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by

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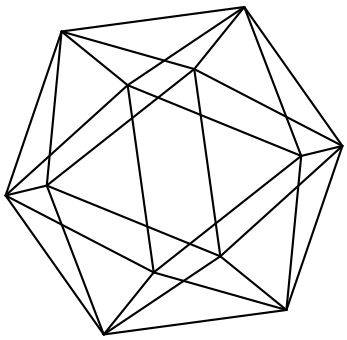

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BROWKIN'S DISCRIMINATOR CONJECTURE

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Dedicated to the memory of Prof. Jerzy Browkin (1934–2015)

ABSTRACT. Let $q \geq 5$ be a prime and put $q^* = (-1)^{(q-1)/2} \cdot q$. We consider the integer sequence $u_q(1), u_q(2), \ldots$, with $u_q(j) = (3^j - q^*(-1)^j)/4$. No term in this sequence is repeated and thus for each n there is a smallest integer m such that $u_q(1), \ldots, u_q(n)$ are pairwise incongruent modulo m. We write $D_q(n) = m$. The idea of considering the discriminator $D_q(n)$ is due to Browkin (2015) who, in case 3 is a primitive root modulo q, conjectured that the only values assumed by $D_q(n)$ are powers of 2 and of q. We show that this is true for $n \neq 5$, but false for infinitely many q in case $n = 5$. We also determine $D_q(n)$ in case 3 is not a primitive root modulo q.

Browkin's inspiration for his conjecture came from earlier work of Moree and Zumalacárregui [12], who determined $D_5(n)$ for $n \geq 1$, thus establishing a conjecture of Sălăjan. For a fixed prime q their approach is easily generalized, but requires some innovations in order to deal with all primes $q \geq 7$ and all $n \geq 1$. Interestingly enough, Fermat and Mirimanoff primes play a special role in this.

1. Introduction

Given a sequence of distinct positive integers $v = \{v(j)\}_{j=1}^{\infty}$ and any positive integer n, the discriminator $D(n)$ of the first n terms of the sequence is defined as the smallest positive integer m such that $v(1), \ldots, v(n)$ are pairwise incongruent modulo m. There are some results regarding the discriminator in case $v(j)$ is a polynomial in j (see [12] and references therein). Beyond the polynomial case, very little is known.

In this paper we determine the discriminator for the following infinite family of second order recurrences.

Definition 1. *Let $q \geq 5$ be a prime and put $q^* = (-1)^{(q-1)/2} \cdot q$. The sequence $u_q(1), u_q(2), \ldots$, with*

$$u_q(j) = \frac{3^j - q^*(-1)^j}{4},$$

we call the Browkin-Sălăjan sequence for q.

(In the sequel p and q will always denote primes.) The sequence u_q satisfies the recursive relation $u_q(j) = 2u_q(j-1) + 3u_q(j-2)$ for $j \geq 3$, with initial values $u_q(1) = (3 + q^*)/4$ and $u_q(2) = (9 - q^*)/4$. In the context of the discriminator, the sequence u_5 $(2, 1, 8, 19, 62, 181, 548, 1639, 4922, \ldots)$ was first considered by Sabin Sălăjan during an internship carried out in 2012 at the Max Planck Institute for Mathematics in Bonn under the guidance of the second author. A generalization of his sequence was introduced by Jerzy Browkin in an e-mail to the second author [3]. In the same e-mail, Browkin made the following conjecture.

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Conjecture 1 (Browkin, 2015)**.** *Let $n \geq 1$ and let $q \geq 5$ be a prime such that 3 is a primitive root modulo q. Then $D_q(n)$ is either a power of 2 or a power of q.*

In formulating Conjecture 1, Browkin was inspired by the following result of Moree and Zumalacárregui [12] establishing a conjecture made by Sălăjan.

Theorem 1 ([12])**.** *Let $n \geq 1$ be an arbitrary integer. Let e be the smallest integer such that $2^e \geq n$ and f be the smallest integer such that $5^f \geq 5n/4$. Then*

$$D_5(n) = \min\{2^e, 5^f\}.$$

This result shows that Browkin's conjecture holds true for $q = 5$.

Our main result, Theorem 2, determines $D_q(n)$ for every $n \geq 1$ and prime $q \geq 5$. It provides, at the same time, the first instance of the determination of the discriminator for an infinite family of second-order recurrences having characteristic equation with rational roots. Very recently, Faye, Luca and Moree [7] determined the discriminator for another infinite family, this time having irreducible characteristic equation. For each member in this family they characterize the discriminator for all n sufficiently large. Making this effective is considered in the follow-up paper [6]. Despite structural similarities, there are considerable differences in the details of the proofs in [7] and ours. For e.g., in our case it is much harder to exclude small prime numbers as discriminator values. However, in the other case one has to work with elements and ideals in quadratic number fields.

In number theory in general, and in our paper in particular, the following primes play a special role.

Definition 2.
A prime q is said to be Artin if 3 is a primitive root modulo q.
A prime q is said to be Fermat if it is of the form $2^m + 1$ with $m \geq 1$.
A prime number q is said to be Mirimanoff if it satisfies $3^{q-1} \equiv 1 \pmod{q^2}$.

The definition of Artin primes is non-standard and used here for brevity. See Section 7 for more on these special primes.

Our main result shows that Browkin's conjecture is true provided that we exclude $n = 5$. Theorem 1 is obtained on setting $q = 5$. We illustrate Theorem 2 by examples in Section 8, Tables 2–6.

Theorem 2. *Let $q \geq 5$ be a prime and $n \geq 1$ an arbitrary integer. Then*

$$D_q(n) = \begin{cases} \min\{2^e, q^f : 2^e \geq n,\ q^f \geq \frac{q}{q-1}n\} & \text{if } q \text{ is Artin, but not Mirimanoff;} \\ \min\{2^e, q : 2^e \geq n,\ q \geq n+1\} & \text{if } q \text{ is Artin, Mirimanoff, but not Fermat;} \\ \min\{2^e : 2^e \geq n\} & \text{if } q \text{ is Artin, Mirimanoff and Fermat;} \\ \min\{2^e : 2^e \geq n\} & \text{if } q \text{ is not Artin,} \end{cases}$$

except for $n = 5$ and $q \equiv \pm 1 \pmod{28}$, in which case $D_q(5) = 7$.

All the powers of 2 and q listed in each of the above subcases occur as values, except that, in case q is Artin but not Mirimanoff, then q^f occurs if and only if

$$\left\{ f \frac{\log q}{\log 2} \right\} > \frac{\log(q/(q-1))}{\log 2}, \tag{1}$$

where $\{x\}$ denotes the fractional part of the real number x.

Example 1. *If q is Artin, Mirimanoff, but not Fermat, then the powers of 2 and q listed are 2^e, $e \geq 0$ and q. All of them occur as values.*

Theorem 2 suggests the following question.

Question 1. *Does there exist a prime that is both Fermat and Mirimanoff?*

If such a prime exists, it is actually automatically Artin by Lemma 32. However, finding it seems a chimera.

Taking into account the value of $D_q(5)$, we see that the theorem leads us to partition the set of all primes $q \geq 5$ into eight subsets. These are considered in detail in Section 8.2, where they are listed with examples in Table 7 and the natural density of each of them is (conditionally) evaluated and listed in Table 8. For example, the primes q such that $D_q(5) = 7$ and q is not Artin have, assuming the Generalized Riemann Hypothesis, the density

$$\frac{5}{6} - \frac{173}{205}A = 0.5177511101327382317\ldots,$$

where

$$A = \prod_{p \text{ prime}} \Big(1 - \frac{1}{p(p-1)}\Big) = 0.373955813619202288\ldots \tag{2}$$

is the Artin constant. By density we mean the asymptotic ratio of the number of primes up to x in a set of primes and the total number of primes up to x.

Definition 3. *The set $\{D_q(n) : n \geq 1\}$ we will denote by $\mathcal{D}_q$. An integer $m \in \mathcal{D}_q$ is said to be a Browkin-Sălăjan value, whereas an integer $m \notin \mathcal{D}_q$ is said to be a Browkin-Sălăjan non-value. (For brevity, we sometimes use the shorter 'value', respectively 'non-value'.)*

The value of q^* does not play a role in comparing terms $u_q(i)$ and $u_q(j)$ with i and j of the same parity. For this reason, various results from [12] can be copied (almost) verbatim. In such a case we say that the proof follows by an equal parity index argument. It is partly for this reason that the proof of our main result has a lot in common with the proof of Theorem 1 given in Moree and Zumalacárregui [12]. Nevertheless, there are various complications to be surmounted. In our proof we will show that, if $9 \nmid d$, the sequence is purely periodic with period $\rho_q(d)$. In [12] the fact that $\rho_5(d)$ is even for $d > 1$ plays a crucial role. For general q it can happen that $\rho_q(d)$ is odd, and this is a source of complications. However, luckily there is at most one exceptional d, namely $d = q$.

For the convenience of the reader we prove our results in detail. The extent to which the proofs are similar to the corresponding ones in [12] is pointed out in the commentaries at the end of the sections. There are also results that have no counterpart in [12].

2. Strategy of the proof of the main result

As in [12], we think it is for the benefit of the reader to describe the strategy of the (now even lengthier) proof of our main result, Theorem 2.

We start by showing that, if $n \leq 2^e$, then $D_q(n) \leq 2^e$, which will give us the crucial upper bound $D_q(n) \leq 2n - 1$.

We then study the periodicity of the sequence modulo d and determine its period $\rho_q(d)$. The information obtained by doing so will be used to exclude many values of d from being Browkin-Sălăjan. In case $9 \nmid d$, the sequence turns out to be (purely) periodic, with a period that we can compute exactly. As we can easily see that $3 \nmid D_q(n)$, this will be enough to serve our purposes.

Restricting our attention now to those d for which $9 \nmid d$ and using that $D_q(n) < 2n$, we see that, if $\rho_q(d) \leq d/2$, then d is a Browkin-Sălăjan non-value. The basic property (4) of the period, together with an analysis of its parity, will then exclude composite values of d. Thus we must have $d = p^m$, with p a prime.

In order for $\rho_q(p^m) > p^m/2$ to hold, we find that we must have $\mathrm{ord}_p(9) = (p-1)/2$, that is, 9 must have maximal possible order modulo p. The set of these primes $p \geq 5$ different from (any fixed) q is denoted by $\mathcal{P}$ and will play an important role. In fact, 9 must have maximal possible order modulo p^m, that is, $\mathrm{ord}_p(9) = \varphi(p^m)/2$, for any $m \geq 1$. (A square cannot have a multiplicative order larger than $\varphi(p^m)/2$ modulo p^m.) This is about as far as the study of the periodicity will get us. To get further we will use a more refined tool, the *incongruence index*, which, for any given integer m, is the largest integer k such that $u_q(1), \ldots, u_q(k)$ are pairwise distinct modulo m. We denote this by $\iota_q(m) = k$. It is easy to see that $\iota_q(d) \leq \rho_q(d)$ if the sequence is purely periodic modulo d. Using again that $D_q(n) < 2n$, one notes that, similarly with the period, if $\iota_q(d) \leq d/2$, then d is a Browkin-Sălăjan non-value.

For the primes $p > 3$ we show by a lifting argument that, if $\iota_q(p) < \rho_q(p)$, then $p^2, p^3, \ldots$ are Browkin-Sălăjan non-values. Likewise, we prove that, if $\iota_q(p) \leq p/2$, then $p, p^2, p^3, \ldots$ are Browkin-Sălăjan non-values. We then show that all primes in $\mathcal{P}$ satisfy $\iota_q(p) < \rho_q(p)$.

At this point, for any fixed prime q, we are left with the primes p in $\mathcal{P}$ as the only possible Browkin-Sălăjan values. Then, using classical combinatorial number theory techniques, we infer that no prime $p > 2060$ different from q can be a Browkin-Sălăjan value. In order to deal with the remaining primes in $\mathcal{P}$, we study the quantity $\upsilon(p) = \max\{\iota_q(p) : q \geq 5,\ q \neq p\}$, which we dub the *universal incongruence index*. It is easy to see that, if $\upsilon(p) \leq (p+1)/2$, then $D_q(n) \neq p$ for $p \neq q$. We provide a simple way to compute $\upsilon(p)$ and use this to check that the inequality $\upsilon(p) \leq (p+1)/2$ holds for $29 < p < 2060$. By a slightly more refined approach we manage to show that, in fact, $p = 7$ is the only prime $p \neq q$ that can arise as Browkin-Sălăjan value; it can be seen directly for which values of n and q it occurs.

Apart from this exception, we are left with $D_q(n) = 2^e$ for some e or $D_q(n) = q^f$ for some f. The first case is trivial. In the analysis of the second case, Artin, Fermat and Mirimanoff primes naturally appear. For instance, if q is Mirimanoff, then powers q^f with $f \geq 2$ can not appear as values, whilst q does. This analysis is not complicated, but rather long-winding and therefore we will not say more about it until later.

3. Preparations for the proof

3.1. **The sequence u_q viewed as an interlacing.** The sequence u_q can be regarded as an interlacing of the sequence $u_{1,q}$ consisting of the odd indexed elements and the sequence $u_{2,q}$ consisting of the even indexed elements. We have

$$u_{1,q}(n) = u_q(2n-1) = (3^{2n-1} + q^*)/4,\ u_{2,q}(n) = u_q(2n) = (3^{2n} - q^*)/4.$$

In order to determine whether a given m discriminates $u_q(1), \ldots, u_q(n)$ modulo m, we separately consider whether $u_q(i) \equiv u_q(j)$ modulo m, with i and j of the same parity and with i and j of different parity. In the first case, the behavior modulo m is determined by that of consecutive powers of 9 modulo m.

Lemma 1. *Suppose that $3 \nmid m$ and $1 \leq \alpha \leq n$. We have $u_q(i) \not\equiv u_q(j) \pmod{m}$ for every pair (i,j) satisfying $\alpha \leq i < j \leq n$ with $i \equiv j \pmod 2$ if and only if $\operatorname{ord}_{4m}(9) > (n-\alpha)/2$.*

Proof. We have $u_q(i) \not\equiv u_q(i+2k) \pmod{m}$ iff $9^k \not\equiv 1 \pmod{4m}$. Thus $u_q(i) \not\equiv u_q(j) \pmod{m}$ for every pair (i,j) with $\alpha \leq i < j \leq n$ and $i \equiv j \pmod 2$ iff $9^k \not\equiv 1 \pmod{4m}$ for $1 \leq k \leq (n-\alpha)/2$. □

`Commentary`. Lemma 1 is proved by an equal index parity argument.

3.2. **The sequence u_q modulo powers of 2.** We will show that 2^e with $2^e \geq n$ discriminates $u_q(1), \ldots, u_q(n)$, that is, we will show that, if $2^e \geq n$, the terms of the sequence $u_q(1), \ldots, u_q(n)$ lie in distinct residue classes modulo 2^e.

Let p be a prime. If $p^a | n$ and $p^{a+1} \nmid n$, then we put $\nu_p(n) = a$. The following result is well-known; for a proof see, e.g., Beyl [2].

Lemma 2. *Let p be a prime, $r \neq -1$ an integer satisfying $r \equiv 1 \pmod p$ and n a natural number. Then*

$$\nu_p(r^n - 1) = \begin{cases} \nu_2(n) + \nu_2(r^2-1) - 1 & \text{if } p = 2 \text{ and } n \text{ is even;} \\ \nu_p(n) + \nu_p(r-1) & \text{otherwise.} \end{cases}$$

A crucial fact about u_q is that its terms have alternating parity. Indeed, we have the following trivial observation (note that $q^* \equiv 1 \pmod 4$).

Lemma 3. *If $q^* \equiv 1 \pmod 8$, the terms of $u_q(1), u_q(2), \ldots$ alternate between odd and even. If $q^* \equiv 5 \pmod 8$, it is the other way around.*

Armed with these two lemmas we are ready to establish the following result.

Lemma 4. *Let $n \geq 2$ be an integer with $n \leq 2^m$. Then $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo 2^m.*

Proof. For $n = 2$ the result is obvious by Lemma 3. So assume that $n \geq 3$. Since by Lemma 3 the terms of the sequence alternate in parity, it suffices to compare the remainders $\pmod{2^m}$ of the terms having an index with the same parity. Thus assume that we have

$$u_q(2j+\alpha) \equiv u_q(2k+\alpha) \pmod{2^m} \text{ with } 1 \leq 2j+\alpha < 2k+\alpha \leq n,\ \alpha \in \{1,2\}.$$

It follows from this that $9^{k-j} \equiv 1 \pmod{2^{m+2}}$. We have $\nu_2(9^{k-j}-1) = \nu_2(k-j)+3$ by Lemma 2. Further, $2k-2j \leq n-1 < 2^m$, so $\nu_2(k-j) \leq m-2$ (here we used that $n \geq 3$). Therefore $\nu_2(9^{k-j}-1) = \nu_2(k-j)+3 \leq (m-2)+3 = m+1$, which implies that $9^{k-j} \not\equiv 1 \pmod{2^{m+2}}$. Contradiction. □

On noting that trivially $D_q(n) \geq n$ and that, for $n \geq 2$, the interval $[n, 2n-1]$ always contains some power of 2, we obtain the following corollary to Lemma 4.

Corollary 1. *We have $n \leq D_q(n) \leq 2n-1$.*

`Commentary`. Lemma 4 is proved by an equal index parity argument.

4. Periodicity and discriminators

4.1. **Generalities.** We say that a sequence of integers $\{v_j\}_{j=1}^{\infty}$ is *(eventually) periodic* modulo d if there exist integers $n_0 \geq 1$ and $k \geq 1$ such that

$$v_n \equiv v_{n+k} \pmod{d} \tag{3}$$

for every $n \geq n_0$. The minimal choice for n_0 is called the *pre-period*. The smallest $k \geq 1$ for which (3) holds for every $n \geq n_0$ is said to be the *period* and denoted by $\rho_v(d)$. In case we can take $n_0 = 1$ we say that the sequence is *purely periodic* modulo d.

Let $\{v_j\}_{j=1}^{\infty}$ be a second order linear recurrence with the two starting values and the coefficients of the defining equation being integers. Note that, for a given d, there must be a pair (a, b) such that $a \equiv v_n$ and $b \equiv v_{n+1}$ modulo d for infinitely many n. Since a pair of consecutive terms determines uniquely all the subsequent ones, it follows that the sequence is periodic modulo d. If we consider n-tuples instead of pairs modulo d, we see that an nth order linear recurrence with the n starting values and the coefficients of the defining equation being integers is always periodic modulo d.

If a sequence v is periodic modulo d_1 and modulo d_2 with $(d_1, d_2) = 1$, then we obviously have

$$\rho_v(d_1 d_2) = \operatorname{lcm}(\rho_v(d_1), \rho_v(d_2)). \tag{4}$$

If the sequence is purely periodic modulo d_1 and modulo d_2 with $(d_1, d_2) = 1$, then it is also purely periodic modulo $d_1 d_2$. Another trivial property of ρ_v is that, if the sequence v is periodic modulo d_2, then for every divisor d_1 of d_2 we have

$$\rho_v(d_1) | \rho_v(d_2). \tag{5}$$

The following result links the period with the discriminator. Its moral is that, if $\rho_v(d)$ is small enough, we cannot expect d to occur as D_v-value, i.e., d does not belong to the image of D_v.

Lemma 5. *Assume that $D_v(n) \leq g(n)$ for every $n \geq 1$ with g non-decreasing. Assume that the sequence v is purely periodic modulo d with period $\rho_v(d)$. If $g(\rho_v(d)) < d$, then d is a D_v-non-value.*

Proof. Since $v_1 \equiv v_{1+\rho_v(d)} \pmod{d}$ we must have $\rho_v(d) \geq n$. Suppose that d is a D_v-value, that is, for some n we have $D_v(n) = d$. Then $d = D_v(n) \leq g(n) \leq g(\rho_v(d)) < d$. Contradiction. □

`Commentary`. This section is taken over verbatim from [12, Section 4].

4.2. **Periodicity of the Browkin-Sălăjan sequence.** The purpose of this section is to establish Theorem 3, which gives an explicit formula for the period $\rho_q(d)$ and the pre-period for the Browkin-Sălăjan sequence. Since it is easy to show that $3 \nmid D_q(n)$, it would be actually enough to study those integers d with $3 \nmid d$ (in which case the Browkin-Sălăjan sequence is purely periodic modulo d). However, for completeness, we discuss the periodicity of the Browkin-Sălăjan sequence for *every* d. In the sequel it is helpful to have in mind the trivial observation that, if $3 \nmid m$, then

$$2 \operatorname{ord}_{4m}(9) = \operatorname{lcm}(2, \operatorname{ord}_{4m}(3)). \tag{6}$$

Theorem 3. *Suppose that $d > 1$. Write $d = 3^\alpha \cdot \delta$ with $(\delta, 3) = 1$. The period $\rho_q(d)$ of the Browkin-Sălăjan sequence modulo d exists and satisfies*

$$\rho_q(d) = \begin{cases} \operatorname{ord}_{4\delta}(9) & \text{if } d = q \text{ and } 2 \nmid \operatorname{ord}_q(3); \\ 2\operatorname{ord}_{4\delta}(9) & \text{otherwise.} \end{cases}$$

The pre-period equals $\max(1, \alpha)$.

Corollary 2. *The Browkin-Sălăjan sequence is purely periodic if and only if $9 \nmid d$.*

In the proof of the next lemma we will use that $\rho_q(3) = 2$. Since modulo 3^f the sequence u_q eventually alternates between $q^*/4$ and $-q^*/4$, it even follows that $\rho_q(3^f) = 2$ for every $f \geq 1$.

Lemma 6. *Write $d = 3^\alpha \cdot \delta$ with $(\delta, 3) = 1$. The Browkin-Sălăjan sequence is purely periodic if and only if $9 \nmid d$. Furthermore, if $9 \nmid d$, then*

$$\rho_q(d) = \begin{cases} 2\operatorname{ord}_{4\delta}(9) & \text{if } 2 \mid \rho(d); \\ \operatorname{ord}_{4\delta}(9) & \text{otherwise.} \end{cases}$$

Proof. Since $u = (3 + q^*)/4, \overline{q^*/4, -q^*/4} \pmod 9$ and $3 + q^* \not\equiv -q^* \pmod 9$, the condition $9 \nmid d$ is necessary for the Browkin-Sălăjan sequence to be purely periodic modulo d.

We will now show that it is also sufficient. Using that $\rho_q(3) = 2$, it follows that $u_q(n) \equiv u_q(n+2k) \pmod d$ iff $3^n \equiv 3^{n+2k} \pmod{4\delta}$. Since there exists k satisfying $3^{2k} \equiv 1 \pmod{4\delta}$, it follows that the Browkin-Sălăjan sequence is purely periodic modulo d. Moreover, we have $\rho_q(d) \,|2\operatorname{ord}_{4\delta}(9)$ with equality if $\rho_q(d)$ is even and $\rho_q(d) = \operatorname{ord}_{4\delta}(9)$ otherwise. □

Corollary 3. *We have* $2\operatorname{ord}_{4\delta}(9) = \operatorname{lcm}(2, \rho_q(d))$.

We next determine the parity of $\rho_q(d)$.

Lemma 7. *Suppose that $9 \nmid d$ and $d > 1$. We have that $2 \nmid \rho_q(d)$ if and only if $d = q$ and $2 \nmid \operatorname{ord}_q(3)$.*

Proof. Suppose that d satisfies the conditions of the lemma and $\rho_q(d)$ is odd. Then $d > 2$. Since $\rho_q(3) = 2$ it follows that $(d, 3) = 1$. We have

$$u_q(n) \equiv u_q(n + \rho_q(d)) \pmod d \tag{7}$$

iff

$$(1 - 3^{\rho_q(d)})/2 \equiv q^*(-3)^{-n} \pmod{2d}. \tag{8}$$

Case 1. $(q, d) = 1$. If (8) is to hold for every $n \geq 1$, then $(-3)^n$ assumes only one value as n ranges over the positive integers. Since $(-3)^{\varphi(2d)} \equiv 1 \pmod{2d}$ we must have $(-3)^n \equiv 1 \pmod{2d}$ for every $n \geq 1$. This has no solution with $d > 2$.
Case 2. $q|d$. If the left-hand side of (8) is not divisible by q, then (8) has no solution and $\rho_q(d)$ must be even. So assume that the left-hand side is divisible by q. Then

$$(1 - 3^{\rho_q(d)})/(2q) \equiv (-1)^{(q-1)/2}(-3)^{-n} \pmod{2d/q}. \tag{9}$$

The only possible solutions here are $d = q$ and $d = 2q$. Since $\rho_q(2) = 2$ we are left with $d = q$. Since $u_q(j) \equiv 3^j/4 \pmod q$ we infer that $\rho_q(q) = \operatorname{ord}_q(3)$, which is odd iff $\operatorname{ord}_q(3)$ is odd. □

Proof of Theorem 3. It is an easy observation that modulo 3^α the Browkin-Sălăjan sequence has pre-period $\max(\alpha,1)$ and period two. From (3) we infer that $\rho_q(q) = \mathrm{ord}_q(3) = \mathrm{ord}_{4q}(9)$ if $\rho_q(q)$ is odd. This then, in combination with Lemmas 6 and 7, completes the proof. □

`Commentary`. Since now $\rho_q(d)$ can be odd, various complications arise and the results become a bit more difficult to formulate. The proofs proceed, however, in the same way as before. Lemmas 6 and 7 taken together cover the same material as Lemmas 6 and 7 of [12]; however, we think we improved the presentation. In Lemma 6 we determine the order in case it is even. In Lemma 7 we determine all cases where the order is odd. This is more logically structured than in [12]. In the earlier version, Theorem 2 and Lemma 7 cover practically the same ground; this is avoided in the new version. Our new approach also avoids having to make the case distinction between $\alpha = 0$ and $\alpha = 1$.

We would also like to point out that, instead of $\mathrm{ord}_\delta(9)$ in [12, Lemma 6], one should read $\mathrm{ord}_{4\delta}(9)$ (but that is also clear from the proof given in [12]).

4.3. **Comparison of $\rho(d)$ with d.** For notational convenience, we will from now on use $\rho(d)$ instead of $\rho_q(d)$, unless the dependence on q is necessary to be pointed out.

Lemma 8. *We have $\rho(2^e) = 2^e$ and $\rho(3^e) = 2$. If p is odd, then $\rho(p^e)|\varphi(p^e)$.*

Proof. From Lemma 2 it follows that $\mathrm{ord}_{2^{e+2}}(9) = 2^{e-1}$ and hence, by Theorem 3, $\rho(2^e) = 2^e$. For n large enough, modulo 3^e the sequence alternates between $-q^*/4$ and $q^*/4$ modulo 3^e. Since these are different residue classes, we have $\rho(3^e) = 2$.

It remains to prove the final claim. If $p = 3$ it is clearly true and thus we may assume that $p > 3$. Note that $\mathrm{ord}_{4p^e}(9) = \mathrm{ord}_{p^e}(9)$ and thus it follows from Lemma 6 that $\rho(p^e)|2\,\mathrm{ord}_{p^e}(9)\,|\,2(\varphi(p^e)/2)$. □

This lemma together with Theorem 3 and (4) yields the following result.

Lemma 9. *We have $\rho(d) \le \mathrm{lcm}(2, \rho(d)) \le d$.*

The sharper bound $\rho(m) \le m/2$ holds in case m is not a prime power.

Lemma 10. *Suppose that $d_1, d_2 > 1$ and $(d_1, d_2) = 1$. Then*

$$\rho(d_1 d_2) \le d_1 d_2/2.$$

Proof. We have $\rho(d_1 d_2) = \mathrm{lcm}(\rho(d_1), \rho(d_2))$. In case both $\rho(d_1)$ and $\rho(d_2)$ are even it thus follows that $\rho(d_1 d_2) \le \rho(d_1)\rho(d_2)/2$. Lemma 9 then gives $\rho(d_1 d_2) \le d_1 d_2/2$. By Lemma 7 it remains, without loss of generality, to deal with the case where $d_1 = q$ and $\rho(q) = \mathrm{ord}_q(3)$ is odd. Since $\varphi(q)$ is even, we see that $\rho(q) \le (q-1)/2$. We then infer that $\rho(qd_2) \le \rho(q)\rho(d_2) \le q\rho(d_2)/2 \le qd_2/2$. □

`Commentary`. The fact that the period can be odd requires some modifications. In Lemma 8 we use that $\rho(p^e)|2\,\mathrm{ord}_{p^e}(9)$ instead of $\rho_5(p^e) = 2\,\mathrm{ord}_{p^e}(9)$. In the proof of Lemma 10 we have to deal with the case $d_1 = q$ separately.

5. Browkin-Sălăjan non-values of D_q

Recall that, if $m = D_q(n)$ for some $n \ge 1$, we call m a Browkin-Sălăjan value and otherwise a Browkin-Sălăjan non-value.

Most of the following proofs rely on the simple fact that for certain sets of integers we have that, if $u_q(1), \dots, u_q(n)$ are in n distinct residue classes modulo m, then $m \geq 2n$ contradicting Corollary 1.

5.1. $D_q(n)$ is not a multiple of 3.

Lemma 11. *We have* $3 \nmid D_q(n)$.

Proof. We argue by contradiction and so assume that $D_q(n) = 3^\alpha m$ with $(m, 3) = 1$ and $\alpha \geq 1$. Since by definition $u_q(\alpha) \not\equiv u_q(\alpha+2t) \pmod{3^\alpha m}$ for $t = 1, \dots, \lfloor (n-\alpha)/2 \rfloor$ and $u_q(\alpha) \equiv u_q(\alpha+2t) \pmod{3^\alpha}$ for every $t \geq 1$, it follows that $u_q(i) \not\equiv u_q(j) \pmod m$ with $\alpha \leq i < j \leq n$ and i and j of the same parity. By Lemma 1 it then follows that $\operatorname{ord}_{4m}(9) > (n-\alpha)/2$. By (in this order) Corollary 3, Lemma 9 and Corollary 1 we then find that $n - \alpha + 1 \leq 2 \operatorname{ord}_{4m}(9) = \operatorname{lcm}(2, \rho(m)) \leq m \leq 2n/3^\alpha$. This implies that $n \leq 3^\alpha(\alpha-1)/(3^\alpha - 2)$. On the other hand, by Corollary 1 we have $3^\alpha m \leq 2n$ and hence $n \geq 3^\alpha/2$. Combining the upper and the lower bound for n yields $3^\alpha \leq 2\alpha$, which has no solution with $\alpha \geq 1$. □

`Commentary`. This proof is quite similar to that of [12, Lemma 11]. The $2 \operatorname{ord}_{4m}(9) = \rho(m)$ there has now been replaced by the identity $2 \operatorname{ord}_{4m}(9) = \operatorname{lcm}(2, \rho(m))$ (Corollary 3). Instead of the earlier $\rho(m) \leq m$, we now need $\operatorname{lcm}(2, \rho(m)) \leq m$, but this is true by Lemma 9. In the earlier proof there is $n \leq 3^\alpha(\alpha-1)/(3^\alpha - 1)$ instead of the correct $n \leq 3^\alpha(\alpha-1)/(3^\alpha - 2)$ and $3^\alpha \leq 2\alpha - 1$ instead of the correct $3^\alpha \leq 2\alpha$.

5.2. $D_q(n)$ is a prime-power.

Lemma 12. *Suppose that d with $9 \nmid d$ satisfies $\rho(d) \leq d/2$. Then d is a Browkin-Sălăjan non-value.*

Proof. Suppose that $d = D_q(n)$ for some integer n. By Lemma 3 the condition $9 \nmid d$ guarantees that the Browkin-Sălăjan sequence is purely periodic modulo d. Therefore $u_q(1) \equiv u_q(1+\rho(d)) \pmod d$ and so $\rho(d) \geq n$. The assumption $\rho(d) \leq d/2$ now implies that $d \geq 2\rho(d) \geq 2n$, contradicting Corollary 1. □

We now have the necessary ingredients to establish the following result. Let p be odd. On noting that in $(\mathbb{Z}/p^m\mathbb{Z})^*$ a square has maximal order $\varphi(p^m)/2$, we see that the following result says that a Browkin-Sălăjan value is either a power of two or a prime power p^m with 9 having maximal multiplicative order in $(\mathbb{Z}/p^m\mathbb{Z})^*$.

Lemma 13. *A Browkin-Sălăjan value greater than 1 must be of the form p^m, with $p = 2$ or $p > 3$ and $m \geq 1$. Further, one must have $\operatorname{ord}_{p^m}(9) = \varphi(p^m)/2$ and $\operatorname{ord}_p(9) = (p-1)/2$. If $m \geq 2$, then p is not Mirimanoff. In case $p^m = q$ we even have that $\operatorname{ord}_q(3) = q - 1$.*

Proof. Suppose that $d > 1$ is a Browkin-Sălăjan value that is not a prime power. Thus we can write $d = d_1 d_2$ with $d_1, d_2 > 1$, $(d_1, d_2) = 1$. By Lemma 11 we have $3 \nmid d_1 d_2$. By Lemma 10 we have $\rho(d_1 d_2) \leq d_1 d_2/2$, which by Lemma 12 implies that $d = d_1 d_2$ is a non-value. Thus d is a prime power p^m. By Lemma 11 we have $p = 2$ or $p > 3$. Now let us assume that $p > 3$. By Lemma 8 we have either $\rho(p^m) = \varphi(p^m)$ or $\rho(p^m) \leq \varphi(p^m)/2$. The latter inequality leads to $\rho(p^m) \leq p^m/2$ and hence to p^m being a non-value. Thus we must have $\rho(p^m) = \varphi(p^m)$.

Case 1. $p^m = q$. Here we note that $\rho(q) = \operatorname{ord}_q(3)$. If 3 is not a primitive root modulo

q, then $\rho(q)|\varphi(q)/2$ and hence is $\leq q/2$ and so a non-value. Thus $\mathrm{ord}_q(3) = q-1$ and hence $\mathrm{ord}_q(9) = (q-1)/2$.

Case 2. $p^m \neq q$. The condition $\rho(p^m) = \varphi(p^m)$ by Theorem 3 can be rewritten as $\mathrm{ord}_{p^m}(9) = \varphi(p^m)/2$. Now, if $\mathrm{ord}_p(9) < (p-1)/2$, this leads to $\mathrm{ord}_{p^m}(9) < \varphi(p^m)/2$ and hence we must have $\mathrm{ord}_p(9) = (p-1)/2$. Finally, suppose that $m \geq 2$ and that p is Mirimanoff, that is, $3^{p-1} \equiv 1 \pmod{p^2}$. Then $\mathrm{ord}_{p^m}(9) \leq \varphi(p^m)/p < \varphi(p^m)/2$. This contradiction shows that, if $m \geq 2$, then p is not Mirimanoff. □

`Commentary`. The statement and proof of Lemma 13 is similar to that of [12, Lemma 13], but with the case $p^m = q$ being considered separately.

5.3. **Powers of q assumed by $D_q(n)$.** In the study of $\mathcal{D}_q$ the powers of q play a special role and require separate consideration. We will use the following simple result.

Lemma 14. *Let p be a Fermat prime. Then $p \notin \cup_{q\geq 5}\mathcal{D}_q$.*

Proof. By contradiction. So suppose that $D_q(n) = p$ for some $q \geq 5$ and $n \geq 1$. Write $p = 2^e + 1$. By Lemma 4 we see that $D_q(n) \leq 2^e$ for $n \leq p-1$. Since $D_q(n) \geq n > p$ for $n > p$, it follows that $n = p$. As $u_q(1) \equiv u_q(p) \pmod{p}$, this is impossible. □

Lemma 15. *Let $q \geq 5$ be a prime.*
a) *If q is Artin, then the integers $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo q if and only if $q \geq n+1$.*
b) *If q is Artin and not Mirimanoff, then the integers $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo q^f if and only if*

$$q^f \geq \frac{qn}{q-1}.$$

Proof. If $q^f < qn/(q-1)$, then $1 + (q-1)q^{f-1} \leq n$. By Lemma 2 we have

$$3^{(q-1)q^f} \equiv 1 \pmod{q^f},$$

which ensures that $u_q(1) \equiv u_q(1 + (q-1)q^f) \pmod{q^f}$. Thus the condition $q^f \geq qn/(n-1)$ is necessary in order that $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo q^f. We next show it is also sufficient. So assume that $q^f \geq qn/(q-1)$. We distinguish the following two cases.

a) We let $f = 1$ and we have to show that $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo q iff $q \geq n+1$. This is a consequence of the sequence being periodic with period $q-1$ (as by assumption q is an Artin prime).

b) The statement in case $f = 1$ is a weaker version of part a). So we may assume that $f \geq 2$. It suffices to show that $u_q(j_1) \not\equiv u_q(k_1) \pmod{q^f}$ with $1 \leq j_1 < k_1 \leq n$ in the same congruence class modulo $q-1$. We will argue by contradiction and so assume that

$$u_q((q-1)j + \alpha) \equiv u_q((q-1)k + \alpha) \pmod{q^f},$$

with $1 \leq (q-1)j + \alpha < (q-1)k + \alpha \leq n$ and $1 \leq \alpha \leq q-1$. From this it follows that

$$3^{(q-1)(k-j)} \equiv 1 \pmod{q^f},$$

where

$$k - j \leq \frac{n-\alpha}{q-1} < \frac{n}{q-1} \leq q^{f-1}$$

by hypothesis and hence $\nu_q(k-j) \le f-2$. The assumption that q is not Mirimanoff prime ensures that $\nu_q(3^{q-1}-1)=1$. On invoking Lemma 2 we now infer that

$$\nu_q(3^{(q-1)(k-j)}-1) = \nu_q(k-j)+\nu_q(3^{q-1}-1) \le f-2+1 = f-1,$$

contradiction. □

The following result allows one to determine precisely which powers of q appear in $\mathcal{D}_q$.

Lemma 16. *Let $q \ge 5$ be a prime.*
a) *We have $q \in \mathcal{D}_q$ if and only if q is Artin and q is not Fermat.*
b) *Let $f \ge 2$. Then $q^f \in \mathcal{D}_q$ if and only if q is Artin and not Mirimanoff, and satisfies, for some natural number e, the inequality*

$$\frac{q}{q-1}(2^e+1) \le q^f < 2^{e+1}. \tag{10}$$

Proof.
a) Note that $u_q(j) \equiv 3^j/4 \pmod p$. Thus, if q is Artin, then $u_q(1),\dots,u_q(q-1)$ are pairwise distinct modulo q and hence $D_q(q-1) \le q$. If q is not Fermat, then $q-1$ is not a power of any prime number $p \ge 2$ and so by Lemma 13 it follows that $D_q(q-1)=q$. If q is Fermat, then $q \notin \mathcal{D}_q$ by Lemma 14. If q is not Artin, then $\rho(q) \le q/2$ and $q \notin \mathcal{D}_q$ by Lemma 12.

b) Let $f \ge 2$ and $q^f \in \mathcal{D}_q$. Note that $\rho(q) = \mathrm{ord}_q(3)$. If q is not Mirimanoff, then $\rho(q^f) \le q^f/q < q^f/2$ and q^f is a non-value. If q is not Artin, then $\rho(q^f) \le q^{f-1}\rho(q) \le q^f/2$ and again q^f is a non-value.

Now assume that q is Artin and not Mirimanoff. By Lemma 15, if $q^f = D_q(n)$, then

$$q^f \ge \frac{qn}{q-1}.$$

Therefore, assuming

$$\frac{q}{q-1}(2^e+1) > q^f,$$

we obtain $n \le 2^e$, which means that $D_q(n) \le 2^e < q^f$, contradiction. Conversely, $D_q(q^{f-1}(q-1)) \le q^f$ by Lemma 15. Since $q^{f-1}(q-1) \ge 2^e+1$, we infer that neither 2^e nor $2^{e+1} > q^f$ can be a discriminator for $q^{f-1}(q-1)$ and hence $q^f \in \mathcal{D}_q$. □

The following proposition gives a reformulation of the inequality (10) which is computationally very easy to work with.

Proposition 1. *Let $q \ge 5$ be a prime. Put*

$$\mathcal{F}_q = \Big\{ b \ge 1 : \Big\{ b\frac{\log q}{\log 2} \Big\} > \frac{\log(q/(q-1))}{\log 2} \Big\}.$$

The set $\mathcal{F}_q$ is the set of integers $b \ge 1$ for which there is an integer e such that

$$\frac{q}{q-1}(2^e+1) \le q^b < 2^{e+1}. \tag{11}$$

Alternatively, it is the set of integers $b \ge 1$ such that the interval $[(q-1)q^{b-1}, q^b]$ does not contain a power of 2.

Proof. The inequality (11) is equivalent with $2^e q/(q-1) < q^b < 2^{e+1}$. By taking logarithms and easy manipulations this is seen to be equivalent with

$$\frac{\log(q/(q-1))}{\log 2} < f\frac{\log q}{\log 2} - e < 1.$$

This inequality can only be satisfied if we take $e = \lfloor f \log q/\log 2 \rfloor$. We are left with the inequality

$$\left\{ f\frac{\log q}{\log 2} \right\} > \frac{\log(q/(q-1))}{\log 2},$$

which finishes the proof.

For the second assertion, we let $\mathcal{G}$ be the set of exponents $k \geq 1$ such that $(q-1)q^{g-1} \leq 2^k \leq q^g$ for some integer $g \geq 1$. We have to show that $\mathcal{G}$ is the complement of $\mathcal{F}_q$ in the natural integers. Note that g is in $\mathcal{G}$ iff $\log(q-1) + (g-1)\log q \leq k \log 2 \leq g \log q$, that is, iff $\log(q-1)/\log 2 + (g-1)\alpha \leq k \leq g\alpha$, where $\alpha = \log q/\log 2$. Since k is an integer, we may replace $g\alpha$ by $\lfloor g\alpha \rfloor$ and the condition becomes $k \in [\lfloor g\alpha \rfloor + \{g\alpha\} + \log(q-1)/\log 2 - \alpha, \lfloor g\alpha \rfloor]$. Note that there can be only one integer k in this interval iff $\{g\alpha\} \leq \log(q/(q-1))/\log 2$. □

The reader might wonder how sparse the set $\mathcal{F}_q$ is. The following result gives an asymptotic answer.

Proposition 2. *As $x \to \infty$, we have*

$$\#\{b \in \mathcal{F}_q : b \leq x\} \sim \frac{\log(2(q-1)/q)}{\log 2} x.$$

Proof. It is easy to see that $\log q/\log 2$ is irrational. Now it is a consequence of Weyl's criterion that, for a fixed $0 < \beta < 1$ and an irrational α, we have

$$\#\{g \leq x : \{g\alpha\} > \beta\} \sim (1-\beta)x, \ x \to \infty.$$

To conclude, apply this result with $\alpha = \log q/\log 2$ and $\beta = \log(q/(q-1))/\log 2$. □

`Remark`. Note that the proportionality constant in Proposition 2 satisfies

$$\frac{\log(2(q-1)/q)}{\log 2} = 1 - \frac{1}{q \log 2} + O\left(\frac{1}{q^2}\right), \ q \to \infty.$$

`Commentary`. Lemma 14 is new. In Lemma 15 it is crucial to have $\nu_q(3^{q-1}-1) = 1$, that is, we need to have that q is not Mirimanoff. The proof also hinges on $u_q(1), \ldots, u_q(q-1)$ being pairwise distinct modulo q, which happens iff q is Artin. With these assumptions on q, the earlier proof for $q = 5$ generalizes. Proposition 2 is a very straightforward generalization of [12, Proposition 1].

5.4. $D_q(n)$ **is a power of** 2 **or** q**.** Put

$$\mathcal{P} = \{p > 3 : p \neq q \text{ and } \operatorname{ord}_p(9) = (p-1)/2\}.$$

Let

$$\mathcal{P}_j = \{p > 3 : p \neq q, \ p \equiv j \pmod 4 \text{ and } \operatorname{ord}_p(3) = p-1\}, \ j \in \{1,3\},$$

and

$$\mathcal{P}_2 = \{p > 3 : p \neq q, \ p \equiv 3 \pmod 4 \text{ and } \operatorname{ord}_p(3) = (p-1)/2\}.$$

We have

$$\mathcal{P}_1 = \{5, 17, 29, 53, 89, 101, 113, 137, 149, 173, 197, 233, 257, 269, 281, 293, \ldots\},$$
$$\mathcal{P}_2 = \{11, 23, 47, 59, 71, 83, 107, 131, 167, 179, 191, 227, 239, 251, 263, \ldots\},$$
$$\mathcal{P}_3 = \{7, 19, 31, 43, 79, 127, 139, 163, 199, 211, 223, 283, \ldots\},$$

where, for any fixed q, if any of the primes listed equals q, it has to be removed from the corresponding set. By (6) we have $2\,\mathrm{ord}_p(9) = \mathrm{lcm}(2, \mathrm{ord}_p(3))$, from where we infer that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$. If a prime $p > 3$ is a Browkin-Sălăjan value, then by Lemma 13 we must have $p \in \mathcal{P}$. If $p \in \mathcal{P}$, then by Theorem 3 we have $\rho(p) = p - 1$. This will be used a few times in the sequel.

The aim of this section is to establish the following result, the proof of which makes use of properties of the incongruence index and is given in Section 5.5.1.

Proposition 3. *Let $d > 1$ be an integer coprime to $2q$. If d is a Browkin-Sălăjan value, then $d \in \mathcal{P}$.*

5.4.1. *The incongruence index.*

Definition 4. *Let $\{v_j\}_{j=1}^{\infty}$ be a sequence of integers and m an integer. Then $\iota_v(m)$, the incongruence index of v modulo m, is the largest number k such that $v_1, \ldots, v_k$ are pairwise incongruent modulo m.*

Note that $\iota_v(m) \le m$. In case the sequence v is purely periodic modulo d, we have $\iota_v(d) \le \rho_v(d)$. A minor change in the proof of Lemma 5 yields the following result.

Lemma 17. *Assume that $D_v(n) \le g(n)$ for every $n \ge 1$ with g non-decreasing. If $d > g(\iota_v(d))$, then d is a D_v-non-value.*

For the Browkin-Sălăjan sequence u_q we write $\iota_q(d)$ to highlight the dependence on q. However, whenever the dependence on q does not play a role, we will write $\iota(d)$ for simplicity.

A minor variation of the proof of Lemma 12 gives the following result, which will be of vital importance in order to discard possible Browkin-Sălăjan values.

Lemma 18. *If $\iota(d) \le d/2$, then d is a Browkin-Sălăjan non-value.*

5.4.2. *The incongruence index for q^2.* In this section we consider the incongruence index for q^2. The result and its corollary are not used in the sequel, but shed some light on the behaviour of the incongruence index.

We define

$$\mathcal{Q} = \{q > 5 : q \equiv 3 \pmod 4 \text{ and } \mathrm{ord}_q(3) = (q-1)/2\},$$

and write

$$\alpha_q = \frac{3^{(q-1)/2} - 1}{q}.$$

Note that, if $q \in \mathcal{Q}$, then α_q is an integer and $q^* = -q$.

Lemma 19. *Let $q \in \mathcal{Q}$ and suppose that $3^{(q-1)/2} \not\equiv 1 \pmod{q^2}$.*
a) *If $2\alpha_q$ is a quadratic residue modulo q, then there exists a smallest integer $1 \le m \le (q-1)/2$ such that $9^m \equiv 2/\alpha_q \pmod q$ has a solution. We have $u_q(2m) \equiv u_q(2m + (q-1)/2) \pmod{q^2}$ and $\iota_q(q^2) = 2m - 1 + (q-1)/2$.*

b) *If $2\alpha_q$ is a quadratic non-residue modulo q, then there exists a smallest integer $1 \le m \le (q-1)/2$ such that $9^m \equiv -6/\alpha_q \pmod q$ has a solution. We have $u_q(2m-1) \equiv u_q(2m-1+(q-1)/2) \pmod{q^2}$ and $\iota_q(q^2) = 2m-2+(q-1)/2$.*

Corollary 4. *If $q \in \mathcal{Q}$ and $3^{(q-1)/2} \not\equiv 1 \pmod{q^2}$, then $\iota_q(q^2) \le 3(q-1)/2 - 1 < q^2/2$.*

Proof of Lemma 19. Our argument uses that $q \nmid \alpha_q$, which is a consequence of the assumption that $3^{(q-1)/2} \not\equiv 1 \pmod{q^2}$.

For part a) we have to show that $3^{2m} + q \equiv 3^{2m+(q-1)/2} - q \pmod{4q^2}$. Since the congruence clearly holds modulo 4, it is enough to show that it holds modulo q^2; in other words, it is enough to show that $2q \equiv 9^m \alpha_q q \pmod{q^2}$. That this holds is a consequence of the identity $2 \equiv 9^m \alpha_q \pmod q$. Our assumption on q implies that $\mathrm{ord}_q(9) = (q-1)/2$. Thus the subgroup of $(\mathbb{Z}/q\mathbb{Z})^*$ generated by 9 is the subgroup of all squares. Since by assumption $2\alpha_q$ is a quadratic residue modulo q, so is $2/\alpha_q$ and hence there is a smallest integer $1 \le m \le (q-1)/2$ such that $2 \equiv 9^m \alpha_q \pmod q$. We thus conclude that $\iota_q(q^2) \le 2m-1+(q-1)/2$. In order to establish equality we notice that, if r is the smallest number such that $u_q(k) \equiv u_q(r) \pmod q$ for some $1 \le k < r$, then for general q we have $k \equiv r \pmod{\mathrm{ord}_q(3)}$, and thus for our choice of q we must have $r \in \{k+(q-1)/2, k+(q-1), \ldots\}$. Since $r-1 = \iota_q(q^2) \le 2m-1+(q-1)/2 \le 3(q-1)/2-1$, we infer that $r = k+q-1$ or $r = k+(q-1)/2$. Two cases must be dealt with.

Case 1. $r = k+q-1$.
Here r and $k+q-1$ are of the same parity and we must have $3^{q-1} \equiv 1 \pmod{q^2}$. In particular, $u_q(1) \equiv u_q(q) \pmod{q^2}$ and so $\iota_q(q^2) \le q-1$.

Case 2. $r = k+(q-1)/2$.
Here k and $r = k+(q-1)/2$ are of different parity. If k is odd, we can write it as $2v-1$ and then infer that $9^v \equiv -6/\alpha_q \pmod q$, which has no solution as

$$\left(\frac{-6\alpha_q}{q}\right) = \left(\frac{-3}{q}\right)\left(\frac{2\alpha_q}{q}\right) = -1.$$

Thus we must have $k = 2v$. We conclude that $\iota_q(q^2) = 2v-1+(q-1)/2$, where v is the smallest positive integer such that $u_q(2v) \equiv u_q(2v+(q-1)/2) \pmod{q^2}$, which yields $v = m$ as we have seen above.

The proof of part b) is very similar and left to the interested reader. □

5.4.3. *Lifting from p^m to p^{m+1}.*

Lemma 20. *Let $p > 3$. Let $1 \le t < m$. Then*

$$\rho(p^m) \mid \mathrm{lcm}(2, \rho(p^t))p^{m-t} \tag{12}$$

and

a) *if $p^t \ne q$, then $\rho(p^m) \mid \rho(p^t)p^{m-t}$;*
b) *if $p^m \ne q$, then either $\rho(p^{m+1}) = \rho(p^m)$ or $\rho(p^{m+1}) = p\rho(p^m)$;*
c) *if $\rho(p^2) = p\rho(p)$, then $\rho(p^m) = p^{m-1}\rho(p)$ for $m \ge 2$.*

Proof. For notational convenience write $\rho_1(p) = \mathrm{lcm}(2, \rho(p^t))$. Since $u_q(k) \equiv u_q(k+\rho_1(p^t)) \pmod{p^t}$ for every $k \ge 1$, it follows that $3^{\rho_1(p^t)} \equiv 1 \pmod{p^t}$ and from this we obtain $3^{\rho_1(p^t)p^{m-t}} \equiv 1 \pmod{p^m}$ and hence we deduce that

$$u_q(k) \equiv u_q(k+\rho_1(p^t)p^{m-t}) \pmod{p^m}$$

for every $k \geq 1$ and so $\rho(p^m) \mid \rho_1(p^t)p^{m-t}$.

Assertion a) follows from (12) since $\rho(p^t)$ is even for $p^t \neq q$ by Theorem 3. Assertion b) follows from assertion a) and the observation that $\rho(p^m) \mid \rho(p^{m+1})$. Finally, assertion c) is a consequence of Theorem 3 and Lemma 2. □

Lemma 21. *If $p > 3$ and $\iota(p^m) < \rho(p^m)$, then $\iota(p^{m+1}) < p^{m+1}/2$.*

Proof. Since $\iota(q) = \rho(q)$ we may assume that $p^m \neq q$. By Theorem 3 this implies that $\rho(p^m)$ is even. It then follows by part b) of Lemma 20 that either $\rho(p^{m+1}) = \rho(p^m)$ or $\rho(p^{m+1}) = p\rho(p^m)$. In the first case

$$\iota(p^{m+1}) \leq \rho(p^{m+1}) = \rho(p^m) \leq p^m < p^{m+1}/2,$$

so we may assume that $\rho(p^{m+1}) = p\rho(p^m)$. This implies that

$$3^{\rho(p^m)} \equiv 1 + kp^m \pmod{p^{m+1}} \tag{13}$$

with $p \nmid k$. From this we infer that $u_q(i + j\rho(p^m))$ assumes p different values modulo p^{m+1} as j runs through $0, 1, \ldots, p-1$. Put $j_1 = \iota(p^m) + 1$. By assumption there exists $1 \leq i_1 < j_1 < \rho(p^m)$ such that $u_q(i_1) \equiv u_q(j_1) \pmod{p^m}$. Modulo p^{m+1} we have

$$\{u_q(i_1 + j\rho(p^m)) : 0 \leq j \leq p-1\} = \{u_q(j_1 + j\rho(p^m)) : 0 \leq j \leq p-1\}.$$

The cardinality of these sets is p. Now let us consider the subsets obtained from the above two sets if we restrict j to be $\leq p/2$. Each contains $(p+1)/2$ different elements. It follows that these sets must have an element in common. Say we have

$$u_q(i_1 + k_1\rho(p^m)) \equiv u_q(j_1 + k_2\rho(p^m)) \pmod{p^{m+1}},\ 0 \leq k_1, k_2 \leq p/2.$$

Since by assumption $i_1 \not\equiv j_1 \pmod{\rho(p^m)}$, we have that

$$i_1 + k_1\rho(p^m) \neq j_1 + k_2\rho(p^m).$$

The proof is completed on noting that $i_1 + k_1\rho(p^m)$ and $j_1 + k_2\rho(p^m)$ are bounded above by

$$\iota(p^m) + 1 + (p-1)\frac{\rho(p^m)}{2} \leq (p+1)\frac{\rho(p^m)}{2} \leq (p+1)\frac{\varphi(p^m)}{2} = p^{m-1}\frac{(p^2-1)}{2} < \frac{p^{m+1}}{2},$$

where we used that, by assumption, $\iota(p^m) + 1 \leq \rho(p^m)$ and Lemma 8. □

Lemma 22. *Let $p > 3$ and $k \geq 1$ an integer. If $\iota(p^k) \leq p^k/2$, then $\iota(p^m) \leq p^m/2$ for every $m > k$.*

Proof. It suffices to prove the result for $m = k+1$ and then apply induction. Note that by Lemma 8 we have $\rho(p^{k+1}) | \varphi(p^{k+1})$.
Case 1. $\rho(p^{k+1}) \leq \varphi(p^{k+1})/2$.
It follows that $\iota(p^{k+1}) \leq \rho(p^{k+1}) \leq \varphi(p^{k+1})/2 \leq p^{k+1}/2$.
Case 2. $\rho(p^{k+1}) = \varphi(p^{k+1})$.
If $p^k = q$, then $\iota(p) = \rho(p)$ and so $k \geq 2$ and by Theorem 3 it follows that $\rho(p^k) = \varphi(p^k)$.
If $p^k \neq q$, then it also follows by Theorem 3 that $\rho(p^k) = \varphi(p^k)$. Since $p > 3$ we have

$$\iota(p^k) \leq p^k/2 < p^{k-1}(p-1) = \varphi(p^k) = \rho(p^k).$$

On applying Lemma 21 we infer that also in this case $\iota(p^{k+1}) \leq p^{k+1}/2$. □

On combining the latter two lemmas with Lemma 18 we arrive at the following more appealing result.

Lemma 23. *Let $p > 3$.*
a) *If $\iota(p) < \rho(p)$, then $p^2, p^3, \ldots$ are all Browkin-Sălăjan non-values.*
b) *If $\iota(p) \leq p/2$, then $p, p^2, p^3, \ldots$ are all Browkin-Sălăjan non-values.*

Proof.
a) If the conditions on p are satisfied, then by Lemma 21 it follows that $\iota(p^2) \leq p^2/2$, which by Lemma 22 implies that $\iota(p^m) \leq p^m/2$ for every $m \geq 2$. By Lemma 18 it then follows that p^m is a non-value.
b) If $\iota(p) \leq p/2$, then $\iota(p^m) \leq p^m/2$ for every $m \geq 1$ by Lemma 22 and by Lemma 18 it then follows that p^m is a non-value. □

`Commentary`. Section 5.4.2 is new. Lemma 20 is a generalization of the trivial [12, Lemma 8], whereas Lemmas 21 and 23 are proved in a similar way as Lemmas 16, respectively 18 from [12].

The set $\mathcal{P}$ in [12] was partitioned in three subsets, $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$. In [12, Lemma 19] it is shown that, if p is in $\mathcal{P}_3$, then $\iota(p) \leq p/2$ and hence $p \notin \mathcal{D}_5$. The argument given there cannot be generalized to arbitrary q. However, we will see that the weaker statement that $\iota(p) < \rho(p)$ is true, which is enough for our purposes and shows that $p^2, p^3, \ldots$ cannot be Browkin-Sălăjan values.

Lemma 22 is patterned after [12, Lemma 17], but a somewhat more elegant proof is given now. In the earlier proof one should read $p^{m-1}(1-1/p)$ instead of $p^{m-2}(1-1/p)$.

5.5. **If $p \in \mathcal{P}$, then $\iota(p) < \rho(p)$.** Lemma 18 in combination with the following lemma shows that every $p \in \mathcal{P}$ is a Browkin-Sălăjan non-value. Recall that, if $p \in \mathcal{P}$, then $\rho(p) = p - 1$ and that by definition $q \notin \mathcal{P}$.

Lemma 24. *If $p \in \mathcal{P}$, then $\iota(p) < \rho(p)$.*

Proof. We will find solutions to the congruence $3^{2i-1} + q^* \equiv 3^{2j} - q^* \pmod{4p}$ with $1 \leq i, j \leq (p-1)/2$, which then gives $u_q(2i-1) \equiv u_q(2j) \pmod{p}$ and yields that $\iota(p) < \max\{2i-1, 2j\} \leq p - 1 = \rho(p)$. The indices are here of different parity as focusing on terms with indices having the same parity will give only $\iota(p) \leq \rho(p)$. As trivially $3^{2i-1} + q^* \equiv 3^{2j} - q^* \pmod 4$, it is enough to consider the congruences only modulo p. We will make use of the fact that $\{3^{2k} \pmod p : 1 \leq k \leq (p-1)/2\}$ swipes out all non-zero squares modulo p and that the set $\{3^{2k-1} \pmod p : 1 \leq k \leq (p-1)/2\}$ swipes out all non-squares modulo p in case $\left(\frac{3}{p}\right) = -1$. This is a consequence of our assumption that $\mathrm{ord}_p(9) = (p-1)/2$.
Case 1. $p \in \mathcal{P}_1 \cup \mathcal{P}_3$.
Note that $\left(\frac{3}{p}\right) = -1$. If q^* is not a quadratic residue mod p, then $q^* \equiv 3^{2i-1} \pmod p$ for some $1 \leq i \leq (p-1)/2$, therefore

$$3^{2i} - 3^{2i-1} = 2 \cdot 3^{2i-1} \equiv 2q^* \pmod p,$$

which yields $u_q(2i) \equiv u_q(2i-1) \pmod p$. If q^* is a quadratic residue mod p, then we have $q^* \equiv 3^{2k} \pmod p$, for some $1 \leq k \leq (p-1)/2$ and we distinguish two subcases:
a) $-q^*$ is a quadratic non-residue mod p. Then $-q^* \equiv 3^{2\ell-1} \pmod p$, for some $1 \leq \ell \leq (p-1)/2$, and we get $u_q(2\ell-1) \equiv u_q(2k) \equiv 0 \pmod p$.
b) $-q^*$ is a quadratic residue mod p. Then $-q^* \equiv 3^{2h} \pmod p$, for some $1 \leq h \leq (p-1)/2$. If $h < (p-1)/2$, then $u_q(2h+1) \equiv -3q^* + q^* = -q^* - q^* \equiv u_q(2h) \pmod p$. If $h = (p-1)/2$, then $-q^* \equiv 1 \pmod p$ and $u_q(1) \equiv u_q(p-1) \equiv 2 \pmod p$.

CASE 2. $p \in \mathcal{P}_2$.
We have

$$u_q(2m-1) \equiv u_q(2m) \pmod{p} \Leftrightarrow 3^{2m} \equiv 3q^* \pmod{p}, \tag{14}$$

and

$$u_q(2m) \equiv u_q(2m+1) \pmod{p} \Leftrightarrow 3^{2m} \equiv -q^* \pmod{p}. \tag{15}$$

Since $\left(\frac{3q^*}{p}\right) = \left(\frac{-3}{p}\right)\left(\frac{-q^*}{p}\right) = -\left(\frac{-q^*}{p}\right)$, it follows that either $3q^*$ or $-q^*$ is a square modulo p. Thus $3^{2k} \equiv -q^* \pmod{p}$ or $3^{2k} \equiv 3q^* \pmod{p}$ holds for some $1 \leq k \leq (p-1)/2$. Note that if $q^* \not\equiv -1, 1/3 \pmod{p}$, then either (14) or (15) is satisfied with $m = k < (p-1)/2$. Otherwise, we have to deal with the following two subcases:
a) $q^* \equiv 1/3 \pmod{p}$. Then $u_q(p-2) \equiv (1/3+1/3)/4 = (1-1/3)/4 \equiv u_q(p-1) \pmod{p}$.
b) $q^* \equiv -1 \pmod{p}$. Then $u_q(1) = (3-1)/4 = (1-(-1))/4 \equiv u_q(p-1) \pmod{p}$. □

5.5.1. *A long overdue proof.* Finally we have developed enough tools to live up to our promise made at the end of Section 5.4 and prove Proposition 3.

Proof of Proposition 3. Suppose that $(d, 2q) = 1$. By Lemma 13 it follows that $d = p^m$ with $p > 3$ and $p \in \mathcal{P}$ (hence $p \neq q$). It follows from Lemma 24 that $\iota(p) < \rho(p)$ for every $p \in \mathcal{P}$, which implies by Lemma 23 that $m = 1$ and $d = p$. □

`Commentary.` The first case of the proof of Lemma 24 is the counterpart of [12, Lemma 19], while the second is that of [12, Lemma 20].

5.6. $D_q(n)$ **is not a 'big' prime.** We will now use classical exponential sum techniques to show that, for sufficiently large primes, the condition given in Corollary 4 is not satisfied. Therefore, big primes are Browkin-Sălăjan non-values.

Let us denote by ψ the additive characters of the group G and ψ_0 the trivial character. For any non-empty subset $A \subseteq G$, let us define the quantity

$$|\widehat{A}| = \max_{\psi \neq \psi_0} \left| \sum_{a \in A} \psi(a) \right|, \tag{16}$$

where the maximum is taken over all non-trivial characters in G.

The following result is Lemma 21 in [12].

Lemma 25. *Let G be a finite abelian group. For any given non-empty subsets $A, B \subseteq G$, whenever $A \cap (B+B) = \emptyset$ we have*

$$|B| \leq \frac{|\widehat{A}||G|}{|A| + |\widehat{A}|},$$

where $|\widehat{A}|$ is the quantity defined in (16).

We will need the following auxiliary result, which can be found in Cilleruelo and Zumalacárregui [5].

Lemma 26. *Let g be a primitive root modulo p and a, b and c be integers such that $p \nmid abc$. Then the set*

$$A_g(p; a, b, c) = \{(x, y) : ag^x - bg^y \equiv c \pmod{p}\} \subset \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}$$

has $p-2$ elements and satisfies $|\widehat{A}_g(p; a, b, c)| < \sqrt{p}$.

Proposition 4. *Let $p > 3$ be a prime with $p \neq q$. Suppose that $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo p. Then $p > \left\lfloor \frac{n}{4} \right\rfloor^{4/3}$.*

Proof. First observe that, if two elements have the same parity index, then $u_q(i) \not\equiv u_q(i+2k) \pmod{p}$ iff $9^k \not\equiv 1 \pmod{p}$, thus $\mathrm{ord}_p(9) \geq n/2$. (Alternatively one might invoke Lemma 1 to obtain this conclusion.) By hypothesis, on comparing elements with distinct parity index, it follows that

$$3 \cdot 9^k - 9^s \equiv 6q^* \pmod{p},\ 1 \leq k, s \leq \left\lfloor \tfrac{n}{2} \right\rfloor \tag{17}$$

has no solution (otherwise $u_q(2k) \equiv u_q(2s-1) \pmod{p}$, with $1 \leq 2k, 2s-1 \leq n$).

We will now show that the non-existence of solutions to equation (17) implies that $p > \lfloor \frac{n}{4} \rfloor^{4/3}$. Let g be a primitive root modulo p and let $A_g(p; 3, 1, 6q^*)$ be the set defined in Lemma 26. Let m be the smallest integer such that $g^m \equiv 9 \pmod{p}$ and put

$$B = \{(mx, my) : 1 \leq x, y \leq \lfloor n/4 \rfloor\} \subset \mathbb{Z}_{p-1} \times \mathbb{Z}_{p-1}.$$

Note that, since $\mathrm{ord}_p(9) \geq n/2$, it follows that $|B| = \left\lfloor \frac{n}{4} \right\rfloor^2$ (since m generates a subgroup of order at least $n/2$ modulo $p-1$).

Observe that the non-existence of solutions to equation (17) implies that

$$3 \cdot g^{mk} - g^{ms} \equiv 6q^* \pmod{p},\ 1 \leq k, s \leq \left\lfloor \tfrac{n}{2} \right\rfloor$$

has no solutions and in particular $A_g(3, 1, 6q^*) \cap (B + B) = \emptyset$ (since clearly $B + B \subseteq \{(mx, my) : 1 \leq x, y \leq \lfloor n/2 \rfloor\}$). It follows from Lemma 25 and Lemma 26 that

$$|B| = \left\lfloor \frac{n}{4} \right\rfloor^2 \leq \frac{|\widehat{A}||G|}{|A| + |\widehat{A}|} \leq \frac{p^{1/2}(p-1)^2}{p - 2 + p^{1/2}} < p^{3/2}, \tag{18}$$

which concludes the proof. □

Corollary 5. *If $p > 2060$ is a prime number with $p \neq q$, then p is a Browkin-Sălăjan non-value.*

Proof. First observe that, if $n \geq 2060$, then it follows from Proposition 4 that if, for some prime $p \geq n$, the elements $u_q(1), \ldots, u_q(n)$ are pairwise distinct modulo p, then

$$p > \left\lfloor \frac{n}{4} \right\rfloor^{4/3} \geq 2n,$$

and by Lemma 4 it follows that p is a Browkin-Sălăjan non-value. □

`Commentary`. In the proof of Proposition 4 we now need to consider the more general sets $A_g(p; 3, 1, 6q^*)$ instead of the sets $A_g(p; 3, 1, 30)$. As these behave in the same way as $A_g(p; 3, 1, 30)$, provided $p \neq q$, the proof is very similar to that of the corresponding Proposition 4 in [12].

5.7. **Primes $p < 2060$ that can occur.** A final step in [12] was to check numerically that no prime $5 < p < 2060$ can occur as discriminator for the Sălăjan sequence u_5. For our more general Browkin-Sălăjan sequence u_q, this is no longer true. Numerical computations reveal, for instance, that $D_q(5) = 7$ for certain values of q. By computer verification we will see, in fact, that 7 is the only such exceptional value, and Lemmas 30 and 31 will clarify when it occurs.

Definition 5. *Given a prime p, we define the universal incongruence index as*

$$\upsilon(p) = \max\{\iota_q(p) : q \neq p,\ q \geq 5\},$$

where q ranges over the primes $q \geq 5$.

The following easy property of the incongruence index allows one to compute $\upsilon(p)$.

Lemma 27. *Let $5 \leq q_1 < q_2$ be two primes such that $q_2 \equiv \pm q_1 \pmod{4p}$, then we have $\iota_{q_1}(p) = \iota_{q_2}(p)$.*

Proof. Follows on noting that $u_{q_1}(n) \equiv u_{q_2}(n) \pmod{p}$ for every $n \geq 1$. □

Lemma 28. *We define*

$$S(p;r) = \{3 \cdot 9^x - 9^y \pmod{p} : 1 \leq 2x,\ 2y-1 \leq r\} \cup \{0\}.$$

If $p \in \mathcal{P}$, then $\upsilon(p) = h(p)$, where

$$h(p) = \max\{r : S(p;r) \neq \mathbb{Z}/p\mathbb{Z}\}$$

is well-defined.

Proof. An equal parity argument only yields that $\upsilon(p) \leq \rho(p)$. By Lemma 24 the assumption $p \in \mathcal{P}$ implies that $\iota(p) < \rho(p)$. Thus the smallest ℓ for which there exists $1 \leq k < \ell$ and

$$u_q(k) \equiv u_q(\ell) \pmod{p} \tag{19}$$

has a distinct parity from k.

Thus we obtain a congruence of the form $u_q(2x) \equiv u_q(2y-1) \pmod{p}$, which is equivalent with

$$3 \cdot 9^x - 9^y \equiv 6q^* \pmod{p}. \tag{20}$$

First suppose that $S(p;r) \neq \mathbb{Z}/p\mathbb{Z}$. If $a \notin S(p;r)$, then for those q satisfying $q^* \not\equiv a/6 \pmod{p}$, we have that (20) is not satisfied with $1 \leq 2x, 2y-1 \leq r$ and so $\iota_q(p) \geq r$. By Dirichlet's theorem for primes in arithmetic progression there are indeed primes q satisfying $q^* \not\equiv a/6 \pmod{p}$. It follows that $\upsilon(p) \geq r$. For $r > \rho(p)$ we have $S(p;r) = \mathbb{Z}/p\mathbb{Z}$. Note that if $S(p;r_0) = \mathbb{Z}/p\mathbb{Z}$ for some r_0, then $S(p;r) = \mathbb{Z}/p\mathbb{Z}$ for every $r > r_0$. We thus conclude that $h(p)$ is well-defined and that $\upsilon(p) \geq h(p)$.

Next suppose that $S(p;r) = \mathbb{Z}/p\mathbb{Z}$. Then, whatever $q \neq p$ we choose, the congruence (20) has a solution with $1 \leq 2x, 2y-1 \leq r$. We conclude that $\iota_q(p) \leq \upsilon(p) < r$ and $\upsilon(p) < h(p)+1$. This inequality, together with $\upsilon(p) \geq h(p)$ finishes the proof. □

Lemma 29. *Let $p \in \mathcal{P}$. If there is a power of 2 in the interval $[h(p), p)$, then p is a Browkin-Sălăjan non-value.*

Proof. By contradiction. Recall that $p \in \mathcal{P}$ implies that $p \neq q$. If $D_q(n) = p$ for some n, then $n \leq \iota_q(p) \leq \upsilon(p) = h(p)$ by Lemma 28. Now if there is a power of two, say 2^e, in the interval $[h(p), p)$, it discriminates the first $h(p)$ values of u_q. As $2^e < p$, it follows that $D_q(n) \leq 2^e$. Contradiction. □

Corollary 6. *Let $p \in \mathcal{P}$. If $h(p) \leq (p+1)/2$, then p is a Browkin-Sălăjan non-value.*

This corollary gives a very powerful and easy to implement criterion to exclude small values of p from the possible Browkin-Sălăjan values. By numerical work done in Maple and Mathematica, we infer that $h(p) \leq (p+1)/2$ for all primes $31 \leq p < 3000$, $p \in \mathcal{P}$, see Table 1. Finally, by Lemma 29 we are left only with $p = 7$ as potential exception.

p	$h(p)$	p	$h(p)$
5	3	31	16
7	5	43	21
11	7	47	20
17	11	53	20
19	11	59	23
23	12	71	25
29	16	79	27

TABLE 1. Values of $h(p)$ for p in $\mathcal{P}$

Lemma 30. *Suppose that $D_q(n) = p$ with $p \neq q$ a prime. Then $n = 5$ and $p = 7$.*

Proof. We note that 7 can only be a discriminator for $n = 5$ and $n = 6$. Namely, 4 is a discriminator for $n \leq 4$ and 7 discriminates at most 6 values as $u_q(1) \equiv u_q(7) \pmod 7$. It is not difficult to show that $D_q(6) = 7$ if $q = 7$ and $D_q(5) = 8$ in all other cases. Thus we conclude that $n = 5$. □

`Commentary`. In [12] the idea was to bound $\iota_5(p)$ by $(p-1)/2$, leading to the conclusion that p is a Browkin-Sălăjan non-value. Here the basic idea is the same, but now with $\upsilon(p) = \{\iota_q(p) : q \neq p,\ q \geq 5\}$. That turns out to be rather more difficult and so this section is mainly new.

5.8. **Discriminator values for small fixed n.** Obviously as n is fixed and q ranges over the primes ≥ 5, $D_q(n)$ can assume only finitely many possible values. Indeed, trivially one has $D_q(1) = 1$, $D_q(2) = 2$, $D_q(3) = 4$ and $D_q(4) = 5$. The values $D_q(5)$ and $D_q(6)$ are slightly more difficult to determine.

Lemma 31. *We have*

$$D_q(5) = \begin{cases} 7 & \text{if } q = 7 \text{ or } q \equiv \pm 1 \pmod{28}; \\ 8 & \text{otherwise,} \end{cases} \quad \text{and } D_q(6) = \begin{cases} 7 & \text{if } q = 7; \\ 8 & \text{otherwise.} \end{cases}$$

Proof. Writing, say, $q = 4k + 1$, the Browkin-Sălăjan sequence reads as

$$k + 1, 2 - k, k + 7, 20 - k, 61 + k, 182 - k, 547 + k, \ldots.$$

Simply by testing all residue classes of k modulo 7 one concludes that $D_q(5) = 7$ iff $k \equiv 0 \pmod 7$. If $q = 4k + 3$, the sequence becomes

$$-k, 3 + k, 6 - k, 21 + k, 60 - k, 183 + k, 546 - k, \ldots$$

and, by the same method, one concludes that $D_q(5) = 7$ iff $k \equiv 6 \pmod 7$ or $q = 7$. Also, one sees that $D_q(6) = 7$ iff $q = 7$. □

`Commentary`. There is no counterpart of this in [12].

6. The proof of the main result

In Section 3, we established that powers of 2 and powers of prime numbers $p > 3$ are candidates for Browkin-Sălăjan values. On fixing the prime q, it is seen by Lemma 15 that powers of q itself are candidates too. Finally, after studying the characteristics of

the period and the incongruence index of the Browkin-Sălăjan sequence, we discarded in Section 5 any other possible candidates, except for the value 7 for certain primes q.

Proof of Theorem 2. It follows from Proposition 3 that, if $d > 1$ is a Browkin-Sălăjan value, then either $(2q, d) > 1$ or $d \in \mathcal{P}$. By Lemma 30, if $D_q(n) = p$ for some integer $n \geq 1$ and some odd prime $p \neq q$, then $n = 5$ and $p = 7$. It is easy to check that the predicted value for $D_q(5)$ in the statement of the theorem matches the actual value given in Lemma 31. Thus from now on, we may assume that $n \neq 5$ and $(2q, d) > 1$. Then, by Lemma 13, d has to be a prime power and hence the discriminator must be a power of 2 or a power of q.

Note that 2^e discriminates $u_q(1), \ldots, u_q(n)$ iff $2^e \geq n$. Our analysis splits into several cases.

Case 1. q is Artin and not Mirimanoff.
By Lemma 15 it follows that q^f discriminates $u_q(1), \ldots, u_q(n)$ iff $q^f \geq qn/(q-1)$. We infer that

$$D_q(n) = \min\{2^e, q^f : 2^e \geq n,\ q^f \geq \frac{q}{q-1}n\}.$$

Case 2. q is Artin and Mirimanoff.
By Lemma 13 we must have $f = 1$. In case q is Fermat, by Lemma 14 there is no n with $D_q(n) = q$. Note that $\iota(q) = q - 1$ if q is Artin and hence we must have $q \geq n + 1$. Suppose that q is not Fermat and 2^e is the largest power of 2 less than $q - 1$. Then $D_q(n) = q$ for $n = 2^e + 1, \ldots, q - 1$. Thus we have showed that $D_q(n)$ equals

$$D_q(n) = \begin{cases} \min\{2^e, q : 2^e \geq n,\ q \geq n+1\} & \text{if } q \text{ is Artin, Mirimanoff, but not Fermat;} \\ \min\{2^e : 2^e \geq n\} & \text{if } q \text{ is Artin, Mirimanoff and Fermat.} \end{cases}$$

Case 3. q is not Artin.
It follows by Lemma 16 that q and its powers are all Browkin-Sălăjan non-values. Thus in this case we have

$$D_q(n) = \min\{2^e : 2^e \geq n\}.$$

This proves that the four part formula for $D_q(n)$, together with the exceptional case given in the statement of the theorem, is correct.

As obviously all powers of 2 occur, it remains to determine which powers of q do occur. This we did in Lemma 16. On invoking Proposition 1, the proof is completed. □

`Commentary`. This proof is considerably more involved than in case $q = 5$, as there are now eight cases to be considered.

7. Special primes

We recapitulate some material on Artin, Fermat and Mirimanoff primes.

7.1. **Artin primes.** Recall that an 'Artin prime' we call a prime q such that 3 is a primitive root modulo q. How special are Artin primes? How many Artin primes $q \leq x$ are there? This is related to the celebrated Artin primitive root conjecture. We refer to the appendix of [12] for more information, or Moree [11] for much more information.

7.2. **Fermat primes.** A Fermat prime is a prime of the form $2^m + 1$ with $m \geq 1$. It is a trivial observation that we must have $m = 2^e$. Currently the only Fermat primes known are $3, 5, 17, 257$ and 65537.

Lemma 32. *If a prime $q > 3$ is Fermat, then q is Artin.*

Proof. Note that it is enough to show that $\left(\frac{3}{q}\right) = -1$. Now apply the law of quadratic reciprocity (details left to the reader). □

7.3. **Mirimanoff primes.** Currently there are only two Mirimanoff primes known, namely 11 and 1006003, see Keller and Richstein [8]. The prime 1006003 is Artin, but 11 is not. The Mirimanoff primes arose in the study of Fermat's Last Theorem, see, e.g., Ribenboim [13] or Ribenboim [14, Chapter 8].

7.4. **Fermat-Mirimanoff primes.** A prime that is both Fermat and Mirimanoff we call a Fermat-Mirimanoff prime. Currently no such prime is known and perhaps they do not exist at all. Note that by Lemma 32 every Fermat-Mirimanoff prime is an Artin-Fermat-Mirimanoff prime.

8. Some numerical results

8.1. **Theorem 2 in action.** In Tables 2–5 we demonstrate Theorem 2 in case $q = 5, 7, 11, 17$ and 29. Highlighted are, in each case, the exceptional value 7 and the powers of q.

n	$D_q(n)$	n	$D_q(n)$
1	1	$129-256$	256
2	2	$257-512$	512
$3-4$	4	$513-1024$	1024
$5-8$	8	$1025-2048$	2048
$9-16$	16	$2049-2500$	$\underline{3125}$
$17-20$	$\underline{25}$	$2501-4096$	4096
$21-32$	32	$4097-8192$	8192
$33-64$	64	$8193-12500$	$\underline{15625}$
$65-100$	$\underline{125}$	$12501-16384$	16384
$101-128$	128	$16385-32768$	32768

Table 2. $q = 5$; q is Artin, Fermat, but not Mirimanoff

8.2. **Prime distribution over the eight possible cases in Theorem 2.** Theorem 2 leads to eight possible cases if we take into account the exceptional case where $D_q(5) = 7$ and $q \neq 7$. These are listed in Table 7. For each case we give the first few examples. In three cases there are no known examples. Coming up with such an example would require finding a Fermat prime larger than 65537 or a Mirimanoff prime larger than 1006003. Beyond examples, we give in Table 7 a conjectural natural density of the primes belonging to each subcase. These are all rational multiples of the Artin constant A defined in (2).

We now explain how Table 7 has to be read. In the first column we indicate whether or not the condition $q \equiv \pm 1 \pmod{28}$ is met. If an entry is empty in, say, the 'Fermat'

n	$D_q(n)$	n	$D_q(n)$
1	1	$129-256$	256
2	2	$257-294$	$\underline{343}$
$3-4$	4	$295-512$	512
$5-6$	$\underline{7}$	$513-1024$	1024
$7-8$	8	$1025-2048$	2048
$10-16$	16	$2049-2058$	$\underline{2401}$
$17-32$	32	$2059-4096$	4096
$33-42$	$\underline{49}$	$4097-8192$	8192
$43-64$	64	$8193-16384$	16384
$65-128$	128	$16385-32768$	32768

TABLE 3. $q=7$; q is Artin, not Fermat and not Mirimanoff

n	$D_q(n)$	n	$D_q(n)$
1	1	$129-256$	256
2	2	$257-512$	512
$3-4$	4	$513-1024$	1024
$5-8$	8	$1025-2048$	2048
$9-16$	16	$2049-4096$	4096
$17-32$	32	$4097-8192$	8192
$33-64$	64	$8193-16384$	16384
$65-128$	128	$16385-32768$	32768

TABLE 4. $q=11$; q is not Artin, not Fermat, but Mirimanoff

n	$D_q(n)$	n	$D_q(n)$
1	1	$257-272$	$\underline{289}$
2	2	$273-512$	512
$3-4$	4	$513-1024$	1024
$5-8$	8	$1025-2048$	2048
$9-16$	16	$2049-4096$	4096
$17-32$	32	$4097-4624$	$\underline{4913}$
$33-64$	64	$4625-8192$	8192
$65-128$	128	$8193-16384$	16384
$129-256$	256	$16385-32768$	32768

TABLE 5. $q=17$; q is Artin, Fermat, but not Mirimanoff

column, then this means that both Fermat and non-Fermat primes are allowed. The final column lists the first few examples.

In Table 8 we list the conditional densities of the sets of primes belonging to each of the eight cases. We can only prove that these densities are true under one or both of the following assumptions:

(G) The Generalized Riemann Hypothesis.

(M) The Mirimanoff primes have natural density zero.

n	$D_q(n)$	n	$D_q(n)$
1	1	$129-256$	256
2	2	$257-512$	512
$3-4$	4	$513-812$	$\underline{841}$
5	$\underline{7}$	$813-1024$	1024
$6-8$	8	$1025-2048$	2048
$9-16$	16	$2049-4096$	4096
$17-28$	$\underline{29}$	$4097-8192$	8192
$29-32$	32	$8193-16384$	16384
$33-64$	64	$16385-23548$	$\underline{24389}$
$65-128$	128	$23549-32768$	32768

TABLE 6. $q = 29$; q is Artin, not Fermat and not Mirimanoff

$\pm 1 \pmod{28}$	Artin	Mirimanoff	Fermat	Examples
yes	yes	no		$29, 113, 197, 223, 281, \ldots$
no	yes	no		$5, 7, 17, 19, 31, 43, 53, 79, \ldots$
yes	yes	yes	no	none known
no	yes	yes	no	$1006003, \ldots$
yes	yes	yes	yes	none known
no	yes	yes	yes	none known
yes	no			$83, 167, 251, 307, 337, \ldots$
no	no			$11, 13, 23, 37, 41, 47, 59, \ldots$

TABLE 7. The eight prime sets arising in Theorem 2

Which assumptions we make in order to establish the density are indicated in the first column. The column 'Empirical' rests on a Maple computation using the first million prime numbers.

We determine the density in the first case given in Table 8. If one assumes G and M, then it is given by Lemma 33. Using that, under GRH, the density of Artin primes is A (see, e.g., [10, Theorem 1.2]) and that, by Dirichlet's theorem on primes in arithmetic

Assumption	Density	Numerical	Empirical
G, M	$32A/205$	$0.05837\ldots$	≈ 0.0584
G, M	$173A/205$	$0.31558\ldots$	≈ 0.3155
M	0	0	
M	0	0	
M	0	0	
M	0	0	
G	$1/6 - 32A/205$	$0.10829\ldots$	≈ 0.1083
G	$5/6 - 173A/205$	$0.51775\ldots$	≈ 0.5178

TABLE 8. Conjectural densities of the eight prime sets arising in Theorem 2

progressions, the density of the primes $q \equiv \pm 1 \pmod{28}$ is $1/6$, the remaining densities are easily obtained.

Lemma 33 (GRH)**.** *The density of primes* $q \equiv \pm 1 \pmod{28}$ *that are Artin equals* $32A/205$.

Sketch of proof. Let K be a number field. Then the natural density of primes p that split completely and have 3 as a primitive root exists and is given by

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{[K(\zeta_n, 3^{1/n}) : \mathbb{Q}]}.$$

Using that the primes p that split completely in $\mathbb{Q}(\zeta_{28})$ are precisely the primes $p \equiv \pm 1 \pmod{28}$ we see that the density we are after equals

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{[\mathbb{Q}(\zeta_{28} + \zeta_{28}^{-1}, \zeta_n, 3^{1/n}) : \mathbb{Q}]}. \tag{21}$$

By some algebraic number theory making use of the fact that $\mathbb{Q}(\zeta_{28} + \zeta_{28}^{-1})$ is the compositum of $\mathbb{Q}(\sqrt{7})$ and the cubic real field $\mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, we are led to the following degree evaluation in case $4 \nmid n$,

$$[\mathbb{Q}(\zeta_{28} + \zeta_{28}^{-1}, \zeta_n, 3^{1/n}) : \mathbb{Q}] = \begin{cases} n\varphi(n) & \text{if } 42|n; \\ 2n\varphi(n) & \text{if } 7|n \text{ and } 6 \nmid n; \\ \varphi(\operatorname{lcm}(28, n))n/2 & \text{if } 7 \nmid n. \end{cases}$$

Note that since the Möbius function is zero for non-squarefree numbers, it is enough to compute the degree in case $4 \nmid n$. After some calculations using the Euler product in the form $\sum_{(n,m)=1} \mu(n)f(n) = \prod_{p \nmid m}(1 - f(p))$, the proof is completed. □

The reader interested in working out the details is referred to Moree [9, 10, 11] for similar computations that are worked out in more detail. Alternatively, we have the following rigorous proof.

Second proof. By [10, Theorem 1.2] we find that, under GRH, the density of the set of primes $q \equiv 1 \pmod{28}$, respectively $q \equiv -1 \pmod{28}$, that are Artin, is $18A/205$, respectively $14A/205$. □

The above proof shows that the Artin primes are not equidistributed over the primitive residue classes modulo 28. Indeed, by Moree [9, Theorem 1] they are not equidistributed over the primitive residue classes modulo d for any $d \geq 3$.

As a curiosity, we point out that the set of primes p such that 2 is a primitive root modulo p and p is in various residue classes modulo 28 appeared in work of Rodier [15] in connection with a coding theoretical problem involving Dickson polynomials.

`Commentary`. This section is new. In [12, Appendix A] the same method was used to deduce that, assuming the Generalized Riemann Hypothesis, $\delta(\mathcal{P}_1) = \delta(\mathcal{P}_2) = 3A/5$ and $\delta(\mathcal{P}_3) = 2A/5$.

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His passing away, a few months after the visit, came as very sad and unexpected news for the authors. This paper is dedicated to his memory.

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