain r.

Definition 3.8. *Let e be an edge of a tree τ. The* **contraction** *of τ along e, denoted τ/e, is given as follows:*

1. *If e is an internal edge, then τ/e is again a tree: it is the same tree as τ except that e is contracted and the incident vertices v and v' of e are identified to a single vertex.*
2. *If e is an external edge not containing the root vertex, then τ/e is obtained as follows: first we contract the edge e to a vertex w and then we perform a splitting at w.*
3. *If the tree consists of a single edge, its contraction gives the empty tree.*

Two typical examples are given below: in Figure 2 we contract a *leaf*, i.e. an external vertex which is not the root vertex, and in Figure 3 the root edge is contracted.

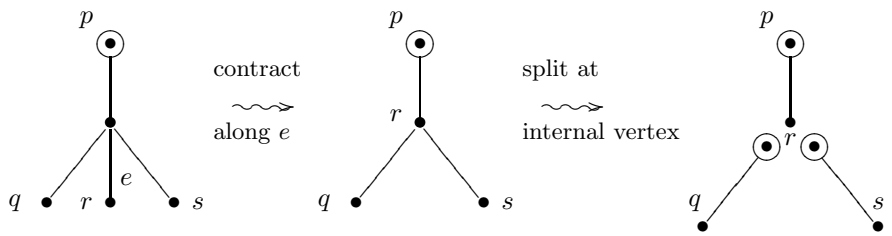


Figure 2: Contracting a leaf.

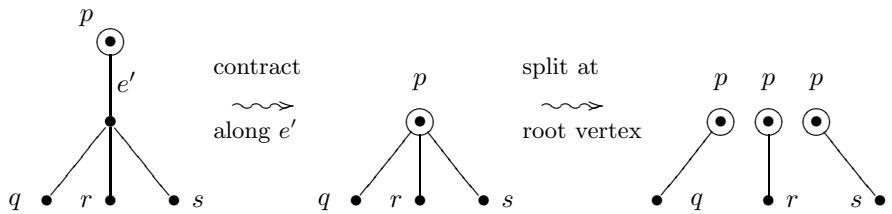


Figure 3: Contracting the root edge.

Remark 3.9: The linear ordering on the edges of τ induces a linear ordering on each of the trees which result from a splitting, and it furthermore induces a linear ordering on the trees in the resulting forest. The notion of an *unshuffle* is appropriate to describe this.

Let S be a finite set and Or_S the orientation torsor of S. We present its elements as $s_1 \wedge ... \wedge s_n$, where $n = |S|$, with the relation $s_{\sigma(1)} \wedge \cdots \wedge s_{\sigma(n)} = \mathrm{sgn}(\sigma)\big(s_1 \wedge \cdots \wedge s_n\big)$ for any permutation σ on n letters. For $|S| = 1$ there is a unique orientation.

Given $s \in S$ and $\omega \in \mathrm{Or}_S$, we define an element $i_s\omega \in \mathrm{Or}_{S-s}$ as follows:

$$i_s\omega := s_2 \wedge ... \wedge s_n \quad \text{if } \omega = s \wedge s_2 \wedge ... \wedge s_n \,.$$

Definition 3.10. *Let τ be a finite tree with set of edges $E(\tau)$, and let ω be an orientation of τ. The* **differential** *on (τ, ω) is defined as*

$$d : (\tau, \omega) \mapsto \sum_{e \in E(\tau)} (\tau/e, i_e\omega) \,.$$

In particular, d maps a tree with at most one edge to zero (which corresponds to the empty tree).

Proposition 3.11. *The map d just defined is in fact a differential, i.e. $d^2 = 0$.*

Proof. For a tree τ with one edge only we already have $d(\tau) = 0$. Applying d to a generator (τ, ω) with $\geqslant 2$ edges, we obtain $\sum_{e \in E(\tau)} (\tau/e, i_e\omega)$, and a second application of d gives (with the obvious notation $\tau/(e \sqcup e')$ for the independent contraction of the edges e and e' to a point each)

$$\begin{aligned} \sum_{e' \in E(\tau/e)} \big((\tau/e)/e', i_{e'}(i_e(\omega))\big) &= \sum_{\substack{(e,e') \in E(\tau)^2 \\ e \neq e'}} \big(\tau/(e \sqcup e'), i_{e'}(i_e\omega)\big) \\ &= \sum_{e < e'} \big(\tau/(e \sqcup e'), i_{e'}(i_e\omega)\big) + \big(\tau/(e \sqcup e'), i_e(i_{e'}\omega)\big) \end{aligned}$$

which vanishes since $i_{e'}i_e\omega = -i_e i_{e'}\omega$. Note that this argument works also if e and e' are contracted to the same point. □

Proposition 3.12. *The differential of Definition 3.10 is compatible with the R-deco structure.*

Proof. We can identify R-deco forests which arise from successively contracting edges in an R-deco tree with trees having external and possibly internal decoration. Thus, as long as the two contracted edges are not adjacent leaves, the same proof as for the previous proposition applies.

If the two contracted edges are adjacent leaves, then the contraction of the second one gives zero, since it corresponds to contracting a tree with a single edge. □

Example 3.13: The simplest non-trivial example for the differential of an $F^\times$-deco tree, where $F^\times$ is the multiplicative group of a field F, can be seen on a tree with one internal vertex, as given in Figure 4. Here we choose the $F^\times$-decoration (x_1, x_2) with $x_1, x_2 \in F^\times$ for the leaves and the decoration 1 for the root:

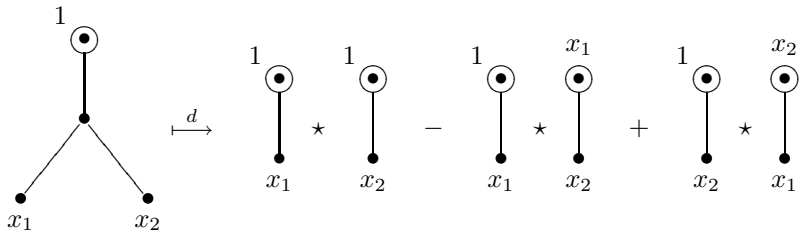


Figure 4: The differential on a tree with one internal vertex.

For the drawings, we use the canonical ordering of edges (cf. Remark 3.3) for $F^\times$-deco trees and the induced ordering of edges for forests which arise from a splitting. The edge ordering of the forest produces its orientation, i.e. the choice of an element in the orientation torsor.

Remark 3.14: There is a $\mathbb{Z}$-bigrading on $\mathcal{T}_\bullet(R)$ given by

$$\mathcal{T}_n^p(R) = \mathbb{Z}[\{R\text{-deco forests with } n \text{ edges and } p \text{ leaves}\}]\,.$$

For $n \leqslant 0$ or $p \leqslant 0$ it is zero, except for $p = n = 0$ where it is $\simeq \mathbb{Q}$. It will correspond to the bigrading on the cycle groups $\mathcal{N}_c^{2p-n}(F, p)$.

For a set R, put

$$\mathcal{T}_\bullet^*(R) := \bigoplus_{p \geqslant 0} \bigoplus_{0 \leqslant n \leqslant p} \mathcal{T}_n^p(R)\,.$$

We have an obvious augmentation map $\varepsilon : \mathcal{T}_\bullet^*(R) \to \mathbb{Q}$ defined by taking the zero$^{\text{th}}$ component.

With the above definitions, we have:

Proposition 3.15. *The algebra $\mathcal{T}_\bullet^*(R)$, together with the differential d above, forms a bigraded DGA with augmentation. The differential lowers the grading indicated with subscripts by 1 and leaves the grading indicated by superscripts unchanged.*

Proof. We have already proved in Proposition 3.11 that $d^2 = 0$. It remains to show that the differential is compatible with the bigrading. In fact, we have more precisely

$$d : \mathcal{T}_n^p(R) \to \mathcal{T}_{n-1}^p(R)\,,$$

where $\mathcal{T}_0^p(R) = 0$ for any $p > 0$.

We distinguish three cases, dealing first with a tree τ in $\mathcal{T}_n^p(R)$:

(1) If we contract an internal edge in τ, we obtain a tree with one edge less (so the subscript grading is lowered by 1) and with the same root and decorations of external edges.

(2) For contractions of an external non-root edge, the vertex to which the edge is contracted serves as a root vertex for all but one tree in the resulting forest. Therefore the number of external non-root vertices does not change under the contraction while the number of edges obviously does.

(3) If the root edge is contracted, none of the external non-root edges changes its properties, except when the tree consists of a single edge in which case the contraction gives the empty tree. In both cases the claim follows immediately.

Using the Leibniz rule, we get the corresponding statement for forests, and by linearity it follows for any element in $\mathcal{T}_n^p(R)$. □

4. Mapping forests to algebraic cycles

In the special case where $R = F^\times$, the multiplicative group of a field F, we can establish the connection between the two differential graded algebras above, and Theorem 4.2 below gives the first main result of this paper.

It turns out that the admissibility condition on algebraic cycles mentioned above forces us to restrict to a subalgebra of $\mathcal{T}_\bullet^*$, which we now describe.

Definition 4.1. *We call an R-deco tree* **generic** *if all the individual decorations of external vertices are different.*

We denote the subalgebra of $\mathcal{T}_\bullet^*(R)$ generated by generic R-deco trees by $\widetilde{\mathcal{T}}_\bullet^*(R)$.

One of our key results is the following statement:

Theorem 4.2. *For a field F, there is a natural map of differential graded algebras*

$$\widetilde{\mathcal{T}}_\bullet^*(F^\times) \to \mathcal{N}_{\mathrm{c}}^{2*-\bullet}(F, *)\,.$$

It is given by the map in Definition 4.3 below.

A proof of this statement is given at the end of this section.

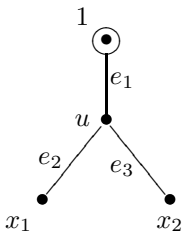
Definition 4.3. *The* **forest cycling map** *for a field F is the map Φ from $\mathcal{T}_\bullet^*(F^\times)$ to (not necessarily admissible) cubical algebraic cycles over F given on generators, i.e. $F^\times$-deco trees τ with orientation ω, as follows:*

(1) *to each internal vertex v of τ we associate a decoration consisting of an independent ("parametrizing") variable;*
(2) *to each oriented edge from a vertex v to another one w equipped with respective decorations y_v and y_w (variables or constants), we associate the expression $[1 - y_v/y_w]$ as a parametrized coordinate in $\mathbb{P}^1_F$;*
(3) *choosing an ordering of edges of τ corresponding to ω, we concatenate all the respective coordinates produced in the previous step.*

The last step in the definition is well-defined due to fact that we take the coinvariants of algebraic cycles with respect to permutation of coordinates: changing the orientation of τ amounts to performing an odd permutation of the coordinates of $\Phi(\tau)$, so in both settings we multiply by -1.

The above somewhat lengthy description of the map Φ is easily understood by looking at an example. We denote the concatenation product for algebraic cycles by $*$, and we encode the expression $1 - \frac{x}{y}$, for x, y in a field, by the following picture

Example 4.4: Let us consider the forest cycling map Φ for the following R-deco tree (τ,ω), where the orientation ω is given by $e_1\wedge e_2\wedge e_3$ (we leave out the arrows since the edges are understood to be directed away from the root):



$$\stackrel{\Phi}{\longmapsto}\ \Big[1-\frac{1}{u}\Big]*\Big[1-\frac{u}{x_1}\Big]*\Big[1-\frac{u}{x_2}\Big]=\Big[1-\frac{1}{u},1-\frac{u}{x_1},1-\frac{u}{x_2}\Big].$$

Figure 5: The forest cycling map on a tree with one internal vertex.

This cycle, as already mentioned, corresponds to the double logarithm $I_{1,1}(x_1,x_2)$, as we will see in Section 6.

Proof (of Theorem 4.2). The bigrading of a tree τ in $\widetilde{\mathcal{T}}_n^p(F^\times)$ translates to the respective bigrading of the cycle $\Phi(\tau)$: since τ has n edges, its image under Φ lands in $\square_F^n$. Furthermore, τ has $n+1$ vertices, $p+1$ of which are external, and so maps to a cycle with $n-p$ independent parameters, i.e., $\Phi(\tau)$ is of codimension p.

It remains to check the admissibility of $\Phi(\tau)$ and the compatibility of the differentials. We prove inductively the following two statements:

Claim 1: Each generic $F^\times$-deco tree in $\widetilde{\mathcal{T}}_n^p(F^\times)$ maps under Φ to an admissible cycle in $\mathcal{N}_{\rm c}^{2p-n}(F,p)$.

Claim 2: The differential on (not necessarily admissible) cycle groups, restricted to $\Phi\big(\widetilde{\mathcal{T}}_n^p(F^\times)\big)$, can be written as $\sum_{j=1}^n(-1)^{j-1}\partial_0^j$.

The case $n=1$: Concerning Claim 1, note that for $p=1$ the only obstruction for a cycle $[a]$ in $\mathcal{N}_{\rm c}^1(F,1)$ to be admissible is that the (constant) coordinate a is not allowed to be equal to 0 or to ∞. Translated back to the $F^\times$-deco trees, this condition means that $1-\frac{y_v}{y_w}\notin\{0,\infty\}$, where y_v and y_w are the elements in $F^\times$ decorating root and non-root vertex, respectively. The second possibility $(1-\frac{y_v}{y_w}=\infty)$ is excluded since $y_w=0$ is not allowed as a decoration, and the first possibility $(1-\frac{y_v}{y_w}=0)$ entails $y_v=y_w$ which is excluded by the genericity condition for $\widetilde{\mathcal{T}}_1^1(F^\times)$. For $p\neq 1$, there is nothing to show.

Concerning Claim 2, note that the differential for $n=1$ (and any p) on both sides turns out to be zero.

The case $n\geqslant 2$: using induction, we show both claims at level n, assuming the validity of both statements for $n-1$.

Concerning Claim 1, note that a tree τ in $\widetilde{\mathcal{T}}_n^p(F^\times)$ produces a cycle $\Phi(\tau)$ which satisfies the admissibility condition everywhere, except possibly at the zeros and poles of its coordinates. Therefore testing for admissibility amounts to testing each term in the differential for admissibility separately.

In case we contract an internal edge, we obtain a single tree in which n is reduced by 1, and which is still generic since the decoration does not change, so that induction applies.

In case we contract an external edge, there are at least two trees (or none) produced, and either tree lies in $\widetilde{\mathcal{T}}_\nu^p(F^\times)$ for some $\nu<n$. The admissibility condition

imposes that all decorations for each individual tree are mutually different. Since decorations are inherited by such a contraction, it is necessary and sufficient for τ to have itself mutually different decorations in order to make $\Phi(\tau)$ admissible: otherwise there would be some sequence of contractions of the tree producing an edge with two identical decorations (in $F^\times$); since this produces under Φ a coordinate equal to the constant 0, admissibility would be violated. This proves Claim 1.

Concerning Claim 2, note that each partial differential ∂^j_∞ for a typical coordinate $1-\frac{y_v}{y_w}$ in some cycle Z produces a non-empty cycle only if $y_v=\infty$ or $y_w=0$. If y_v and y_w are both constants, i.e., lie in $F^\times$, this is not possible. If, on the other hand, y_v or y_w are variables, i.e., correspond to an internal vertex each, they can attain those values in $\mathbb{P}^1_F$, but for each such instance the tree structure ensures that there is a further coordinate in Z of the form $1-\frac{x}{y_v}$ or $1-\frac{y_w}{x}$, respectively (with some x either constant or variable). This will cause the resulting cycle for this partial differential to vanish, since one coordinate becomes equal to 1. Thus any ∂^j_∞ produces only the empty cycle, which proves Claim 2.

Now each of the ∂^j_0 corresponds to the contraction $\big(\tau/e_j, i_{e_j}(\omega)\big)$ of the jth edge e_j, in the tree differential of the generator (τ,ω), where the ordering of edges corresponds to the ordering of coordinates induced via Φ from ω. This shows the compatibility of differentials. □

5. Tree sums with decomposable boundary and polygons

In this section, we single out a number of formal linear combinations over R-deco trees whose boundary consists of decomposable terms only. Each such combination can be encoded by an R-decorated polygon, which suggests to pull back the DGA structure on trees to polygons.

It is convenient to introduce a second differential $\overline{\partial}$ on polygons. Its tree realization coincides with the main one. We relate the Lie coalgebra corresponding to the differential $\overline{\partial}$ to the dihedral Lie coalgebra from [12].

5.1. **Tree sums with decomposable boundary.**

Definition 5.1. *Let $\{x_1,\dots,x_{m+1}\}$ be a collection of distinct elements of R. Then $\tau(x_1,\dots,x_{m+1})\in\mathcal{T}^m_{2m-1}(R)$ is the sum of all trivalent R-deco trees with m leaves whose R-decoration is given by $(x_1,x_2,\dots,x_{m+1})$, the last one decorating the root.*

Recall that the number of such trees is given by the Catalan number $\frac{1}{m}\binom{2(m-1)}{m-1}$.

Example 5.2: 1. For $m=2$, the tree $\tau(x_1,x_2,1)$ is given by the tree in Figure 5.

2. For $m=3$, the sum of trees has already more than a single term:

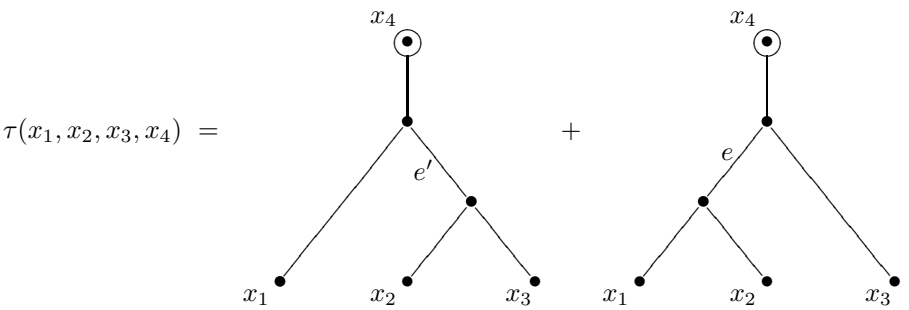


Figure 6: The sum of trees corresponding to the quadruple (x_1, x_2, x_3, x_4).

Applying Φ to the sum of trees $\tau(x_1, x_2, x_3, 1)$ with mutually different $x_i \in F^\times \setminus \{1\}$ for a field F, we get the following cycle:

(14)
$$Z_{x_1,x_2,x_3} = \Big[1-\frac{1}{t}, 1-\frac{t}{x_1}, 1-\frac{t}{u}, 1-\frac{u}{x_2}, 1-\frac{u}{x_3}\Big] + \Big[1-\frac{1}{t}, 1-\frac{t}{u}, 1-\frac{u}{x_1}, 1-\frac{u}{x_2}, 1-\frac{t}{x_3}\Big].$$

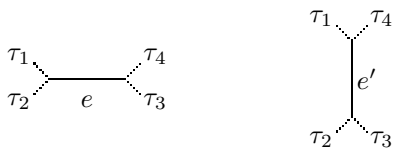
Here we have two parametrizing variables, t and u, and $Z_{x_1,x_2,x_3} \in \mathcal{N}^1_{\rm c}(F, 3)$.

Proposition 5.3. *Let $\{x_1, \ldots, x_m\}$ be a collection of distinct elements of $F^\times$. The image of the tree sum $\tau(x_1, \ldots, x_m)$ under the forest cycling map is an admissible cycle with decomposable boundary.*

Proof. The admissibility of the resulting cycle follows immediately from Lemma 4, since the decorations $x_1, \ldots, x_m$ are mutually different.

Concerning the statement about the boundary, note first that the contraction of an *external edge* in a trivalent tree produces a forest which, if non-empty, is not a tree, and thus maps to a decomposable cycle under Φ.

The contributions of *internal edges* in the tree sum cancel. Indeed, there are exactly two trivalent trees, τ and τ', containing the same subtrees τ_i, $i = 1, \ldots, 4$, but joined in a different way, via edges e and e', as shown in the picture:



We can write the canonical orientations $\omega(\tau)$ of τ and $\omega(\tau')$ of τ' in terms of the subtrees τ_i $(i = 1, \ldots, 4)$, where τ_1 contains the root of τ, as follows:

$$\omega_\tau = \omega_{\tau_1} \wedge \omega_{\tau_2} \wedge e \wedge \omega_{\tau_3} \wedge \omega_{\tau_4}, \qquad \omega_{\tau'} = \omega_{\tau_1} \wedge e \wedge \omega_{\tau_2} \wedge \omega_{\tau_3} \wedge \omega_{\tau_4}$$

where ω_{τ_i} denotes the canonical orientation for τ_i. The essential point is that each of the trees τ_i have an odd number of edges, and thus the corresponding orientations anticommute. Thus $\omega_{\tau'} = -\omega_\tau$. Therefore the terms in the differential corresponding to the contraction of e in τ and of e' in τ' produce the same tree with opposite orientation torsor and thus cancel. □

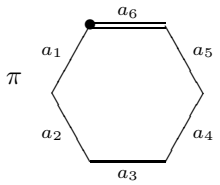
5.2. **The algebra of R-deco polygons.** A polygon is an **oriented** convex polygon with $N \geqslant 2$ sides. It inherits a cyclic order of the sides. Let R be a set.

Definition 5.4. *An R-deco polygon is a polygon whose sides are decorated by elements of R, with a distinguished side, called its* **root side**.

The orientation of the polygon induces a linear order of its sides such that the root side is the last one. The **first vertex** of the polygon is the common vertex of the root side and the first side. Once the root side is determined, the orientation determines the first vertex, and vice versa. So we indicate the orientation in pictures by marking the first vertex by a bullet.

We use the notation $[a_1, \ldots, a_N]$ for an R-deco N-gon with linearly ordered sides decorated by $a_1, \ldots, a_N$, where a_N is the root decoration.

Example 5.5: Here is an R-deco 6-gon $\pi = [a_1, \dots, a_6]$. The root side is drawn by a double line, the first vertex is marked by a bullet, and the orientation is counterclockwise.



The **weight** $\chi(\pi)$ of an R-deco polygon π is the number of its non-root sides.

We define the graded vector space $V_\bullet^{\mathrm{pg}}(R) = \bigoplus_{n=0}^{\infty} V_n^{\mathrm{pg}}(R)$, where $V_n^{\mathrm{pg}}(R)$, for $n \geqslant 1$, denotes the vector space of R-deco $(n+1)$-gons, and $V_0^{\mathrm{pg}}(R) := \mathbb{Q}$.

Definition 5.6. *The* **polygon algebra** $\mathcal{P}_\bullet^{(\bullet)} = \mathcal{P}_\bullet^{(\bullet)}(R)$ *is the exterior algebra of the graded vector space* $V_\bullet^{\mathrm{pg}}(R)$. *It is bigraded: the second grading, denoted via superscripts, comes from the exterior power.*

As we will often ignore the second grading, we mostly denote this algebra by $\mathcal{P}_\bullet$.

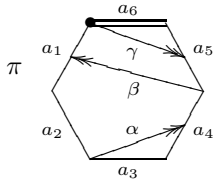
To define a differential on $\mathcal{P}_\bullet$, we introduce arrows and dissections:

Definition 5.7. *An* **arrow in an** R**-deco polygon** π *is a line segment beginning at a vertex and ending at the interior of a side of* π. *An arrow connecting a vertex with an incident side is trivial.*

An arrow in π *is a* **backward arrow** *if, in the linear order of the sides of* π, *its end is before its beginning. Otherwise, it is a* **forward arrow**.

We identify arrows with the same vertex ending at the same side.

Example 5.8: The following picture shows two forward arrows α and γ and a backward arrow β in the R-deco 6-gon of the previous example.

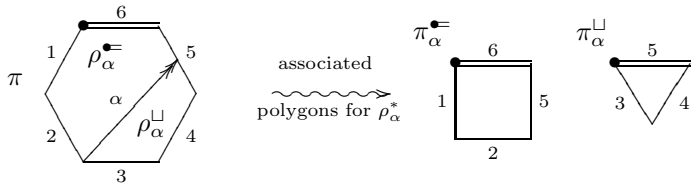


Any non-trivial arrow dissects a polygon into two regions.

Definition 5.9. *Let* π *be an* R*-deco polygon with* $\geqslant 3$ *sides, and* α *a non-trivial arrow in* π. *A* **dissection** *of* π *by* α *is the ordered pair* $(\rho_\alpha^{\bullet=}, \rho_\alpha^{\sqcup})$ *of a* **root region** $\rho_\alpha^{\bullet=}$ *and a* **cut-off region** $\rho_\alpha^{\sqcup}$, *where* $\rho_\alpha^{\bullet=} \cup \rho_\alpha^{\sqcup} = \pi$. *The root region is the one containing the first vertex and part of the root side.*

We assign to these regions R*-deco polygons* $\pi_\alpha^{\bullet=}$ *and* $\pi_\alpha^{\sqcup}$ *by contracting the arrow* α *in either region. The polygon* $\pi_\alpha^{\bullet=}$ *inherits the root and the orientation from* π. *In the polygon* $\pi_\alpha^{\sqcup}$, *the root side is the one on which* α *ends. Its orientation is inherited from the one of* π *if the dissecting arrow is forward, and is the opposite one otherwise.*

Example 5.10: The arrow α dissects the hexagon π in the picture into the regions $\rho_\alpha^{\bullet=}$ and $\rho_\alpha^{\sqcup}$. On the right hand side, we draw the R-deco polygons associated to the two regions.



Let $\chi(\alpha) := \chi(\pi_\alpha^{\sqcup})$ be the number of the non-root edges of the polygon $\pi_\alpha^{\sqcup}$.

Definition 5.11. *To an arrow α in an R-deco polygon π we assign its sign:*

$$\mathrm{sgn}(\alpha) = \begin{cases} (-1)^{\chi(\alpha)} & \text{if } \alpha \text{ is backward,} \\ 1 & \text{otherwise.} \end{cases}$$

The sum over all the dissections of π provides a differential $\partial\,\pi$:

Definition 5.12. *Given a non-trivial arrow α of an R-deco polygon π, set*

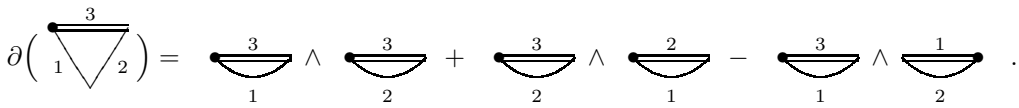
$$\partial_\alpha \pi = \pi_\alpha^{\bullet=} \wedge \pi_\alpha^{\sqcup} \in \mathcal{P}_\bullet^{(2)},$$

where $\pi_\alpha^{\bullet=}$ and $\pi_\alpha^{\sqcup}$ are as in Definition 5.9.

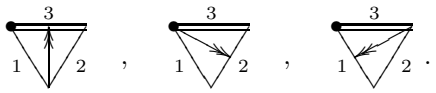
The **differential** *$\partial\,\pi$ of π is the sum $\sum_\alpha \mathrm{sgn}(\alpha)\partial_\alpha \pi$ over the arrows α in π.*

It extends to a **differential on** *$\mathcal{P}_\bullet^{(\bullet)}$ via the Leibniz rule $\partial(a \wedge b) = (\partial a) \wedge b + (-1)^{\deg(a)}\, a \wedge (\partial b)$. Here* deg *is the degree in the exterior algebra.*

Example 5.13: The differential applied to a triangle gives three terms:



The summands correspond respectively to the following three dissections:



The justification of the name "differential" is given by Proposition 5.15 below.

Another differential on R-deco polygons. Given an arrow α in an R-deco polygon, set

$$\varepsilon_\alpha = \begin{cases} -1 & \text{if } \alpha \text{ is backward,} \\ 1 & \text{otherwise.} \end{cases}$$

Definition 5.14. *In the set-up of Definition 5.9, a dissecting arrow α gives rise to two R-deco polygons $\overline{\pi}_\alpha^{\bullet=}$ and $\overline{\pi}_\alpha^{\sqcup}$, whose orientations are inherited from the one of π, and whose decorations are as in Definition 5.9.*

A differential $\overline{\partial}$ is defined as in Definition 5.12, with $\overline{\partial}\pi = \sum_\alpha \varepsilon_\alpha \overline{\pi}_\alpha^{\bullet=} \wedge \overline{\pi}_\alpha^{\sqcup}$.

Note that $\overline{\pi}_\alpha^{\bullet=} = \pi_\alpha^{\bullet=}$, and that $\overline{\pi}_\alpha^{\sqcup} = \pi_\alpha^{\sqcup}$ for a forward arrow α. If α is backward, then $\overline{\pi}_\alpha^{\sqcup}$ has the opposite orientation.

Proposition 5.15. *1. One has $\partial^2 = 0$. The polygon algebra $\mathcal{P}_\bullet^{(\bullet)}(R)$ is an augmented DGA with the differential ∂.*

2. One has $\overline{\partial}^2 = 0$*, providing the same polygon algebra with another augmented DGA structure, denoted* $\overline{\mathcal{P}}_\bullet^{(\bullet)}(R)$*.*

Remark 5.16: Proposition 5.15 is equivalent to the following statement: Each of the maps $\overline{\partial}$ and ∂ provides a Lie cobracket

$$V_\bullet^{\mathrm{pg}}(R) \to V_\bullet^{\mathrm{pg}}(R) \wedge V_\bullet^{\mathrm{pg}}(R) \tag{15}$$

So there are two graded Lie coalgebra structures on the graded vector space $V_\bullet^{\mathrm{pg}}(R)$.

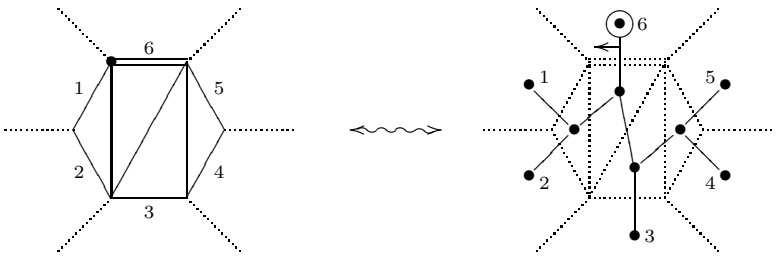
The graded-commutative DGA $\overline{\mathcal{P}}_\bullet^{(\bullet)}(R)$ (respectively $\mathcal{P}_\bullet^{(\bullet)}(R)$) is just the standard cochain complex of the Lie coalgebra $(V_\bullet^{\mathrm{pg}}(R), \overline{\partial})$ (respectively $(V_\bullet^{\mathrm{pg}}(R), \partial)$).

We will give a direct proof of Proposition 5.15 in Section 6.

On the other hand, there is another proof. Namely, we identify in Proposition 8.1 the map ∂ with the cobracket in a Lie coalgebra of formal iterated integrals defined in [15]. The latter Lie coalgebra was interpreted in loc. cit. as the dual to the Lie algebra of all derivations of a certain structure. So the property $\partial^2 = 0$ is valid on the nose. Similarly, we identify in Subsection 5.4 the map $\overline{\partial}$ with the cobracket in a rooted version of one of the dihedral Lie coalgebras defined in [12].

5.3. **Relating the polygon algebra to the tree algebra.** The standard duality between triangulated polygons and trivalent trees carries over to the respective R-deco objects (to each side we first associate a region outside of the polygon, and then each of its sides and diagonals corresponds to an edge of the tree, while each region—inside or outside—corresponds to a vertex):

Example 5.17: We give a picture of an $\mathbb{N}$-deco hexagon with triangulation, on the left (the dotted lines indicate the outside regions of the polygon), and of its dual $\mathbb{N}$-deco tree, on the right. Note the little arrow attached to the root edge which indicates the orientation (counterclockwise for the given example).



Proposition 5.18. *The standard duality induces a map from* $\mathcal{P}_\bullet(R)$ *to the subalgebra* $\mathcal{T}_{3-\mathrm{val}}(R)$ *of the tree algebra* $\mathcal{T}_\bullet(R)$ *which is generated by* trivalent *trees.*

Proof. Let ∇ be a triangulation of an R-deco polygon π with dual tree τ. For each internal vertex of τ, the orientation of π induces a cyclic ordering on the incident edges, and the root of π induces a root of τ. Putting both data together, τ inherits an orientation (in fact, even a linear ordering on its edges).

We associate to π the following sum, where ∇ runs through all triangulations of π and $\tau(\pi, \nabla)$ denotes the (oriented) R-deco tree dual to the triangulation ∇ of π,

$$\pi \mapsto \sum_{\nabla \text{ triangulation}} \tau(\pi, \nabla)\,. \tag{16}$$

By the duality above, we identify each term in the sum with the corresponding oriented R-deco tree. □

Theorem 5.19. *The map from (16) induces a map of differential graded algebras* $\mathcal{P}_\bullet(R) \to \mathcal{T}_\bullet(R)$ *as well as* $\overline{\mathcal{P}}_\bullet(R) \to \mathcal{T}_\bullet(R)$.

Proof. We start from the second claim, which is a bit more straightforward. Let π be an R-deco polygon with triangulation ∇, and let $\tau = \tau(\pi, \nabla)$ be the dual R-deco tree. Let us calculate the effect of contracting an edge e.

Internal edges: The proof of Proposition 5.3 shows that the terms in the differential for τ arising from internal edges cancel.

External edges: Let e be an external edge. Recall that the dual tree for a triangulation of an oriented polygon has a canonical orientation (that is, an element of the orientation torsor of the tree).

Remark 5.20: Changing the orientation of an N-gon π amounts to changing the corresponding orientation of the dual tree of any triangulation of π by $(-1)^N$. (Flip the two branches at each of the $N-2$ internal vertices.)

The edge e is dual to a side s_e of the polygon π. Let us cut out the triangle t_e of the triangulation containing the side s_e from the triangulation of the polygon π. We obtain two polygons, π_R and π_O, where π_R is the one containing the root side if s_e is not a root, and the first vertex in the latter case. Let ω_R and ω_O be the canonical orientations of the dual trees assigned to the induced triangulations of these polygons.

Let α be the dissecting arrow corresponding to the triangle t_e: it ends at the side s_e, and its vertex coincides with the one of t_e opposite to the side s_e. Then, if α is a forward (respectively a backward) arrow, the canonical orientation of the tree τ is $\omega_R \wedge \omega_O \wedge e = e \wedge \omega_R \wedge \omega_O$ (respectively $\omega_R \wedge e \wedge \omega_O = -e \wedge \omega_R \wedge \omega_O$). Here we used the fact that the dual tree always has an odd number of edges. In particular, the root edge corresponds to a forward arrow and produces the orientation $e \wedge \omega_R \wedge \omega_O$. In each case this matches the definition ε_α.

A proof of the first claim is deduced from this and Remark 5.20. Indeed, this is obvious for a forward dissecting arrow. For a backward dissecting arrow α we have a different orientation of the cut-off polygon, which thanks to Remark 5.20 amounts to the sign $(-1)^{\chi(\alpha)+1}$, and $\mathrm{sgn}(\alpha) = (-1)^{\chi(\alpha)}$. So the total sign difference will be -1, matching ε_α. □

An R-deco polygon is **generic** if its decorations are all different. Now we can connect the polygons to algebraic cycles.

Corollary 5.21. *There is a map of differential graded algebras from the generic part* $\widetilde{\mathcal{P}}_\bullet(F^\times)$ *of* $\mathcal{P}_\bullet(F^\times)$ *to the cubical algebraic cycles* $\mathcal{N}_c^{2*-\bullet}(*)$ *over* F.

More precisely, $\widetilde{\mathcal{P}}_{N+1}(F^\times)$ *gets mapped to* $\mathcal{N}_c^{2*-N}(*)$.

Proof. We just need to combine the above theorem with Theorem 4.2. □

5.4. **Appendix: Relation with the dihedral Lie coalgebra from** [12]**.** Let us identify the graded Lie coalgebra of R-deco polygons $(V_\bullet^{\mathrm{pg}}(R), \overline{\partial})$ with a rooted version of one of the Lie coalgebras introduced in [12] under the name of dihedral Lie coalgebras. Let G be a commutative group. Recall the bigraded dihedral Lie coalgebra $\widetilde{\mathcal{D}}_{\bullet,\bullet}(G) = \oplus_{w,m} \widetilde{\mathcal{D}}_{w,m}(G)$ defined in Section 4.5 of loc. cit. Consider the

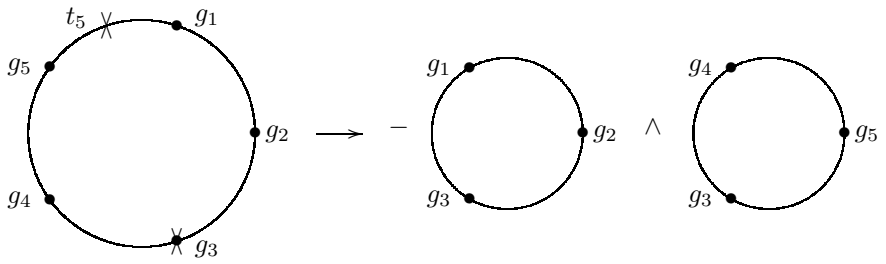
corresponding graded Lie sub-coalgebra $\widetilde{\mathcal{D}}_\bullet(G)$:

$$\widetilde{\mathcal{D}}_\bullet(G) := \bigoplus_{w>0} \widetilde{\mathcal{D}}_{w,w}(G), \tag{17}$$

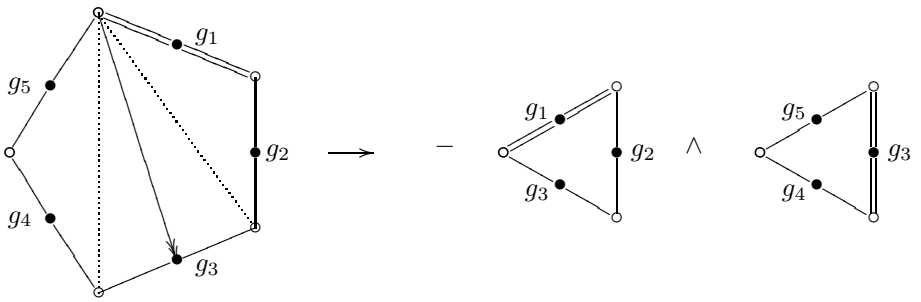
called the diagonal part of the dihedral Lie coalgebra $\widetilde{\mathcal{D}}_{\bullet,\bullet}(G)$. We relate it to $V^{\mathrm{pg}}_\bullet(R)$ as follows. Put $R = G$. Then the group G acts on $V^{\mathrm{pg}}_\bullet(G)$ by simultaneous multiplication on the G-decoration, so we can pass to the Lie coalgebra of G-coinvariants. Killing in the latter coalgebra, for each integer $n > 0$, the generator $(\underbrace{e : \cdots : e}_{n+1})$, where e is the unit in G, we get the Lie coalgebra (17).

The correspondence on pictures. In [12] a typical generator of (17) is given as a cyclically ordered set of n "black" points on the circle. The complement to these points is the union of n arcs. The cobracket is the sum of several terms, each of which is obtained via the following three step procedure:

(1) Cut the circle at a black point and at an arc not adjacent to that point.
(2) Complete each of the two resulting half-circles to a circle by gluing its ends.
(3) Take the wedge product of the above two circles, so that in the left factor the path from the black point cut to the arc cut follows the cyclic order.



The corresponding term in the differential on polygons looks as follows: we assign to the circle with black points a polygon whose vertices are centers of the arcs, and whose sides are labelled by the black points. Then the cutting of the circle corresponds to an arrow from the middle of an arc to the respective black point. The dissection induced by this arrow gives a term in the differential and corresponds precisely to the wedge product of the two smaller circles above.



We will see in Section 8 that the other graded Lie coalgebra $(V^{\mathrm{pg}}_\bullet(R), \partial)$ is related to the Lie coalgebra of motivic iterated integrals.

6. Cycles in the bar construction of the polygon algebra

6.1. **Higher dissections of polygons.** To deal with the bar construction, we introduce n-fold dissections.

Definition 6.1. *We say that two arrows in a polygon* **do not intersect** *if they have no point in common, except possibly the starting point.*

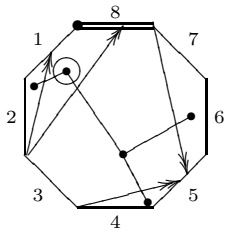
Example 6.2: Two intersecting arrows on the left, four non-intersecting arrows on the right.

Definition 6.3. *Let π be an R-deco polygon and n a positive integer. An* **n-fold dissection** *D* **of** *π is the set of n regions provided by $n-1$ non-intersecting arrows in π: the root region of D is defined in the same way as for 2-fold dissections.*

There is a partial order on the set of regions of a dissection, induced by the root region and by the adjacency of regions. A 1-fold dissection of π is equal to π itself.

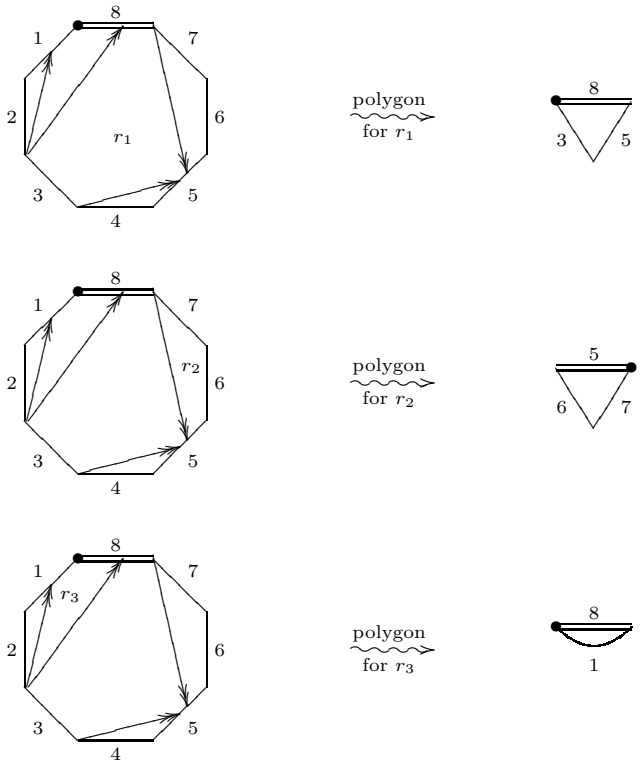
An n-fold dissection D of an R-deco polygon π gives rise to its **dual tree** $\tau(D)$ whose vertices correspond to the regions of the dissection, and whose $(n-1)$ edges are dual to the arrows of D. The root region of D determines the root of $\tau(D)$.

Example 6.4: Here is a polygon with a 5-fold dissection, and its dual tree.



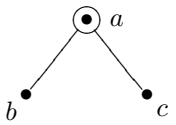
Each n-fold dissection D of π gives rise to n R-deco polygons: the root region inherits the root of π, and for any other region, there exists a unique arrow in its boundary which is the closest one to the root of π. Contracting arrows in the boundary of a region, we get an R-deco polygon. Its root side is the one where the arrow specified above ends.

Example 6.5: We illustrate the above on an $\mathbb{N}$-deco octagon π, where the sides are decorated according to their place in the linear ordering of π. The regions r_1, r_2 and r_3 below get equipped with the root sides 8, 5 and 8, respectively. To each of the first two regions we associate an R-deco 3-gon, decorated by $[3,5,8]$ and $[7,6,5]$, respectively, while to region r_3 there is the 2-gon $[1,8]$ associated. Finally, there are two more regions r_4 and r_5 leading to the 2-gons $[2,1]$ and $[4,5]$, respectively. We illustrate the above by giving the polygons associated to r_i, $i=1,2,3$.



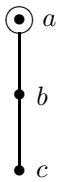
6.2. **Proof of Proposition 5.15.** We first prove the second claim. A typical term in $\overline{\partial}^2\pi$ for an R-deco polygon π arises from two non-intersecting arrows α and β. There are two cases to consider, depending on whether the dual tree τ of the 3-fold dissection associated to (α, β) is linear or not.

In case τ is non-linear, it has the form



for some subpolygons a, b and c, where a is the root polygon. In $\overline{\partial}^2\pi$ both terms $(a \wedge b) \wedge c$ and $(a \wedge c) \wedge b$ occur with the same sign, regardless of the type of the arrows α and β. So they cancel in the wedge product.

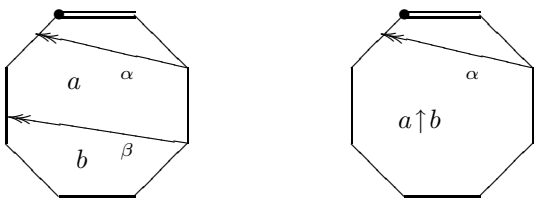
If τ is linear, it has the form



Let us first exclude the case when both arrows α and β are backward, and end at the same side. Then, by the Leibniz rule, cutting first the upper edge and then the lower one we obtain the same term $a \wedge b \wedge c$ with the opposite sign compared to

cutting the lower edge first. This does not depend on the directions of the arrows. In the excluded case we get $a \wedge c \wedge b + a \wedge b \wedge c = 0$. Finally, the augmentation is defined by taking the zero$^{\text{th}}$ component. This proves the second claim.

Let us prove the first claim. Given an arrow α in an R-deco polygon π, we use the notation $\varepsilon_{\alpha,\pi} := \varepsilon_\alpha$ to emphasize π. Let α and β be non-intersecting arrows in an R-deco polygon π, and α is closer to the root than β. From now on, given a collection D of non-intersecting arrows in π including an arrow α, we denote by $\pi^{\sqcup}_\alpha$ the unique polygon sitting right under the arrow α, i.e., the one corresponding to the bottom vertex of the edge of the tree dual to α. Denote by a and b the corresponding cut-off polygons, $a = \pi^{\sqcup}_\alpha$, $b = \pi^{\sqcup}_\beta$, and let $a \uparrow b$ be the cut-off polygon for the single dissecting arrow α in π.



In the above notation, one has the following obvious lemma.

Lemma 6.6. *(1).* $\chi(a) + \chi(b) = \chi(a \uparrow b)$.
(2). $\varepsilon_{\alpha,\pi}\varepsilon_{\beta,\pi} = \varepsilon_{\beta,a\uparrow b}$.

Note that, if α and β are backward in π, then β is not necessarily forward in $\overline{\pi}^{\sqcup}_\alpha$. Thus the straightforward analog of (2) may not be correct for $\overline{\pi}^{\sqcup}_\alpha$ polygons.

We compare the same two triple wedge products as above. The corresponding R-deco polygons match each other, so we have to worry only about the extra signs which we pick up using ∂ instead of $\overline{\partial}$. After these remarks, the proof is identical to the one above in the first case (i.e., for τ non-linear), and in the second case (i.e., if τ is linear) if the top arrow α is forward. Let us prove it in the second case, when α is backward.

Assume that β is forward. Then cutting out α first and β second we pick up an extra sign $(-1)^{\chi(\alpha)+\chi(\beta)}$: indeed, by Lemma 6.6(2) after the first cut β becomes a backward arrow in $a \uparrow b$. By Lemma 6.6(1) the weight of the middle polygon is $\chi(\alpha) - \chi(\beta)$, so cutting in the other order we pick up the same sign.

Now assume β is backward. Then cutting out α and then β we pick up an extra sign $(-1)^{\chi(\alpha)}$: indeed, after the first cut β becomes a forward arrow in $a \uparrow b$. Cutting in the other order we pick up the sign $(-1)^{\chi(\beta)}(-1)^{\chi(\alpha)-\chi(\beta)} = (-1)^{\chi(\alpha)}$. □

6.3. **The sign of a dissection.** The last ingredient needed for description of the cocycles in the bar construction of polygons is the sign of a dissection. Given a dissection D of an R-deco polygon π, and a dissecting arrow α, there are two regions defined by the dissection D containing α. Take the one which is further away from the root on the dual tree, and denote by $\pi^{\sqcup}_{D,\alpha}$ the R-deco polygon obtained by shrinking its arrow sides. Recall the weight $\chi(\pi)$ of an R-deco polygon π.

Definition 6.7. *Let D be a dissection of an R-deco polygon π. The* **sign** *of D is*

$$\mathrm{sgn}(D) := \prod_{\text{backward arrows } \alpha \text{ of } D} (-1)^{\chi(\pi^{\sqcup}_{D,\alpha})} = \prod_{\text{arrows } \alpha \text{ of } D} \varepsilon_\alpha^{\chi(\pi^{\sqcup}_{D,\alpha})}\,.$$

Example 6.8: The dissection in Example 6.5 has two backward arrows: the arrow ending in 1 cuts off a polygon of weight 1 and thus contributes a minus sign, while the arrow starting from between 7 and 8 contributes a plus sign: the cut-off polygon has weight 2. Thus the sign of the dissection is equal to -1.

6.4. **The cocycle attached to a polygon in the bar construction.** We define the **Adams grading** of an R-deco N-gon as $2(N-1)$. Since it is even, the shuffle product in the bar construction below is commutative.

Recall that the **bar construction** $\mathrm{B}(\mathcal{A})$ associated to an augmented DGA $\mathcal{A}_\bullet = \oplus_{m\geqslant 1}\mathcal{A}_m$, with graded-symmetric multiplication $\wedge$ and differential ∂ of degree $+1$, is the tensor coalgebra $\oplus_i \mathcal{A}_\bullet^{\otimes i}$ with differential $D_1 + D_2$, where for homogeneous elements $a_1, \ldots, a_N$ in $\mathcal{A}_\bullet$ one defines, denoting any tensor sign by a $\big|$,

$$D_1\big([a_1|a_2|\cdots|a_N]\big) = \sum_{j=1}^{N-1}(-1)^{\sum_{i\leq j}(\deg(a_i)-1)}[a_1|a_2|\cdots|a_j\wedge a_{j+1}|\cdots|a_N]\,,$$

$$D_2\big([a_1\,\big|\,\ldots\,\big|\,a_N]\big) = \sum_{j=1}^{N}(-1)^{\sum_{i<j}(\deg(a_i)-1)}[a_1|a_2|\cdots|\partial(a_j)|\cdots|a_N]\,.$$

Furthermore, $\mathrm{B}(\mathcal{A}_\bullet)$ carries an algebra structure, given by the graded-commutative shuffle product Ш.

Considering $\mathrm{B}(\mathcal{A}_\bullet)$ as a double complex with respect to D_1 and D_2, we are particularly interested in the cohomology of the "main diagonal", i.e., in $\mathrm{H}^0\mathrm{B}(\mathcal{A}_\bullet)$, where the terms live which have only bars and no wedges.

The bar construction $\mathrm{B}(\mathcal{P}_\bullet)$ for the polygon algebra $\mathcal{P}_\bullet = \mathcal{P}_\bullet(R)$ is a commutative Hopf algebra. The unit in $\mathrm{B}(\mathcal{P}_\bullet)$ is given by $\mathbb{Q} \to \mathbb{Q}\cdot\mathbb{1} \hookrightarrow \mathrm{B}(\mathcal{P}_\bullet)$, the counit is the augmentation, and the antipode is induced by the remaining structures. By the Milnor-Moore theorem, there exists a Lie algebra whose universal enveloping algebra is dual to $\mathrm{B}(\mathcal{P}_\bullet)$.

Definition 6.9. *Let π be an R-deco polygon. We associate to it an element $\mathrm{B}(\pi)$ in the bar construction $\mathrm{B}(\mathcal{P}_\bullet)$. Its component in $\mathcal{P}_\bullet^{|n}$ is the sum*

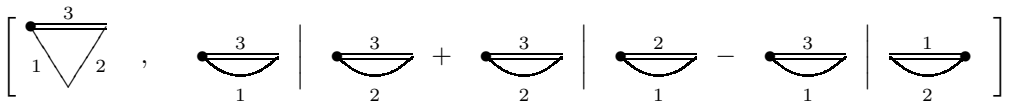
$$\sum_D \mathrm{sgn}(D)\sum_\lambda [\,\pi_D^{\lambda(1)}\,\big|\,\ldots\,\big|\,\pi_D^{\lambda(n)}\,]$$

over all n-fold dissections D of π, where the inner sum runs through all linear orders λ of the associated subpolygons π_D^i compatible with the partial order on $\tau(D)$.

Example 6.10: The case $n=2$: the cocycle $\mathrm{B}([\,1\,2\,3\,])$ associated to a triangle

$[\,1\,2\,3\,] =$

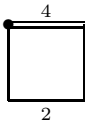
is given as



Note that the terms of the second component of $\mathrm{B}([\,1\,2\,3\,])$ cancel pairwise under D_1 with the terms of the differential on polygons $\partial\big([\,1\,2\,3\,]\big)$.

Example 6.11: The case $n = 3$: the cocycle associated to a quadrangle

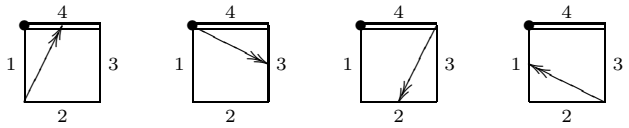
$$[\,1\,2\,3\,4\,] \;=\;$$



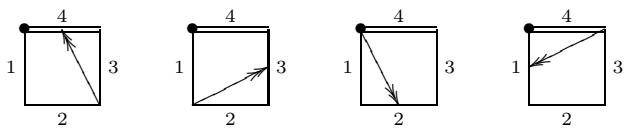
has the following component in $\mathcal{P}_\bullet \,|\, \mathcal{P}_\bullet$ (we omit [and])

$$\begin{aligned} &14\,|\,234 \;+\; 34\,|\,123 \;-\; 124\,|\,32 \;-\; 134\,|\,21 \\ &+\; 124\,|\,34 \;+\; 134\,|\,23 \;+\; 234\,|\,12 \;+\; 14\,|\,321\,, \end{aligned}$$

corresponding to the following respective 2-fold dissections:



$$14\,|\,234 \;+\; 34\,|\,123 \;-\; 124\,|\,32 \;-\; 134\,|\,21$$

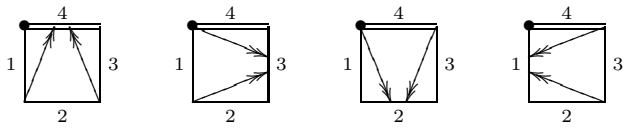


$$+\,124\,|\,34 \;+\; 134\,|\,23 \;+\; 234\,|\,12 \;+\; 14\,|\,321$$

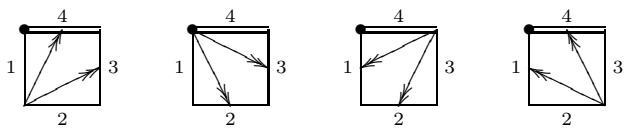
The component of the cocycle in $\mathcal{P}_\bullet \,|\, \mathcal{P}_\bullet \,|\, \mathcal{P}_\bullet$ (here we use, e.g., the shorthand $12\,\text{Ш}\,34$ for $12\,|\,34 + 34\,|\,12$, the shuffle product of $12\,|\,34$ and $34\,|\,12$) is

$$\begin{aligned} &14\,|\,24\,|\,34 \;+\; 34\,|\,13\,|\,23 \;-\; 24\,|\,12\,\text{Ш}\,32 \;+\; 14\,|\,31\,|\,21 \\ &+\; 14\,|\,34\,|\,23 \;+\; 34\,|\,23\,|\,12 \;+\; 14\,|\,21\,|\,32 \;-\; 14\,|\,21\,\text{Ш}\,34 \\ &-\; 14\,|\,24\,|\,32 \;-\; 34\,|\,13\,|\,21 \;+\; 24\,|\,12\,\text{Ш}\,34 \;-\; 14\,|\,31\,|\,23\,. \end{aligned}$$

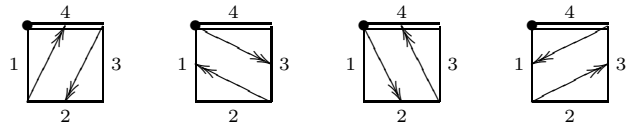
On pictures, the above twelve summands look as follows:



$$14\,|\,24\,|\,34 \;+\; 34\,|\,13\,|\,23 \;-\; 24\,|\,12\,\text{Ш}\,32 \;+\; 14\,|\,31\,|\,21$$



$$+\,14\,|\,34\,|\,23\;+\;34\,|\,23\,|\,12\;+\;14\,|\,21\,|\,32\;-\;14\,|\,21\,Ш\,34$$



$$-\,14\,|\,24\,|\,32\;-\;34\,|\,13\,|\,21\;+\;24\,|\,12\,Ш\,34\;-\;14\,|\,31\,|\,23$$

Again, all the terms in the differential cancel because they appear twice with different signs; we give two examples:

(1) $\partial(14\,|\,234)$ gives a term $-\,14\,|\,24\wedge 34$ (the sign comes from the fact that we take the differential in the second factor) from taking the differential in $\mathcal{P}_\bullet$, while $14\,|\,24\,|\,34$ gives a term $14\,|\,24\wedge 34$ (the sign is $+$ since we replace the second $|$ by a $\wedge$).

(2) $\partial(134\,|\,23)$ gives a term $+\,14\wedge 34\,|\,23$ (no sign since we take the differential in the first factor), while $14\,|\,34\,|\,23$ gives $-\,14\wedge 34\,|\,23$ (the sign is $-$ since we replace the first $|$ by a $\wedge$).

We need the multiplicativity of the sign of a dissection, formulated as follows. Let $D'\supset D$ be an overdissection of a dissection D. So for each arrow α of D there is the cut-off polygon $\pi_\alpha^{\sqcup}$. We include here a "root arrow" providing the root polygon of the dissection as a cut-off polygon. Then there is a dissection D_α of the polygon $\pi_\alpha^{\sqcup}$ induced by D'.

Lemma 6.12. *One has*

$$\mathrm{sgn}(D')=\mathrm{sgn}(D)\prod_{\alpha}\mathrm{sgn}(D_\alpha,\pi_\alpha^{\sqcup}) \tag{18}$$

where the product is over all arrows of D, including the root arrow.

Further, let α be the unique predecessor of β in the dual tree of D. Then

$$\mathrm{sgn}(D-\{\beta\})=\mathrm{sgn}(D)(\varepsilon_\alpha\varepsilon_\beta)^{\chi(\pi_\beta^{\sqcup})}, \tag{19}$$

Proof. We first prove the second claim. Denote by a and b the subpolygons of D whose root arrows are α and β, respectively. Then one has

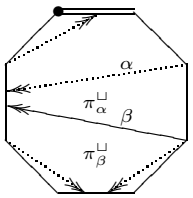
$$\mathrm{sgn}(D)=\frac{\mathrm{sgn}(D-\{\beta\})}{\varepsilon_\alpha^{\chi(a\uparrow b)}}\varepsilon_\alpha^{\chi(a)}\varepsilon_\beta^{\chi(b)}\,.$$

So we have to check that

$$\frac{\varepsilon_\alpha^{\chi(a)}\varepsilon_\beta^{\chi(b)}}{\varepsilon_\alpha^{\chi(a\uparrow b)}}=(\varepsilon_\alpha\varepsilon_\beta)^{\chi(b)}$$

which follows from Lemma 6.6(1). This proves the second claim.

We pass to the first claim. Given a dissection D', we proceed by the induction on the number of arrows in D. If D has just one arrow, it is the root arrow, and it is a forward arrow. So $\mathrm{sgn}(D)=1$, and (18) is just the definition of $\mathrm{sgn}(D')$.



Let us declare an arrow β of $D' - D$ as a new arrow of the dissection D. Denote by α the dissecting arrow of D just above β, i.e., preceding it in the dual tree. Then $\mathrm{sgn}(D')$ stays the same. By the second claim, $\mathrm{sgn}(D)$ is multiplied by

$$\varepsilon_\beta^{\chi(\pi_\beta^{\sqcup})}\varepsilon_\alpha^{\chi(\pi_\beta^{\sqcup})}.$$

The $\mathrm{sgn}(D_\alpha, \pi_\alpha^{\sqcup})$ is multiplied by $\prod_\gamma(\varepsilon_\alpha\varepsilon_\gamma)^{\chi(\varphi_\gamma^{\sqcup})}$, where γ runs through all arrows in $\pi_\alpha^{\sqcup}$, and $\varphi_\gamma^{\sqcup}$ denotes the polygon cut-off by the arrow γ in $\pi_\beta^{\sqcup}$. The $\mathrm{sgn}(D_\beta, \pi_\beta^{\sqcup})$ is multiplied by $\prod_\gamma(\varepsilon_\beta\varepsilon_\gamma)^{\chi(\varphi_\gamma^{\sqcup})}$. Multiplying these two products, we see that the total change of $\prod_\alpha \mathrm{sgn}(D_\alpha, \pi_\alpha^{\sqcup})$ in (18) is

$$\prod_\gamma(\varepsilon_\alpha\varepsilon_\beta)^{\chi(\varphi_\gamma^{\sqcup})} = (\varepsilon_\alpha\varepsilon_\beta)^{\sum_\gamma \chi(\varphi_\gamma^{\sqcup})} = (\varepsilon_\alpha\varepsilon_\beta)^{\chi(\pi_\beta^{\sqcup})}.$$

Here in the last step we used Lemma 6.6(1). The lemma is proved. □

Proposition 6.13. *The element* $\mathrm{B}(\pi)$ *associated to the R-deco polygon* π *is a 0-cocycle in* $\mathrm{B}(\mathcal{P}_\bullet)$.

Proof. We check that the terms in the differential $(D_1 + D_2)\mathrm{B}(\pi)$ cancel pairwise.

A typical term of $\mathrm{B}(\pi)$ is of the form $\mathrm{sgn}(D)\,[\,a_1\,|\,\ldots\,|\,a_n\,]$ $(n \geqslant 1)$ where the order $(a_1, \ldots, a_n)$ on the regions a_n of an n-fold dissection D of π is compatible with the partial order $\prec$ on the dissection.

Therefore, a typical term of $D_1\big(\mathrm{B}(\pi)\big)$ is of the form ($\deg(a_i)$ is even)

$$(-1)^j\,\mathrm{sgn}(D)\;[\,a_1\,|\,\ldots\,|\,a_j \wedge a_{j+1}\,|\,\ldots\,|\,a_n\,] \tag{20}$$

with the same requirements on the a_i (i.e., the compatibility of orders). Since the differential ∂ respects the partial order of a dissection, the terms of $D_2\big(\mathrm{B}(\pi)\big)$ are also ordered compatibly with $\prec$.

We distinguish two cases:

Case 1: The regions a_j and a_{j+1} are comparable with respect to $\prec$. Then the same term arises, up to sign, from $D_2\big(\mathrm{B}(\pi)\big)$: we must have $a_j \prec a_{j+1}$, and they have to be neighbouring regions (any γ with $a_j \prec \gamma \prec a_{j+1}$ would have to occur between a_j and a_{j+1} in any compatible linear order which would exclude a term with component $a_j \wedge a_{j+1}$).

Let β be the arrow shared by a_j and a_{j+1}, and let α be its predecessor in the dual tree, which is also the root arrow of a_j. Set $D' = D - \{\alpha_{j+1}\}$. The term $\mathrm{sgn}(D')\,[\,a_1\,|\,\ldots\,|\,a_n\,]$ of $\mathrm{B}(\pi)$ gives, via D_2, a term

$$(-1)^{j-1}\mathrm{sgn}(D')(\varepsilon_\alpha\varepsilon_\beta)^{\chi(a_{j+1})}\,[\,a_1\,|\,\ldots\,|\,a_j \wedge a_{j+1}\,|\,\ldots\,|\,a_n\,]. \tag{21}$$

Equation (19) of the above lemma shows that

$$\mathrm{sgn}(D) = \mathrm{sgn}(D')(\varepsilon_\alpha\varepsilon_\beta)^{\chi(a_{j+1})}. \tag{22}$$

Therefore the terms (21) and (20) cancel.

Note that all terms in $D_2\big(\mathrm{B}(\pi)\big)$ are thus taken care of.

Case 2: Suppose the regions a_j and a_{j+1} are *not* comparable with respect to $\prec$. Then there is a summand $\operatorname{sgn}(D)\,[\,a_1\,|\,\dots\,|\,a_{j+1}\,|\,a_j\,|\,\dots\,|\,a_n\,]$ whose D_1-image gives a term (20) with opposite sign, due to antisymmetry. This proves Case 2.

In both cases the terms in $(D_1+D_2)\big(\mathrm{B}(\pi)\big)$ cancel pairwise, so $\mathrm{B}(\pi)$ is a cocycle. □

7. Recognizing the coproduct for $\mathrm{B}(\pi,\partial)$

In this section, we show that the algebra generated by the $\mathrm{B}(\pi)$ for R-deco polygons π forms a sub-Hopf algebra of $\mathrm{B}(\mathcal{P}_\bullet)$. Let us formulate the main result.

Recall ([6]) that an **admissible cut** in a rooted tree τ is given by a collection C of edges of τ such that each simple path from the root to a leaf vertex contains at most one element of C. We use the same notion for the rooted *plane* trees. We call a dissection D **admissible** if the set of those edges of $\tau(D)$ corresponding to the arrows in D form an admissible cut of $\tau(D)$.

An admissible dissection D gives rise to a region R^D containing the root of π, and the remaining regions P_i^D. There are two particular cases: the empty cut of $\tau(D)$ corresponding to $R^D=\emptyset$, and the "full cut" corresponding to $R^D=\pi$.

Theorem 7.1. *The coproduct $\Delta\big(\mathrm{B}(\pi)\big)$ for a polygon π has the following expression*

$$\Delta\big(\mathrm{B}(\pi)\big)=\sum_{D\ \mathrm{adm}}\operatorname{sgn}(D)\ \mathrm{B}(R^D)\otimes\coprod_i\mathrm{B}\big(P_i^D\big)\,, \tag{23}$$

where D runs through all admissible dissections in π.

Thus the subalgebra generated by the $\mathrm{B}(\pi)$ is a Hopf subalgebra of $\mathrm{B}(\mathcal{P}_\bullet)$.

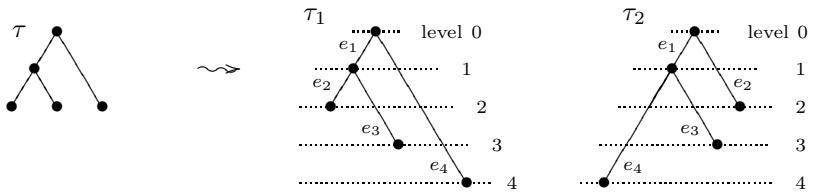
Here is a rough idea of the proof. The terms of $\mathrm{B}(\pi)$ for an R-deco polygon π are parametrized by dissections, together with a linear ordering on the ensuing smaller polygons. The terms in the coproduct arise from splitting each such linearly ordered set into two connected parts, the left and the right. The left one—if non-empty—contains the root region of the underlying dissection.

We regroup those terms in the form $\sum_i\pm\mathrm{B}(\pi_i^1)\otimes\coprod_{j>1}\mathrm{B}(\pi_i^j)$ for certain polygons π_i^j, where on the right we have a (commutative) shuffle product.

7.1. Level structure on plane trees.

Recall that a dissection of an R-deco polygon π provides a rooted *plane* tree whose vertices are labeled by the regions.

Let $\mathrm{B}(\pi)$ be the element in the bar construction of $\mathrm{B}(\mathcal{P}_\bullet)$, let D be a dissection of π and $\tau=\tau(D)$ the associated rooted plane tree. For the terms in $\mathrm{B}(\pi)$ we need to control the linear orders compatible with this tree. It is convenient to replace the notion of "linear orders compatible with the partial order on τ" by the equivalent one of a "tree level structure" (this notion was used by Loday in [17]): each compatible linear order corresponds to a (metrically distorted) version of the tree where each vertex lies on a different level. A linear order on the edges of τ arises from a level structure by identifying the vertex at level i with the unique edge ending at that vertex. For example, we give a simple tree τ together with two of its eight different level structures τ_i:



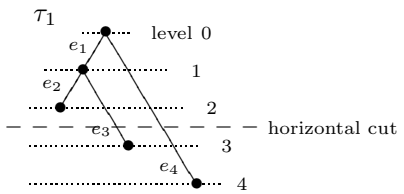
A tree τ, and two of its eight level trees, denoted τ_1 and τ_2.

7.2. **Regrouping the terms in the coproduct $\Delta\big(\mathrm{B}(\pi)\big)$.** Let π be an R-deco polygon. Its associated element $\mathrm{B}(\pi)$ is given in terms of a double sum $\sum_{D \text{ diss.}} \sum_{\lambda \text{ lin.order}}$, where the first sum runs through all dissections D of π while the second sum runs through all linear orders λ compatible with the induced partial order on (the regions π_i of) D. The coproduct of $\mathrm{B}(\pi)$ is given by a triple sum

$$\sum_{D \text{ diss.}} \sum_{\lambda \text{ lin.order}} \sum_{\text{splitting of } \lambda} , \tag{24}$$

where the first two sums are as above and the third sum runs through all the $|D|+1$ splittings of the sequence of polygons $\pi = (\pi_1, \ldots, \pi_{|D|})$ into a "left" part $(\pi_1, \ldots, \pi_n)$ and "right" part $(\pi_{n+1}, \ldots, \pi_{|D|})$. The empty parts are treated as the unit $\mathbb{1} \in \mathrm{B}(\mathcal{P}_\bullet)$.

Now replace each compatible linear order λ by a level structure of $\tau(D)$. The levels are ordered from top to bottom (as in the pictures above). Then a splitting of λ corresponds to a horizontal cut (up to isotopy) of $\tau(D)$ avoiding the vertices.



A horizontal cut of the level tree τ_1 with edge sequence (e_3, e_4).

The "left" part of the splitting of λ corresponds to the upper (=root) part, while the "right" part corresponds to the lower part of the horizontal cut. Therefore the above triple sum (24) is rewritten as

$$\sum_{D \text{ diss.}} \sum_{\text{level tree } \tau' \text{ for } \tau(\mathrm{D})} \sum_{\text{horizontal cut of } \tau'} . \tag{25}$$

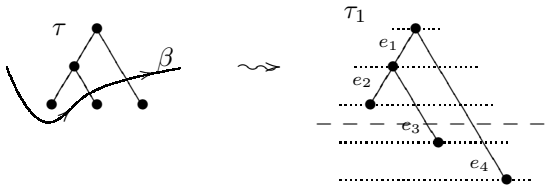
Each horizontal cut is described by the sequence $E(\lambda)$ of edges in $\tau(D)$ which it meets, together with the induced level structure on the remaining parts of $\tau(D) - E(\lambda)$. The set $E(\lambda)$ gives rise to an admissible dissection of D.

Horizontal cuts and admissible dissections are related as follows:

(i) Each horizontal cut of a level tree for τ obviously produces an admissible cut for τ by forgetting the level structure.

(ii) Each admissible dissection $\overline{D}$ of τ can be connected by a path β from "left of tree" to "right of tree" intersecting the edges in τ precisely once and avoiding the remaining parts of the tree. The pair (τ, β) can be "straightened out", not

necessarily uniquely, via a plane isotopy such that β becomes a horizontal cut and τ obtains a level structure.



An admissible cut of τ and an associated horizontal cut of a level tree τ_1 for τ.

Using these remarks, we can rewrite (25) as follows:

$$\sum_{D \text{ diss.}} \quad \sum_{\text{level tree } \tau' \text{ for } \tau(\mathrm{D})} \quad \sum_{\text{adm diss } \overline{D} \subset D} .$$

We can reverse the order of the sum in this triple sum, first summing over the *admissible* dissections $\overline{D}$, then summing over all dissections D of π *containing* $\overline{D}$, and finally summing over all level trees for $\tau(D)$, i.e.

$$\sum_{\overline{D} \text{ adm. diss.}} \quad \sum_{\text{diss } D \supset \overline{D}} \quad \sum_{\text{level tree for } \tau(D)} .$$

Recall that an admissible dissection D separates a "root piece" R^D from the remaining regions P_i^D. Each overdissection $D' \supset D$ gives both a dissection of the root piece $R^D_{D'-D}$ and a dissection of the remaining regions $P^D_{i,D'-D}$. The separated parts can be resummed independently: more precisely, a plane tree τ with horizontal cut C cuts τ into a root piece τ_R and a forest $\phi = \sqcup_i \tau_{P_i}$ of remaining pieces, which provides a bijection between

(1) the level structures of the pair (τ, C) and
(2) the product $\{\text{level structures on } \tau_R\} \times \{\text{level structures on } \phi\}$.

Decomposing $D' - D = D_1 \sqcup D_2$ into the dissections D_1 and D_2 of the root piece and of the remaining pieces, respectively, and using Lemma 6.12, which implies the full multiplicativity for an *admissible* dissection, we get

$$\begin{aligned}
\Delta\big(\mathrm{B}(\pi)\big) &= \operatorname{sgn}(D') \sum_{D \text{ adm}} \sum_{D' \supset D} \sum_{\text{level str. on } \tau(D')} R^D_{D'-D} \otimes \coprod_i P^D_{i,D'-D} \\
&= \sum_{D \text{ adm}} \operatorname{sgn}(D) \left(\sum_{D_1} \operatorname{sgn}(D_1) \sum_{\text{level str. for } \tau(D_1)} R^D_{D_1} \right) \otimes \\
&\qquad \otimes \left(\sum_{D_2} \sum_{\text{level str. for } \tau(D_2)} \coprod_i \operatorname{sgn}(D_{2,i}) P^D_{i,D_2} \right) \\
&= \sum_{D \text{ adm}} \operatorname{sgn}(D)\ \mathrm{B}(R^D) \otimes \coprod_i \mathrm{B}\big(P_i^D\big) .
\end{aligned}$$

We need the shuffle product $\coprod$ since precisely those linear orders on the remaining regions which are induced from such a shuffle do occur. Theorem 7.1 is proved. □

The vector space V^{pg} maps to $\mathrm{H}^0\mathrm{B}(\mathcal{P})$ via $\pi \to \mathrm{B}(\pi)$ for a polygon π. Theorem 7.1 implies that this map extends to a map of the symmetric algebra $\mathrm{S}^\bullet V^{\mathrm{pg}}$ of V^{pg} to $\mathrm{H}^0\mathrm{B}(\mathcal{P})$ sending the product in $\mathrm{S}^\bullet V^{\mathrm{pg}}$ to the shuffle product in $\mathrm{H}^0\mathrm{B}(\mathcal{P})$.

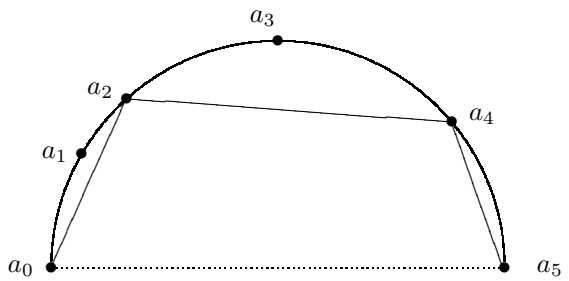
We can view $\mathrm{S}^\bullet V^{\mathrm{pg}}$ as a Hopf algebra via pullback of the coproduct formula (23). So there is a Hopf algebra map $\mathrm{S}^\bullet V^{\mathrm{pg}} \to \chi_{\mathrm{cycle}} = \mathrm{H}^0\mathrm{B}(\mathcal{N})$, defined via $\mathrm{H}^0\mathrm{B}(\mathcal{P})$.

8. Comparing the coproduct for $\mathrm{B}(\pi)$ and the coproduct for the Hopf algebra of iterated integrals from [15]

8.1. **The Hopf algebra $\mathcal{I}_\bullet(R)$ of iterated integrals.** Let R be a set. In [15], the second author described a commutative, graded Hopf algebra $\mathcal{I}_\bullet(R)$. It is generated, as a graded commutative algebra, by elements $\mathrm{I}(a_0; a_1, \dots, a_n; a_{n+1})$ of degree n, $a_i \in R$. Their properties resemble the ones of the iterated integrals.

The coproduct is given via polygons inscribed in a semicircle. For the above generator it is written as a sum indexed by all—possibly empty—subsequences of $(a_1, \dots, a_n)$. A picture describing the combinatorics of the terms arises when we place the a_i on a semicircle, in the order dictated by their index. In particular, a_0 and a_{n+1} are located at the endpoints of the semicircle.

Precisely, each subsequence $(a_{i_1}, \dots, a_{i_r})$ of $(a_1, \dots, a_n)$ gives rise to a single "main" polygon with vertex set $\{a_0, a_{i_1}, \dots, a_{i_r}, a_{n+1}\}$ and to a set of $r+1$ remaining polygons with $\geqslant 2$ sides and set of vertices $\{a_{i_0}, a_{i_0+1}, \dots, a_{i_1}\}$, $\{a_{i_1}, a_{i_1+1}, \dots, a_{i_2}\}$, $\dots$, $\{a_{i_r}, a_{i_r+1}, \dots, a_{i_{r+1}}\}$, where $a_{i_0} = a_0$ and $a_{i_{r+1}} = a_{n+1}$. We give an example for $n = 4, r = 2$.



A 4-gon for the subsequence (a_2, a_4) of $(a_1, \dots, a_4)$ inscribed in the semicircle

The corresponding term in the coproduct is then

$$\mathrm{I}(a_0; a_{i_1}, \dots, a_{i_r}; a_{n+1}) \otimes \prod_{k=0}^{r} \mathrm{I}(a_{i_k}; a_{i_k+1}, \dots, a_{i_{k+1}-1}; a_{i_{k+1}}), \tag{26}$$

These generators are subject to the following relations:
the *path composition formula* with respect to an element $x \in R$:

$$\mathrm{I}(a_0; a_1, \dots, a_n; a_{n+1}) = \sum_{k=0}^{n} \mathrm{I}(a_0; a_1, \dots, a_k; x)\, \mathrm{I}(x; a_{k+1}, \dots, a_n; a_{n+1}), \tag{27}$$

the *inversion formula*

$$\mathrm{I}(a_{n+1}; a_n, \dots, a_1; a_0) = (-1)^n \mathrm{I}(a_0; a_1, \dots, a_n; a_{n+1}). \tag{28}$$

and the *unit identity*

$$\mathrm{I}(a; b) = 1 \quad \text{for any } a, b \in R. \tag{29}$$

Let us replace the k-th factor of the right hand product in (26) by the sum arising from the path composition formula for $x = 0$:

$$\sum_{j=i_k}^{i_{k+1}} \mathrm{I}(a_{i_k}; a_{i_k+1}, \ldots, a_j; 0)\, \mathrm{I}(0; a_{j+1}, \ldots, a_{i_{k+1}-1}; a_{i_{k+1}})\,. \tag{30}$$

Using this and the inversion relation (28), we write (26) as

$$\mathrm{I}(a_0; a_{i_1}, \ldots, a_{i_r}; a_{n+1}) \otimes$$

$$\prod_{k=0}^{r} \sum_{j=i_k}^{i_{k+1}} (-1)^{j-i_k} \mathrm{I}(0; a_j, \ldots, a_{i_k+1}; a_{i_k})\, \mathrm{I}(0; a_{j+1}, \ldots, a_{i_{k+1}-1}; a_{i_{k+1}})\,. \tag{31}$$

We denote by $\mathcal{I}_\bullet(R)$ the so obtained Hopf algebra. In [15] several similar Hopf algebras were defined: they differ by the relations added to the basic relations (27)-(29). The Hopf algebra $\mathcal{I}_\bullet(R)$ is the biggest one of them.

8.2. **The Lie coalgebra of iterated integrals and polygons.** Recall that given a commutative Hopf algebra $\mathcal{I}_\bullet$, graded by the integers $n = 0, 1, ...$, with $\mathcal{I}_0 = \mathbb{Q}$, the graded $\mathbb{Q}$-vector space of indecomposables

$$Q(\mathcal{I}_\bullet) := \frac{\mathcal{I}_{>0}}{\mathcal{I}_{>0}^2}$$

has a natural graded Lie coalgebra structure with the cobracket $\delta : Q(\mathcal{I}_\bullet) \longrightarrow \Lambda^2 Q(\mathcal{I}_\bullet)$ induced by the coproduct in the Hopf algebra. Applying this to the Hopf algebra $\mathcal{I}_\bullet(R)$, we arrive at the graded Lie coalgebra $Q(\mathcal{I}_\bullet(R))$. We denote by $\overline{\mathrm{I}}(a_0; a_{i_1}, \ldots, a_{i_r}; a_{n+1})$ the projection of the generator $\mathrm{I}(a_0; a_{i_1}, \ldots, a_{i_r}; a_{n+1})$ to it.

Recall the R-deco $(n+1)$-gon $[a_1, ..., a_{n+1}]$ decorated by the set $(a_1, ..., a_{n+1})$ so that the root side is decorated by a_{n+1}.

Proposition 8.1. *Let R be a set. The map*

$$\overline{\mathrm{I}}(0; a_1, \ldots, a_n; a_{n+1}) \to [a_1, ..., a_{n+1}]$$

gives rise to an isomorphism of the graded Lie coalgebras

$$Q(\mathcal{I}_\bullet(R)) \xrightarrow{\sim} (V_\bullet^{\mathrm{pg}}(R), \partial)\,.$$

Proof. It follows from (27) - (28) that the elements $\overline{\mathrm{I}}(0; a_1, \ldots, a_n; a_{n+1})$ form a basis of the vector space $Q(\mathcal{I}_\bullet(R))$.

Lemma 8.1. *The Lie cobracket is given by*

$$\delta\overline{\mathrm{I}}(0; a_1, \ldots, a_n; a_{n+1}) = \sum_{0 \leqslant k < l \leqslant n+1} \overline{\mathrm{I}}(0; a_1, \ldots, a_k, a_l, \ldots, a_n; a_{n+1}) \wedge \tag{32}$$

$$\Big(\overline{\mathrm{I}}(0; a_{k+1}, \ldots, a_{l-1}; a_l) + (-1)^{l-k-1} \overline{\mathrm{I}}(0; a_{l-1}, \ldots, a_{k+1}; a_k)\Big)\,. \tag{33}$$

Proof. To get a non-zero term, the sequence of elements $\{0, a_{i_1}, \ldots, a_{i_r}, a_{n+1}\}$ determining a term in the coproduct must have exactly one gap, i.e. all sides of the corresponding inscribed polygon except just one must be of length 1. Thus only sequences $\{0, a_1, ..., a_k, a_l, \ldots, a_n, a_{n+1}\}$ can appear where $0 \leqslant k < l \leqslant n+1$. Given such a sequence, the corresponding term of the coproduct is

$$\overline{\mathrm{I}}(0; a_1, \ldots, a_k, a_l, \ldots, a_n; a_{n+1}) \wedge \overline{\mathrm{I}}(a_k; a_{k+1}, \ldots, a_{l-1}; a_l)\,.$$

Using (27) and (28), we get the required formula. □

The formula (32) can be interpreted via R-deco polygons as follows. The two terms in (33) correspond to two dissections of the polygon $[a_1, ..., a_{n+1}]$, given by the forward arrow starting from the vertex sharing a_k and a_{k+1} and ending at the side a_l, and, if $k > 0$, the backward arrow starting at the vertex sharing a_{l-1} and a_l and ending at a_k. The sign of this dissection is $(-1)^{l-k-1}$ for the backward arrow. □.

8.3. **Comparing the two coproducts.**

Theorem 8.2. *There is a map of coalgebras from* $\langle \mathrm{B}(\pi) \mid \pi \in \mathcal{P} \rangle$ *to* $\mathcal{I}_\bullet(R)$.

Proof. The map is given by sending a generator $\mathrm{B}(\pi)$, assigned to an R-deco polygon π decorated by $(a_1, \dots, a_{n+1})$, the generator $\mathrm{I}(0; a_1, \dots, a_n; a_{n+1})$.

To show that the coproduct on $\mathrm{B}(\pi)$ is compatible with $\Delta\mathrm{I}(0; a_1, \dots, a_n; a_{n+1})$, note that any factor $\mathrm{I}(a_{i_0}; a_{i_1}, \dots, a_{i_r}; a_{i_{r+1}})$ which occurs in the latter on the right of the tensor product can be written, using path composition and inversion formula, as a sum of products of terms $\mathrm{I}(0; a_{i_1}, \dots, a_{i_k}; a_{i_{k+1}})$. The factors $\mathrm{I}(0; b) = 1$ are left out. We will show that after these reductions the coproducts are the same.

Let us establish a bijection of terms in (31) with the terms in the coproduct for polygons. We identify each semicircle polygon with an R-deco polygon. Let P be an $(n+2)$-gon ($n \geqslant 1$) with "black" vertices $a_0, \dots, a_{n+1}$ and a base side a_0a_{n+1}, inscribed in a semicircle. We put a "white" vertex, denoted by a $\circ$, on each of the $n+2$ arcs between the a_i's. We order the points *counterclockwise*. The white vertices determine an R-deco $(n+2)$-gon $P^\vee$ with decorations given by the labels of the intermediate black vertices. From this, we obtain an R-deco polygon π: the root arrow of π emanates from the white vertex between the sides a_0 and a_1, and it ends in the side a_{n+1} (so that the side a_0 in $P^\vee$ is cut off).

Let us identify semicircle subpolygons with admissible dissections of π. Each subpolygon P' of P with $r+2$ sides, one of which must be a_0a_{n+1}, gives rise to a number of circumscribing $2(r+2)$-gons Q with alternating black and white vertices. Both P' and Q inherit the orientation, and Q inherits a direction of its edges $\alpha_1, \dots, \alpha_{2(r+2)}$ compatible with the orientation.

Such pairs (P', Q) are in bijection with the terms in (30). Precisely, the $2r+2$ directed sides α_i correspond to the factors on the right of the tensor product.

Every second directed side goes from black to white vertex and thus does not correspond to the direction of arrows in an R-deco polygon (going from vertex to side in an R-deco polygon corresponds to going from white vertex to black vertex in the polygon). Thus we invert each such side α_i which becomes a backward arrow in the R-deco polygon. The sign which is assigned to such a backward arrow in the bar construction for the R-deco polygon coincides with the sign which results from applying the inversion relation to the term for α_i in (31).

Finally, we use further relations for the integrals $\mathrm{I}(\dots)$: the directed sides α_i which connect a black vertex with an adjacent white one correspond to trivial arrows in π and can thus be dropped—in accordance with the fact that $\mathrm{I}(a; b) = 1$. Furthermore, the leftover directed sides α_i which end in a_0 are set to zero—they correspond to terms of the form $\mathrm{I}(0; \dots; 0)$ in (31) which are set to zero if the sequence between the two zeros is not empty.

Let us show how such a pair (P', Q) arises from a term

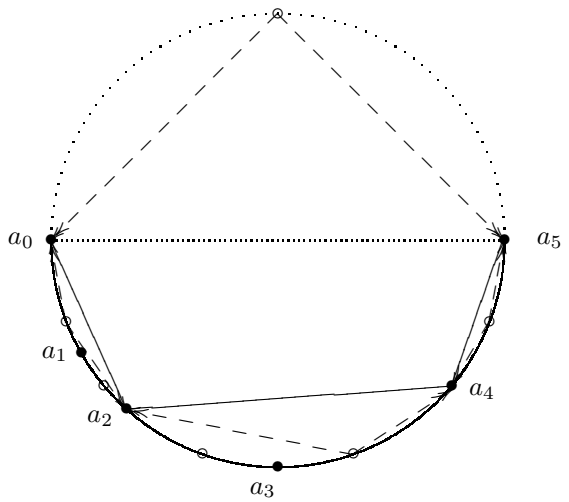
$$\mathrm{I}(a_0; a_{i_1}, \dots, a_{i_r}; a_{n+1}) \ \otimes \ \prod_k \mathrm{I}(a_{i_k}; a_{i_k+1}, \dots, \dots; 0)\, \mathrm{I}(0; \dots, \dots, a_{i_{k+1}-1}; a_{i_{k+1}})$$

The left tensor factor of this term is pictured by a sub-(r+2)-gon P' with vertices $a_0, a_{i_1}, \dots, a_{i_r}, a_{n+1}$, while the right hand factor, multiplied with the unit $\mathrm{I}(a_{n+1}; 0)\mathrm{I}(0; a_0)$, is encoded by a sequence of $2(r+2)$ "arrows" which go from a black point a_{i_k} to a white point 0 or from a white point 0 to a black point $a_{i_{k+1}}$. These arrows form a polygon Q with alternating black and white vertices. □

Example 8.3: The situation corresponding to the term

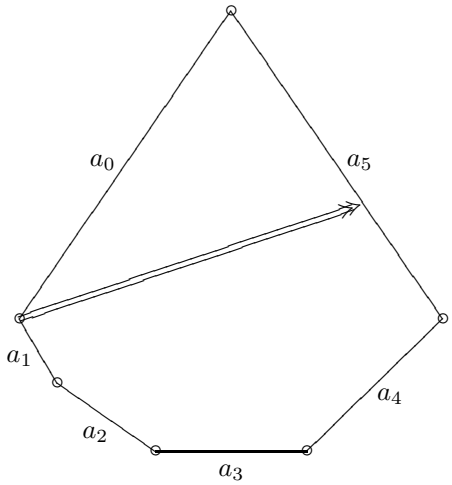
$$\mathrm{I}(a_0; a_2, a_4; a_5) \otimes \mathrm{I}(a_0; a_1; a_2)\, \mathrm{I}(a_2; a_3; a_4)\, \mathrm{I}(a_4; a_5)$$

is illustrated by four pictures below.



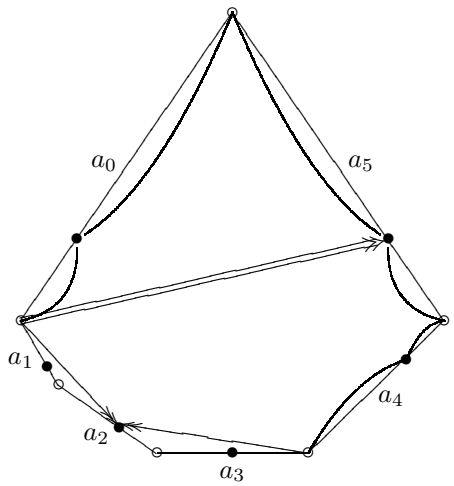
An 8-gon circumscribing the 4-gon (a_0, a_2, a_4, a_5) inscribed in the semicircle

The inscribed polygon with vertices at the black points labeled by a_0, a_2, a_4 and a_5, together with one of the terms arising from the path composition formula, gives rise to a dissection of the R-deco polygon, i.e., to a term in its coproduct.



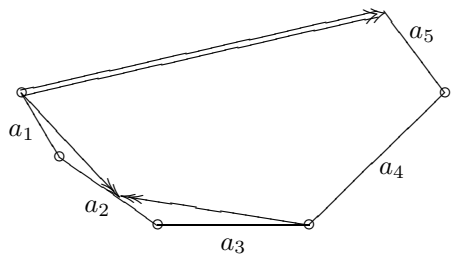
The R-deco polygon associated to the 5-gon $(a_0, a_1, a_2, a_3, a_4, a_5)$

The second picture produces the third.



The picture after pushing arrows inside.

Dropping trivial arrows connecting adjacent vertices (necessarily of different color), we produce a picture of an R-deco polygon with a 3-fold dissection:



A dissection associated to the term $\mathrm{I}(a_0; a_2, a_4; a_5) \otimes \mathrm{I}(0; a_1; a_2)\mathrm{I}(0; a_3; a_2)$.

The root arrow cuts off three trivial arrows and the extra side labeled by a_0.

8.4. **Comparison with the coproduct from** [6]**.** In [6], Connes and Kreimer considered the coproduct on a Hopf algebra on rooted non-planar trees, which has precisely the same form as our coproduct in Theorem 7.1, except that we consider plane trees and have signs. Furthermore, they use the convention of writing the "root part" R^C on the right of the tensor product, while the remaining (cut-off) trees are denoted by P_i^C.

$$\Delta_{\mathrm{CK}}(t) = \sum_{\text{adm cut } C} \prod_i P_i^C(t) \otimes R^C(t)\,.$$

The commutative product in this case is simply the unordered disjoint union.

This connection between the coproduct formula in Theorem 7.1 and the one from [6] is very similar to the relation between the coproduct in the Hopf algebra of iterated integrals in [15] and the one from [6].

9. Associating integrals to the multiple logarithm cycles

In this section we give another reason why the algebraic cycles associated to certain trees are "avatars" of multiple logarithms. We first indicate how to associate, when $R = F^\times$ is the multiplicative group of a field F, an integral to the cycles Z_{x_1,x_2} (cf. (12)) and Z_{x_1,x_2,x_3} (cf. (14)) which arise from the polygons $[\,x_1, x_2, 1\,]$ and $[\,x_1, x_2, x_3, 1\,]$. They deliver the integral presentations for the double and triple logarithms. After these examples, we pass to the general case of any generic R-deco polygon $[\,x_1, x_2, \ldots, x_m\,]$ and associate to it an element in a certain bar construction, from which the Hodge realization can be read off.

9.1. **Brief review of the Hodge realization construction from** [3]**, §8.** Let $\mathcal{N}'$ be a DG-subalgebra of $\mathcal{N}$. In our case it is the image of the polygon algebra $\mathcal{P}$. Let $\mathcal{D}P$ be a larger DGA consisting of *topological* cycles in $\square^n_{\mathbb{C}}$ which satisfy a number of conditions; we mention the ones relevant for us, keeping the notation of loc.cit.:

i) For any cycle of degree n in $\mathcal{D}P$, the integral against the standard volume form $\omega_n = \bigwedge_{j=1}^{n} (2\pi\sqrt{-1})^{-1} \dfrac{dz_j}{z_j}$ converges; denote this integration map by $r : \mathcal{D}P \to \mathbb{C}$.

ii) $\mathcal{N}'$ belongs to $\mathcal{D}P$; denote the embedding by $\sigma : \mathcal{N}' \to \mathcal{D}P$.

iii) For any cycle $\sum_i n_i\,[\,a_1^i|\cdots|a_{k_i}^i\,]$ in $\mathrm{H}^*\mathrm{B}(\mathcal{N}')$, the element $\sum_i n_i\,[\,\sigma(a_1^i)|a_2^i|\cdots|a_{k_i}^i\,]$ in $\mathrm{H}^*\mathrm{B}(\mathcal{D}P, \mathcal{N}')$ vanishes; this is a stronger version of the claim that any cycle in $\mathcal{N}'$ becomes a boundary in $\mathcal{D}P$.

Under their assumptions Bloch and Kriz show that the map

$$\lambda : \mathrm{H}^0\mathrm{B}(\mathcal{D}P, \mathcal{N}') \quad \stackrel{\mathrm{B}(r,\mathrm{id})}{\longrightarrow} \quad \mathrm{H}^0\mathrm{B}(\mathbb{C}, \mathcal{N}') = \mathbb{C} \otimes \mathrm{H}^0\mathrm{B}(\mathcal{N}')$$

is an embedding. Using λ, they define a rational structure on $\mathbb{C} \otimes \mathrm{H}^0\mathrm{B}(\mathcal{N}')$, and the Hodge and weight filtrations are two natural filtrations corresponding to the Adams grading on this space.

As in [3] §9, we construct topological *cycles* by the following trick. Denote by δ the topological boundary. Let Γ be a small disk around 0 in $\square_{\mathbb{C}}$ with the standard orientation. Consider an (oriented) topological *chain* η of real dimension $2n-i$ in $\square_{\mathbb{C}}^{2n-k}$ (in the notation of loc.cit. it is either $\eta_*(*)$ for $i=k$ or $q_*(*)$ for $i=k+1$). Associate to it a *cycle*

$$\theta_n(\eta) = \delta(\eta \times \Gamma) \times (\delta\Gamma)^{\times(i-1)} = \delta\eta \times \Gamma \times (\delta\Gamma)^{\times(i-1)} + (-1)^i \eta \times (\delta\Gamma)^{\times i}$$

of dimension $2n$ in $\square_{\mathbb{C}}^{2n+i-k}$ (in the notation of [3], (9.6), it corresponds to the cycle $\tau_*(*)$ or $v_*(*)$).

For the chains under consideration below we have the following:

$$\partial(\theta_n(\eta)) = \theta_{n-1}(\partial\eta) - \theta_{n-1}(\delta\eta).$$

Indeed, $\delta\Gamma$ does not intersect the facets 0 and ∞. Therefore intersections of $\theta_n(\eta)$ with facets come from the intersections of $\delta(\eta \times \Gamma)$ with the respective facets. The chain Γ intersects the facet 0. For our chains η the intersection of $\delta\eta$ with a facet coincides with the topological boundary δ of the intersection of η with the facet. Hence

$$\partial(\delta(\eta \times \Gamma)) = \partial(\delta\eta \times \Gamma + (-1)^i \eta \times \delta\Gamma) = \partial(\delta\eta) \times \Gamma + (-1)^k \delta\eta + (-1)^i \partial\eta \times \delta\Gamma$$

$$= \delta\partial\eta \times \Gamma + (-1)^k \delta\eta + (-1)^i \partial\eta \times \delta\Gamma = \delta(\partial\eta \times \Gamma) + (-1)^k \delta\eta\,.$$

After multiplication by $(\delta\Gamma)^{\times(i-1)}$ we get

$$\partial(\delta(\eta \times \Gamma)) \times (\delta\Gamma)^{\times(i-1)} = \delta(\partial\eta \times \Gamma) \times (\delta\Gamma)^{\times(i-1)} + (-1)^i \delta\eta \times \delta\Gamma \times (\delta\Gamma)^{\times(i-2)}$$

$$= \delta(\partial\eta \times \Gamma) \times (\delta\Gamma)^{\times(i-1)} + (-1)^{k-i+1} \delta(\delta\eta \times \Gamma) \times (\delta\Gamma)^{\times(i-2)}\,.$$

In the special case $i=k$, where the cycle $\theta_n(\eta)$ is of real dimension $2n$ in $\square_{\mathbb{C}}^{2n}$, we can integrate it against the volume form ω_{2n}, which gives—up to a factor $(-1)^i$—the integral of ω_{2n-i} against the chain η, as the restriction of ω_{2n-i} to $\delta(\eta)$ vanishes for dimension reasons and the integral of $\frac{dz}{z}$ against $\delta\Gamma$ equals $2\pi\sqrt{-1}$.

Therefore the map θ_n assigns to each chain in $\mathcal{D}P$ a cycle, and we can reduce calculations to the case of topological chains η equipped with two differentials: the algebraic one ∂ (intersection with facets) and the topological one δ.

9.2. **Algebraic-topological chains.** Following Bloch and Kriz, we embed the algebraic cycles into a larger set-up of "topological" chains which have both algebraic and topological coordinates as well as both types of differentials, and then apply the bar construction. Those chains are referred to below as *algebraic-topological chains*. We only consider "topological" variables $s_i \in [0,1] \subset \mathbb{R}$ subject to the condition $s_i \leqslant s_j$ if $i<j$, and taking the topological boundary δ for a chain with topological dimension r amounts to taking the formal alternating sum over the subvarieties where either $s_k = s_{k+1}$ for some $k = 1, \ldots, r-1$ or $s_1 = 0$ or $s_r = 1$.

9.3. **An example: the double logarithm case.** 1. In order to bound Z_{x_1,x_2}, consider the algebraic-topological chain parametrized by $t \in \mathbb{P}^1_F$ and $s_1 \in \mathbb{R}$, $0 \leqslant s_1 \leqslant 1$, as

$$\Big[1-\frac{s_1}{t}, 1-\frac{t}{x_1}, 1-\frac{t}{x_2}\Big]\,,$$

whose topological boundary terms are obtained by putting $s_1 = 0$ (which produces the empty cycle) or by $s_1 = 1$ which yields Z_{x_1,x_2}. Its algebraic boundary is given by

$$[1-\frac{s_1}{x_1}, 1-\frac{s_1}{x_2}] - [1-\frac{s_1}{x_1}, 1-\frac{x_1}{x_2}] + [1-\frac{s_1}{x_2}, 1-\frac{x_2}{x_1}]\,, \quad 0 \leqslant s_1 \leqslant 1\,,$$

where the last two terms are "negligible" for the following.

2. Consider the topological chain parametrized by $0 \leqslant s_1 \leqslant s_2 \leqslant 1$, $s_i \in \mathbb{R}$, as

$$[1-\frac{s_1}{x_1}, 1-\frac{s_2}{x_2}]\,,$$

whose boundary terms arise from setting $s_1 = 0$, $s_1 = s_2$ or $s_2 = 1$, giving the empty cycle, $[1-\frac{s_1}{x_1}, 1-\frac{s_1}{x_2}]$ or $[1-\frac{s_1}{x_1}, 1-\frac{1}{x_2}]$, respectively. What we are after is a cycle η in this larger (algebraic-topological) chain complex which bounds the cycle Z_{x_1,x_2}, i.e., such that $Z_{x_1,x_2} = (\partial+\delta)\eta$. The "bounding process" will give rise to a purely topological cycle against which we can then integrate the standard volume form ω_2.

In fact, working modulo the "negligible" term $[1-\frac{s_1}{x_1}, 1-\frac{1}{x_2}]$ above we get

$$Z_{x_1,x_2} = (\partial+\delta)\Big([1-\frac{s_1}{t}, 1-\frac{t}{x_1}, 1-\frac{t}{x_2}] + [1-\frac{s_1}{x_1}, 1-\frac{s_2}{x_2}]\Big) \tag{34}$$

(two of the boundary terms cancel), and we associate to Z_{x_1,x_2} the integral

(35)

$$\frac{1}{(2\pi\sqrt{-1})^2} \int\limits_{\substack{[1-\frac{s_1}{x_1}, 1-\frac{s_2}{x_2}] \\ 0 \leqslant s_1 \leqslant s_2 \leqslant 1}} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} = \frac{1}{(2\pi\sqrt{-1})^2} \int\limits_{0 \leqslant s_1 \leqslant s_2 \leqslant 1} \frac{d(1-\frac{s_1}{x_1})}{1-\frac{s_1}{x_1}} \wedge \frac{d(1-\frac{s_2}{x_2})}{1-\frac{s_2}{x_2}}$$

$$= \frac{1}{(2\pi\sqrt{-1})^2} \int\limits_{0 \leqslant s_1 \leqslant s_2 \leqslant 1} \frac{ds_1}{s_1 - x_1} \wedge \frac{ds_2}{s_2 - x_2}\,.$$

Therefore we see that the algebraic cycle Z_{x_1,x_2} corresponds—in a rather precise way—to the iterated integral $I_{1,1}(x_1, x_2)$.

Note that the three "negligible" terms encountered above can also be covered as part of a boundary if we introduce, following [3], yet another differential $\overline{d}$ (coming from the bar construction), and in the ensuing tricomplex all the terms above are taken care of. With the usual bar notation $\big|$ for a certain tensor product, the "correct" cycle combination is given by

$$\begin{aligned} C_{1,1} := {} & [1-\frac{s_1}{t}, 1-\frac{t}{x_1}, 1-\frac{t}{x_2}] + [1-\frac{s_1}{x_1}, 1-\frac{s_2}{x_2}] \\ & + \Big([1-\frac{s_1}{x_1}\,\big|\,1-\frac{1}{x_2}] - [1-\frac{s_1}{x_1}\,\big|\,1-\frac{x_1}{x_2}] + [1-\frac{s_1}{x_2}\,\big|\,1-\frac{x_2}{x_1}]\Big)\,, \end{aligned} \tag{36}$$

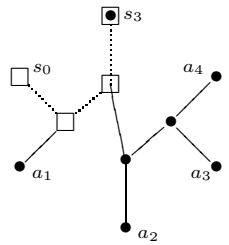
and its image under the boundary $\partial+\delta+\overline{d}$ is precisely the "bar version" of $-Z_{x_1,x_2}$, i.e. $\mathrm{B}(Z_{x_1,x_2}) = -Z_{x_1,x_2} + \big([1-\frac{1}{x_1}\,\big|\,1-\frac{1}{x_2}] - [1-\frac{1}{x_1}\,\big|\,1-\frac{x_1}{x_2}] + [1-\frac{1}{x_2}\,\big|\,1-\frac{x_2}{x_1}]\big)$.

A similar treatment of the triple logarithm can be found in [9].

9.4. **Enhanced polygons as a comodule over the polygon algebra.** The above examples suggest to encode the algebraic-topological chains using trees in a similar way as the algebraic cycles, except that we need to distinguish the topological variables s_i above (which are allowed to run through some real interval $[s_0, s_n]$) from the algebraic variables t_i (which parametrize $\mathbb{P}^1_F$). For this reason, we introduce "enhanced trees" with two types of vertices, the first type encoding the algebraic variables and the second type encoding the topological ones.

9.4.1. *Enhanced trees.* Let $\overline{R} = R \cup \{s_0\}$ for some element s_0. An **enhanced $\overline{R}$-deco tree** is an $\overline{R}$-deco tree with two types of vertices. All non-root vertices are **of first type** except the ones which lie on the path from the first external vertex to the root vertex. These vertices are **of second type**. The root vertex has *both* types; this allows to identify certain enhanced trees after contraction of the root edge with trees in $\mathcal{T}(R)$. External vertices of first type have decoration in R, while the unique external vertex of second type is decorated by s_0. Typically, we choose $R = F^\times$ for a field F, and $s_0 = 0$.

Example 9.1: An enhanced tree with six vertices of first type (marked by a bullet), three vertices of second type (marked by a square) and the root vertex (marked by both a bullet and a square). The first external vertex is decorated by s_0, while the root vertex is decorated by s_3. The path from s_0 to s_3 is drawn with dotted lines.



Definition 9.2. *The differential on enhanced R-deco trees is defined similalry to the one on R-deco trees, except that if we contract an edge whose incident vertices are of different type, then the resulting vertex, if it is not the root, is set to be of second type.*

In analogy with the DG-module $\mathcal{D}P$ over $\mathcal{N}'$, we can define a DG-module $\overline{\mathcal{T}}_\bullet(\overline{R})$ over $\mathcal{T}_\bullet(R)$, where $\overline{R} = R \cup \{s_0\}$ for some element s_0. Its generic part is denoted $\overline{\mathcal{T}}^{\mathrm{g}}_\bullet(\overline{R})$, where genericity means that all decorations (including s_0) are different.

9.4.2. *The enhanced forest cycling map.* Let F be a subfield of $\mathbb{C}$ and $s_0 \in F \cap \mathbb{R}$. We pass from $\mathrm{B}\big(\overline{\mathcal{T}}_\bullet(F^\times \cup \{s_0\})\big)$ to $\mathrm{B}(\mathcal{D}P, \mathcal{N}^\bullet)$ as follows. We modify the forest cycling map Φ from Definition 4.3 by introducing "topological" parameters s_i (cf. §9.3).

Definition 9.3. *The* **enhanced** *forest cycling map $\overline{\Phi}$ is given as follows. Let τ be an enhanced $F^\times$-deco tree.*

1. *To each of the two external vertices of second type we associate elements s_0 and s_{n+1} from $F \cap \mathbb{R}$, while for the external vertices of first type we proceed as in Definition 4.3;*
2. *for each internal vertex of first type, we associate a variable in $\mathbb{P}^1_F$;*

(3) *for each internal vertex of second type we associate a "simplicial" variable s_i which runs through the interval $[s_0, s_{n+1}]$; to each edge between two vertices of second type, oriented from s_0 to s_{n+1}, we associate a "$\leqslant$" constraint;*
(4) *to each oriented edge of τ from v to w, decorated by y_v and y_w, respectively, we associate the expression $\left[1-\frac{y_v}{y_w}\right]$;*
(5) *in the linear ordering of the edges, we concatenate all the coordinates in the previous step, except for the edges connecting vertices of second type.*

The image of the tree with a single edge under $\overline{\Phi}$ is the unique point in $\square^0$.

Example 9.4: To the above example, we associate the chain

$$\eta = \Big[1-\frac{s_1}{a_1}, 1-\frac{s_2}{t_1}, 1-\frac{t_1}{a_2}, 1-\frac{t_1}{t_2}, 1-\frac{t_2}{a_3}, 1-\frac{t_2}{a_4}\Big]$$

of real dimension 6 in $\square^6_{\mathbb{C}}$, with $s_0 \leqslant s_1 \leqslant s_2 \leqslant s_3$, and $t_j \in \mathbb{P}^1_{\mathbb{C}}$. For the standard evaluation $s_0 = 0$ and $s_3 = 1$, the map θ assigns to η the following cycle in $\square^8_{\mathbb{C}}$

$$\theta_4(\eta) = \Big[1-\frac{s_1}{a_1}, 1-\frac{1}{t_1}, 1-\frac{t_1}{a_2}, 1-\frac{t_1}{t_2}, 1-\frac{t_2}{a_3}, 1-\frac{t_2}{a_4}, r\,e^{i\vartheta_1}, e^{i\vartheta_2}\Big] \,+\, \eta \times [e^{i\vartheta_1}, e^{i\vartheta_2}]\,,$$

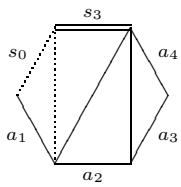
where $0 \leqslant \vartheta_j < 2\pi$ and $|r| < \varepsilon$ for some small ε. Here we parametrize the small disk Γ of radius ε by polar coordinates.

The map $\overline{\Phi}$ induces a morphism of DGAs. The following Theorem is completely similar to Theorem 4.2 and thus its proof is omitted. We put $R = F^\times$ and $s_0 = 0$.

Theorem 9.5. *For a subfield $F \hookrightarrow \mathbb{C}$, the enhanced forest cycling map $\overline{\Phi}$, followed by the map θ above, provides a natural map of differential graded algebras*

$$\overline{\mathcal{T}}^{\mathrm{g}}_\bullet(F) \to \mathcal{D}P\,. \quad \square$$

9.4.3. *Enhanced polygons.* By duality, we are led to introduce "enhanced polygons". Consider, e.g., the triangulated polygon dual to the enhanced tree in Example 9.1 above, where the second type side is drawn by a dotted line:



Forgetting the triangulation we are led to:

Definition 9.6. *An* **enhanced R-deco polygon** $\overline{\pi}$ *is an R-deco polygon with one distinguished side s of the second type adjacent to the root side r. A linear order on the sides of $\overline{\pi}$ is given by starting at s and ending at r,. It induces an orientation of $\overline{\pi}$. The decoration of s and r is in some ordered set S, and $s \leqslant r$.*

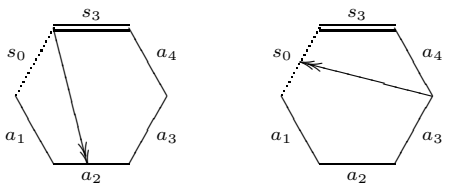
The graded vector space $V^{\overline{\mathrm{pg}}}$ is freely generated by enhanced polygons. The grading is the same as for polygons (i.e., the side of second type does not contribute).

In order to construct a differential and a resulting bar complex, we need the analogous notion of dissection for enhanced polygons.

A **dissecting arrow** of an enhanced R-deco polygon $\overline{\pi}$ is defined as for R-deco polygons, with two exceptions:

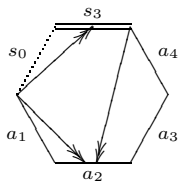
i) it is not allowed to start at the common vertex of s and r;
ii) it is not allowed to end in s.

We give two examples of non-allowed arrows, violating i) and ii), respectively.



An n-fold **dissection** of $\overline{\pi}$ is given by $n-1$ dissecting arrows (i.e., subject to conditions i) and ii) above) which do not intersect.

Example 9.7: We draw a 4-fold dissection of an enhanced polygon



We can view $\overline{\mathcal{P}} := \overline{\mathcal{P}}(\overline{R}) := V^{\overline{\mathrm{pg}}}(\overline{R}) \otimes \bigwedge^{\bullet} V^{\mathrm{pg}}(R)$ (this should not be confused with $\overline{\mathcal{P}}$ in §5) as a module over the algebra $\mathcal{P} = \bigwedge^{\bullet} V^{\mathrm{pg}}(R)$. Similarly to the cobracket κ of $\mathcal{P}$, there is a comodule map

$$\overline{\kappa} : V^{\overline{\mathrm{pg}}}(\overline{R}) \to V^{\overline{\mathrm{pg}}}(\overline{R}) \otimes \bigwedge^{\bullet} V^{\mathrm{pg}}(R)$$

$$\overline{\kappa}(\overline{\pi}) = \sum_{\alpha \text{ arrow}} \mathrm{sgn}(\alpha)\, \overline{\pi}_{\alpha}^{\bullet =} \otimes \overline{\pi}_{\alpha}^{\sqcup},$$

where $\mathrm{sgn}(\alpha)$, $\overline{\pi}_{\alpha}^{\bullet =}$ and $\overline{\pi}_{\alpha}^{\sqcup}$ are as in Definition 5.9.

The comodule map $\overline{\kappa}$ induces a differential on $\overline{\mathcal{P}}$.

Definition 9.8. *There is a differential $\overline{\partial}$ on $\overline{\mathcal{P}}$ given by*

$$\overline{\partial}(\overline{\pi} \otimes \pi') = \overline{\kappa}(\overline{\pi}) \wedge \pi' \ - \ \overline{\pi} \otimes \kappa(\pi') \qquad (\overline{\pi} \in V^{\overline{\mathrm{pg}}}(\overline{R}),\ \pi' \in \bigwedge^{\bullet} V^{\mathrm{pg}}(R)).$$

9.4.4. *A cocycle in* $\mathrm{H}^0\mathrm{B}(\overline{\mathcal{P}}, \mathcal{P})$. The differential on $\overline{\mathcal{P}}$ gives rise, via a right bar resolution, to a right DG-comodule.

Definition 9.9. *The right* $\mathrm{B}(\mathcal{P})$*-comodule* $\mathrm{B}(\overline{\mathcal{P}}, \mathcal{P})$ *is generated as a vector space by the elements*

$$[\overline{\pi}_1 \,|\, \pi_2 \,|\, \ldots \,|\, \pi_n] \qquad (\overline{\pi}_1 \in \overline{\mathcal{P}}, \pi_i \in \mathcal{P}).$$

It has a differential $D_1 + D_2$, where

$$\begin{aligned} D_1([\,\overline{\pi}_1\,|\,\pi_2\,|\,\ldots\,|\,\pi_n\,]) &= -[\,\overline{\pi}_1\otimes\pi_2\,|\,\ldots\,|\,\pi_n\,] \\ &\quad +\sum_{j=2}^{n-1}(-1)^j\,[\,\overline{\pi}_1\,|\,\pi_2\,|\,\ldots\,|\,\pi_j\wedge\pi_{j+1}\,|\,\ldots\,|\,\pi_n\,], \\ D_2([\,\overline{\pi}_1\,|\,\pi_2\,|\,\ldots\,|\,\pi_n\,]) &= [\,\overline{\kappa}(\overline{\pi}_1)\,|\,\pi_2\,|\,\ldots\,|\,\pi_n\,]+ \\ &\quad \sum_{j=2}^{n}(-1)^{j-1}\,[\,\overline{\pi}_1\,|\,\pi_2\,|\,\ldots\,|\,\kappa(\pi_j)\,|\,\ldots\,|\,\pi_n\,]. \end{aligned}$$

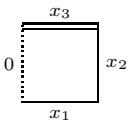
As in §5.2, each enhanced polygon provides a 0-cocycle in the bar complex $B(\overline{\mathcal{P}},\mathcal{P})$.

Definition 9.10. *To an enhanced polygon $\overline{\pi}$ we associate the following element, where π_D^i the associated subpolygons of a dissection D of $\overline{\pi}$ (we include the empty dissection, corresponding to $\overline{\pi}$ itself):*

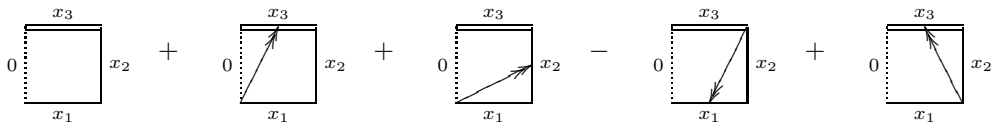
$$\mathrm{B}(\overline{\pi}) = \sum_{\textit{diss. } D \textit{ of } \overline{\pi}} \mathrm{sgn}(D) \quad \sum_{\lambda} [\overline{\pi}_D^{\lambda(1)}\,|\,\pi_D^{\lambda(2)}\,|\,\ldots\,|\,\pi_D^{\lambda(|D|)}] \;\in \mathrm{B}(\overline{\mathcal{P}},\mathcal{P}),$$

where the inner sum runs through all linear orders λ on the π_D^i compatible with the partial order on the dual tree $\tau(D)$.

Example 9.11: For the following enhanced 3-gon $\overline{\pi}$



we get $\mathrm{B}(\overline{\pi})$ as the following combination:



They correspond, in this order, to the terms for $C_{1,1}$ in (36).

Proposition 9.12. *Any enhanced polygon $\overline{\pi}$ provides a 0-cocycle, i.e.* $\mathrm{B}(\overline{\pi})$ *gives a class in* $\mathrm{H}^0\mathrm{B}(\overline{\mathcal{P}},\mathcal{P})$.

Proof. Analogous to Proposition 6.13. □

9.4.5. *Verifying the condition iii) from §9.1.* Recall from [3], (6.12), that there is a shuffle product on the circular bar construction $\mathrm{B}(\mathcal{D}P,\mathcal{N})$ defined by sending

$$(a'\otimes a_1\otimes\cdots\otimes a_m)\otimes(a''\otimes a_{m+1}\otimes\cdots\otimes a_{m+n})$$

to the sum of shuffle terms (σ denotes a shuffle of $\{1,\ldots,m\}$ with $\{1,\ldots,n\}$)

$$\pm(a'a''\otimes a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(m+n)}).$$

Remark 9.13: Let π be an R-deco polygon and $\overline{\pi}$ the associated enhanced polygon where the first vertex of π has been replaced by a side of second type.

Then there is a bijection of the dissections of π and those dissections of $\overline{\pi}$ which have one arrow going from the end of s to the root side of $\overline{\pi}$. Therefore $\mathrm{B}(\overline{\pi})$ breaks up into two parts, one of them being $[\overline{1}\,|\,\mathrm{B}(\pi)]$. But the bar differential on this expression is $\overline{1}\wedge\mathrm{B}(\pi)$, and since $\mathrm{B}(\overline{\pi})$ is a cocycle, we can conclude that $\mathrm{B}(\pi)$ (which is identified with $\overline{1}\wedge\mathrm{B}(\pi)$) is the boundary of the other part of $\mathrm{B}(\overline{\pi})$.

Using the shuffle on the circular bar construction, we can proceed similarly with terms of the form $[\overline{1}\,|\,\coprod\mathrm{B}(\pi_i)]$, i.e., we can bound $\coprod\mathrm{B}(\pi_i)$. In this way we see that property (iii) of §9.1 is satisfied for the cycles under consideration below.

9.4.6. *The comodule map.* We can now explicitly describe the comodule structure on the 0-cocycle representatives $\mathrm{B}(\overline{\pi})$, again with a Connes-Kreimer like description. We need a notion of admissible cut for this setting.

Definition 9.14. *Let D be a dissection of an enhanced polygon $\overline{\pi}$. A subdissection D' of D is* **admissible** *if it corresponds to a cut in the dual tree of D, which*

i) *is admissible and*
ii) *avoids the edges of second type.*

Remark 9.15: 1. The conditions i) and ii) in the definition rule out the edges in $\tau(D)$ which correspond to the arrows of D ending in s or starting at $s\cap r$. These are precisely the ones which are ruled out in the conditions for dissecting arrows.

2. The second condition ensures that only the root polygon of a dissection has edges of second type and therefore is the only polygon which produces an algebraic-topological chain in the associated element in the bar construction.

3. There are two distinguished admissible dissections: the empty one and the one cutting off the side of second type.

The bar functor induces a comodule map $\mathrm{B}(\overline{\kappa})$ from $\mathrm{B}(\overline{\mathcal{P}},\mathcal{P})$ to $\mathrm{B}(\overline{\mathcal{P}},\mathcal{P})\otimes\mathrm{B}(\mathcal{P})$.

Theorem 9.16. *The image of $\mathrm{B}(\overline{\pi})$ under the comodule map $\mathrm{B}(\overline{\kappa})$ has the form*

$$\sum_{D\text{ adm. diss of }\overline{\pi}}\mathrm{sgn}(D)\,\mathrm{B}(\overline{\pi}_R^D)\otimes\coprod_i\mathrm{B}(\pi_{P_i}^D)\,, \tag{37}$$

where $\overline{\pi}_R^D$ and $\pi_{P_i}^D$ denote the root polygon and the remaining polygons arising from the dissection D.

Proof. Along the same lines as the proof of (23). □

We can reinterpret the statement of the theorem as a $\mathrm{S}^\bullet V^{\mathrm{pg}}$-comodule structure on $V^{\overline{\mathrm{pg}}}$: in view of Proposition 6.13 we have a map

$$\mathrm{S}^\bullet V^{\mathrm{pg}}\to\mathrm{H}^0\mathrm{B}(\mathcal{P})\,,$$

and similarly a map

$$V^{\overline{\mathrm{pg}}}\otimes\mathrm{S}^\bullet V^{\mathrm{pg}}\to\mathrm{H}^0\mathrm{B}(\overline{\mathcal{P}},\mathcal{P})\,.$$

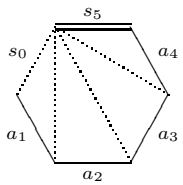
9.5. **The Hodge realization for polygons.** We finally relate $\mathrm{B}(\overline{\mathcal{P}},\mathcal{P})$ to the circular bar construction $\mathrm{B}(\mathcal{DP},\mathcal{N})$ of [3]. Mapping $\overline{\mathcal{P}} \to \mathcal{D}P$ and $\mathcal{P} \to \mathcal{N}$ induces by functoriality a map

$$\mathrm{H}^0\mathrm{B}(\overline{\mathcal{P}},\mathcal{P}) \to \mathrm{H}^0\mathrm{B}(\mathcal{D}P,\mathcal{N})$$

The right hand term carries a mixed Hodge structure. It induces a one on the left.

The comodule structure (37) carries over to algebraic-topological cycles, via trees. The Hodge realization is obtained by integrating the first bar component of the left factor of each term in (37) against the appropriate volume form on $\square^\bullet$.

Note that the realization map r (cf. §9.1) is non-trivial only on those trees whose internal vertices are all of second type. For the associated triangulated polygons, this corresponds to the unique triangulation for which every triangle is incident with the end of the root arrow, see the picture below.



The integration map λ from §9.1 evaluates each enhanced polygon in the coproduct expression (37) for a given element, and the Hodge weight can be read off from the number of sides of the polygon. Note that λ applies only to the first entry $\overline{\pi}_1$ of any component $[\overline{\pi}_1 \,|\, \pi_2 \,|\, \ldots \,|\, \pi_n]$ of the bar construction.

The Adams grading defines two filtrations on V^{pg} (recall that $\deg(\pi)$ is the number of non-root sides of π):

$$\begin{aligned} W_{-2(N-1)}(V^{\mathrm{pg}}) &= \bigoplus_{\deg(\pi)\geqslant N} \mathbb{Q}\,\pi \qquad \text{(the weight filtration)}, \\ F^{-(N-1)}(V^{\mathrm{pg}}\otimes\mathbb{C}) &= \bigoplus_{\deg(\pi)\leqslant N} \mathbb{C}\,\pi \qquad \text{(the Hodge filtration)}. \end{aligned}$$

They induce filtrations on $\mathrm{S}^\bullet V^{\mathrm{pg}}$.

We now define a Hodge-Tate structure on $\mathrm{S}^\bullet V^{\mathrm{pg}}$ as follows: there is an obvious map embedding V^{pg} into $V^{\overline{\mathrm{pg}}}$ by simply adding a side of second type, with a fixed decoration s_0, to a polygon (as in Remark 9.13 above). Further, there is a map

$$V^{\overline{\mathrm{pg}}} \otimes \mathrm{S}^\bullet V^{\mathrm{pg}} \to \mathrm{H}^0\mathrm{B}(\mathcal{D}P,\mathcal{N})$$

where the right hand term is an $\mathrm{H}^0\mathrm{B}(\mathcal{N})$-module and the left hand term is an $\mathrm{S}^\bullet V^{\mathrm{pg}}$-module. More precisely, we can embed $\mathrm{S}^\bullet V^{\mathrm{pg}} \hookrightarrow V^{\overline{\mathrm{pg}}} \otimes \mathrm{S}^\bullet V^{\mathrm{pg}}$ by the map

$$\prod_i \pi_i \mapsto \sum_i \overline{\pi}_i \prod_{j\neq i} \pi_j\,.$$

Composing the above with the integration map $r : V^{\overline{\mathrm{pg}}} \to \mathbb{C}$ now provides a map

$$\mathrm{S}^\bullet V^{\mathrm{pg}} \to \mathbb{C} \otimes \mathrm{S}^\bullet V^{\mathrm{pg}}\,,$$

which gives us the desired rational structure on $\mathrm{S}^\bullet V^{\mathrm{pg}}$.

Example 9.17: We illustrate the above for the example of the double logarithm. The topological cycle which we need to consider is $C_{1,1} - [\overline{1} \,|\, \mathrm{B}(Z_{x_1,x_2})]$, where $C_{1,1}$

is as in (36) and Z_{x_1,x_2} as in (12). Applying $\lambda \circ \theta$ to it annihilates the first term in $C_{1,1}$ (for reasons of type), while the second (purely topological) term gives

$$I \otimes 1 \ \in \mathbb{C} \otimes \mathrm{H}^0\mathrm{B}(\mathcal{P})$$

where I denotes the iterated integral in (35). The three further terms in $C_{1,1}$ give us, up to a factor $(2\pi\sqrt{-1})^{-1}$, the combination

$$\int_{[1-\frac{s_1}{x_1}]} \frac{dz}{z} \otimes \Big[1-\frac{1}{x_2}\Big] - \int_{[1-\frac{s_1}{x_1}]} \frac{dz}{z} \otimes \Big[1-\frac{x_2}{x_1}\Big] + \int_{[1-\frac{s_1}{x_2}]} \frac{dz}{z} \otimes \Big[1-\frac{x_1}{x_2}\Big],$$

and finally integrating $[\overline{1}\,\big|\,\mathrm{B}(Z_{x_1,x_2})]$ yields $1 \otimes \mathrm{B}(Z_{x_1,x_2})$.

10. Algebraic cycles for multiple polylogarithms

In this section, we sketch how to produce algebraic cycles for multiple *poly*logarithms, or rather the associated iterated integrals $I_{n_1,\ldots,n_r}(a_1,\ldots,a_r)$, in a similar fashion using $F^\times$-deco polygons, where we have to allow a third type of side which is undecorated. Roughly, the new type corresponds to factors $\frac{dz}{z}$ in the integral representation of polylogarithms. Accordingly, we need to modify the forest cycling map to form the associated admissible algebraic cycles. The following condition are required: the first and the last side (i.e., the sides which are incident with the first vertex) are of the first type.

The iterated integral $I_{n_1,\ldots,n_r}(a_1,\ldots,a_r)$ corresponds to the polygon with the following sequence of sides (we denote the undecorated sides by an $\emptyset$):

$$\pi_{n_1,\ldots,n_r}(a_1,\ldots,a_r) := [\,a_1,\ \underbrace{\emptyset,\ldots,\emptyset}_{n_1-1 \text{ times}},\ldots,a_r,\ \underbrace{\emptyset,\ldots,\emptyset}_{n_r-1 \text{ times}},1\,]\,.$$

The forest cycling map in this setting is modified as follows: let τ be the dual tree for a triangulation of a polygon π with both types of sides, then we only need to address the undecorated external edges of τ. In this case, the associated coordinate of the algebraic cycle is simply given by the parametrizing variable t itself (as opposed to, e.g., $1-\frac{t}{y}$ for some edge with external decoration y).

If we want to associate an *admissible* algebraic cycle, we need to mod out by the ideal generated by triangulations containing triangles with more than one undecorated side of the original polygon.

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