LARGE SCALE PERTURBATIONS OF THE PLANE DIPOLE ARE NOT HESSIAN

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Abstract. The principal foliations of a surface in $\mathbb{R}^3$ correspond under the Gauss map (into the plane) to Hessian foliations and conversely [3]. A plane dipole foliation is not Hessian [3]. Indeed, no C^2 diffeomorphism of this foliation is Hessian [2]. A smooth singular foliation is an ε-perturbation of another if they have the same singularities and the angles between their leaves are $\leq \varepsilon$ at all nonsingular points. Our main result is that no ε-perturbation of the plane dipole foliation is Hessian and ε can be as large as $\frac{\pi}{4}$. This class of foliations includes many which are not diffeomorphic to a plane dipole (although all foliations in this class are locally homeomorphic to it).

1. Introduction

Carathéodory's Conjecture asserts that a 2-sphere immersed in $\mathbb{R}^3$ has at least two umbilics. An *umbilic* on a surface M immersed in $\mathbb{R}^3$ is a point where the principal curvatures coincide. Outside the set of umbilics the (mutually orthogonal) directions of extremal bending define the pair of *principal line fields* whose integral curves give the *principal foliations* of the immersion. Through Bonnet coordinates there is a simple equivalence between principal foliations in surface theory and Hessian foliations in the plane (see [3]). Roughly speaking, a continuous foliation $\mathcal{F}$ is *Hessian* if it is tangent to one of the line fields diagonalizing the Hessian operator Hess$[\omega]$ of a C^2 real-valued function ω (see §2). Through the above mentioned equivalence the umbilics, as singularities of the principal foliations, correspond to singularities of Hessian foliations (points where the Hessian operator is proportional to the identity). Thus, Carathéodory's Conjecture would be a consequence of Loewner's Conjecture that the index of an isolated singularity of a Hessian foliation is ≤ 1 (see [3]).

Any counterexample to the Carathéodory Conjecture on the sphere would have a single umbilic of index 2 and all principal curves circles, born and dying at the umbilic; this is an easy consequence of Bendixson's work (see [1], Theorems II, III, IV, pp. 11–18). The very simplest example of such a foliation is that consisting of all circles on S^2 with a common tangent – known as the spherical dipole foliation – and its significance for the conjecture was first recognised by Smyth and Xavier [3]. The proof that no diffeomorph (C^2 is enough) of this foliation ever occurs as a principal foliation on a surface in $\mathbb{R}^3$ is given by Lazarovici [2], and the result is even local in a neighborhood of the singularity. By the equivalence referred to above this is the same as saying that a foliation which is locally C^2-diffeomorphic to a

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plane dipole foliation (the image of a spherical dipole foliation under a stereographic projection) cannot be Hessian.

In this note we identify a large class of plane foliations with a single singularity which are not Hessian. These include examples which are far from being diffeomorphic to the dipole foliation.

Theorem. *A smooth orientable foliation on a punctured neighborhood of the origin in the plane which is a local ε-perturbation of a plane dipole cannot be Hessian if $\varepsilon \leq \frac{\pi}{4}$.*

The technique depends on a result from [3], which was also the source of the first observation that the standard plane dipole foliation is not Hessian and also of the first examples of smooth singular foliations of every index $j \in \frac{1}{2}\mathbb{Z}$ which are not Hessian.

2. Preliminaries

Plane vector fields may be identified with $\mathbb{C}$-valued functions and with this in mind the square of a plane vector field is unambiguously defined.

Let Ω be a neighborhood of the origin $o \in \mathbb{R}^2$. A C^1 unit vector field $X\colon \Omega \setminus \{o\} \to S^1$ with a singularity at o is said to be *Hessian at o* if there exists a C^2 real-valued function $\omega\colon \Omega' \to \mathbb{R}$ defined on a neighborhood $\Omega' \subset \Omega$ of o such that

$$\omega_{\bar{z}\bar{z}} = |\,\omega_{\bar{z}\bar{z}}\,| X^2$$

on $\Omega' \setminus \{o\}$ and $\omega_{\bar{z}\bar{z}}$ does not vanish in $\Omega' \setminus \{o\}$. Note that necessarily $\omega_{\bar{z}\bar{z}}(o) = 0$.

A C^1 unit vector field $X\colon \Omega \setminus \{o\} \to S^1$ is Hessian at o if and only if the foliation $\mathcal{X}$ of Ω generated by X is a *singular Hessian foliation* on some neighborhood $\Omega' \subset \Omega$ of o; that is, if and only if there exists a C^2 real-valued function $\omega\colon \Omega' \to \mathbb{R}$ defined on a neighborhood $\Omega' \subset \Omega$ of o whose Hessian operator $\mathrm{Hess}[\omega]$ is proportional to the identity only at o and for all $o \neq p \in \Omega'$, $\mathcal{X}_p$ is an eigenspace of $\mathrm{Hess}_p[\omega]$ (see [3]).

The vector field $D\colon \mathbb{C}^* \to S^1$ defined by

$$D(z) = z^2/|\,z\,|^2$$

is called the *standard plane dipole*. It generates the *standard plane dipole foliation* $\mathcal{D}$ whose leaves are the circles of all radii – including ∞ – passing through o tangent to the x-axis (see Figure 1).

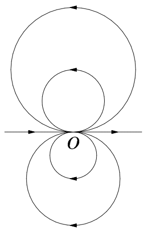

FIGURE 1. Standard plane dipole foliation

Let $\mathcal{D}_\alpha$ be the oriented foliation obtained from $\mathcal{D}$ by rotation through an angle α. The unit vector field D_α tangent to $\mathcal{D}_\alpha$ is then given by

$$D_\alpha(z) = e^{-i\alpha} z^2/|\, z\,|^2.$$

Any of the vector fields D_α is called a *plane dipole*; likewise, any of the foliations $\mathcal{D}_\alpha$ is called a *plane dipole foliation*.

If $\mathcal{X}$ is the oriented foliation generated by a unit vector field X, then its orthogonal is the oriented foliation generated by $iX = e^{i\frac{\pi}{2}}X$. Therefore the orthogonal of $\mathcal{D}_\alpha$ is $\mathcal{D}_{\alpha-\frac{\pi}{2}}$.

For $z \neq o$ denote the oriented leaf of $\mathcal{D}_\alpha$ through z by $\mathcal{D}_{\alpha,z}$ and denote by $\mathcal{D}^+_{\alpha,z}$ ($\mathcal{D}^-_{\alpha,z}$) the oriented part of $\mathcal{D}_{\alpha,z}$ that starts (ends) at z. Denote by $L_{\alpha,z}$ any bounded component of the complement of $\mathcal{D}_{\alpha,z} \cup \mathcal{D}_{\alpha-\frac{\pi}{2},z}$ whose boundary is either $\mathcal{D}^+_{\alpha,z} \cup \mathcal{D}^+_{\alpha-\frac{\pi}{2},z}$ or $\mathcal{D}^-_{\alpha,z} \cup \mathcal{D}^-_{\alpha-\frac{\pi}{2},z}$ (see Figure 2).

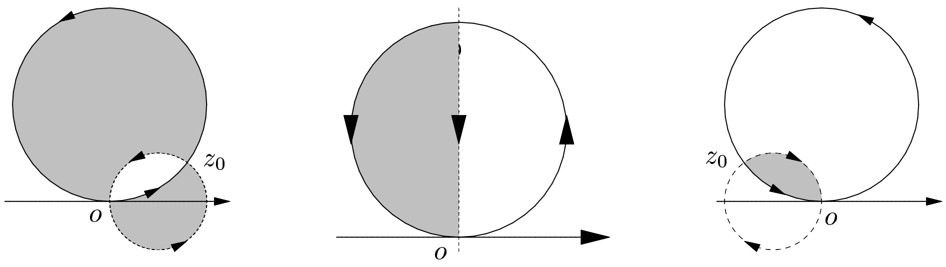

FIGURE 2. The L_{0,z_0} regions.

For each z in a disk of radius r about o at least one of the regions $L_{\alpha,z}$ is contained in a disk of radius $r\sqrt{2}$. If Ω is a neighborhood of o and r_Ω is its in–radius relative to o then we denote by $\Delta(\Omega)$ the open disk of radius $r_\Omega/\sqrt{2}$ centered at o. Then for any $z \in \Delta(\Omega) \setminus \{o\}$ and $\alpha \in \mathbb{R}$ at least one of the regions $L_{\alpha,z}$ is contained in Ω.

Lemma 2.1. *Let ω be a C^2 real-valued function which is $O(r^3)$ on a neighborhood Ω of o. Then for any $z_0 \in \Delta(\Omega) \setminus \{o\}$ and any $\alpha \in \mathbb{R}$ we have*

$$\omega(z_0) = 2|\, z_0\,|^2 \iint\limits_{L_{\alpha,z_0}} \mathbf{Im}\left\{ e^{2i\alpha} \frac{\bar{z}^4}{|\, z\,|^6}\, \omega_{\bar{z}\bar{z}} \right\} dx\, dy \tag{2.1}$$

Proof. The proof is almost a repetition of the first part of the proof of Lemma 2.2 in [3]: apply Riemann's method to the imaginary part of the identity

$$e^{2i\alpha}\frac{\bar{z}^4}{|z|^4}u_{\bar{z}\bar{z}} + e^{2i\alpha}\frac{2\bar{z}^3}{|z|^4}u_{\bar{z}} = e^{2i\alpha}\frac{\bar{z}^4}{|z|^6}\omega_{\bar{z}\bar{z}},$$

where $\omega = z\bar{z}u$. The only thing to note is that the $\pm$ characteristics through $z_0 \neq o$ of the resulting hyperbolic equation are, respectively, $\mathcal{D}_{\alpha,z_0}$ and $\mathcal{D}_{\alpha-\frac{\pi}{2},z_0}$. $\square$

One should note that (2.1) above is not a mere change of coordinates in the first identity of Lemma 2.2 of [3].

3. Proof of the Theorem

For $u, v \in S^1$ we define the *oriented angle* from u to v to be the unique $\phi \in [-\pi, \pi)$ such that $v = e^{i\phi}u$.

A unit vector field $X: \Omega \setminus \{o\} \to S^1$ is called a *local ε-perturbation* of another unit vector field $U: \Omega' \setminus \{o\} \to S^1$ if $\varepsilon < \pi$ and the oriented angle $\phi(z)$ from $U(z)$ to $X(z)$ is in absolute value $\leq \varepsilon$ for all $z \neq o$ in some neighborhood of o. If $\mathcal{X}$ and $\mathcal{U}$ are two orientable foliations of a punctured neighborhood of o, we say that $\mathcal{X}$ is a *local ε-perturbation* of $\mathcal{U}$ if the respective oriented unit vector fields X and U of these two foliations are ε-perturbations of each other, or, if X and $-U$ are ε-perturbations of each other.

Proof of the Theorem. Let $\mathcal{X}$ be an orientable smooth plane foliation on a neighborhood Ω of o with an isolated singularity at o which is an ε-perturbation of a plane dipole, $\varepsilon < \frac{\pi}{4}$, and let X be a unit vector field tangent to $\mathcal{X}$.

Suppose to the contrary that X is Hessian at o.

Note first that we may assume X to be a local ε-perturbation of the standard plane dipole; if X is an ε-perturbation of D_α then we can rotate the coordinates by α and the image of $\mathcal{X}$ is an ε-perturbation of the standard plane dipole. However the property of being a Hessian foliation is invariant under rotation, thus we may assume $\mathcal{X}$ is Hessian and an ε-perturbation of the standard plane dipole.

Then, by restricting Ω if necessary, there exists a C^2 function $\omega: \Omega \to \mathbb{R}$ such that

$$\omega_{\bar{z}\bar{z}} = |\,\omega_{\bar{z}\bar{z}}\,| X^2 \tag{3.1}$$

on $\Omega \setminus \{o\}$ and $\omega_{\bar{z}\bar{z}}$ does not vanish in $\Omega \setminus \{o\}$. Since X has a singularity at o we must have $\omega_{\bar{z}\bar{z}}(o) = 0$. Thus, the 2-jet ω_2 of ω has the form $\omega_2 = \omega(o) + \omega_x(o)x + \omega_y(o)y + k(x^2 + y^2)$ so $\omega - \omega_2$ satisfies (3.1). We may therefore assume that ω is $O(r^3)$ at o.

Let $\phi: \Omega \setminus \{o\} \to \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ be the oriented angle from the standard plane dipole to X so that $|\phi(z)| \leq \varepsilon$.

Using $X = e^{i\phi}D$ and (3.1) in (2.1) we obtain that

$$\omega(z_0) = 2|\,z_0\,|^2 \iint\limits_{L_{\alpha, z_0}} \frac{|\,\omega_{\bar{z}\bar{z}}\,|}{|\,z\,|^2} \sin(2\alpha + 2\phi)\, dx\, dy$$

for all $z_0 \in \Delta(\Omega) \setminus \{o\}$ and all $\alpha \in \mathbb{R}$. Choosing $\alpha = \pm\frac{\pi}{4}$ we obtain that

$$\iint\limits_{L_{\frac{\pi}{4}, z_0}} \frac{|\,\omega_{\bar{z}\bar{z}}\,|}{|\,z\,|^2} \cos(2\phi)\, dx\, dy = -\iint\limits_{L_{-\frac{\pi}{4}, z_0}} \frac{|\,\omega_{\bar{z}\bar{z}}\,|}{|\,z\,|^2} \cos(2\phi)\, dx\, dy.$$

Now note that the interiors of $L_{\frac{\pi}{4}, z_0}$ and $L_{-\frac{\pi}{4}, z_0}$ are disjoint and that $L_{\frac{\pi}{4}, z_0} \cup L_{-\frac{\pi}{4}, z_0}$ is a disk $\Delta(z_0)$ whose boundary is a circle passing through z_0 and tangent at o to either $y = x$ or $y = -x$ (the orthogonal of $\mathcal{D}_{-\frac{\pi}{4}}$ is $\mathcal{D}_{-\frac{3\pi}{4}}$ which has the same leaves as $\mathcal{D}_{\frac{\pi}{4}}$ but with the opposite orientation; see Figure 3).

We therefore obtain that

$$\iint\limits_{\Delta(z_0)} \frac{|\,\omega_{\bar{z}\bar{z}}\,|}{|\,z\,|^2} \cos(2\phi)\, dx\, dy = 0,$$

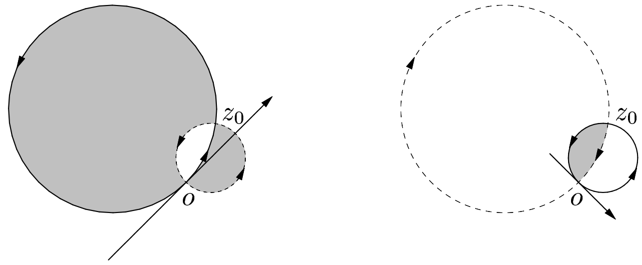

FIGURE 3. $L_{\frac{\pi}{4},z_0}$ and $L_{-\frac{\pi}{4},z_0}$

for all $z_0 \in \Delta(\Omega) \setminus \{o\}$.

Note that as z_0 varies in $\Delta(\Omega)\setminus\{o\}$ the disks $\Delta(z_0)$ cover a neighborhood $\Omega' \subset \Omega$ of o. Now by assumption the integrand in this integral is non-negative, so, if there exists a point $z \in \Omega'$ such that $|\phi(z)| < \frac{\pi}{4}$ then the double integral above is strictly positive for some z_0 with $z \in \Delta(z_0)$ and therefore X could not be Hessian. If $\phi \equiv \frac{\pi}{4}$ ($\phi \equiv -\frac{\pi}{4}$) then we actually have $X = D_{-\frac{\pi}{4}}$ ($X = D_{\frac{\pi}{4}}$) which again, since no plane dipole can be Hessian, shows that X cannot be Hessian. $\square$

4. Examples

The examples show that neither the class of local diffeomorphisms of plane dipoles nor the class of ε-perturbations of plane dipoles, where $\varepsilon < \frac{\pi}{4}$, contains the other. This is to be viewed from the perspective of our result in [2] where we showed that a local C^2-diffeomorphism of a plane dipole cannot be Hessian.

Example 4.1. In this example we construct a unit vector field $X: \mathbb{R} \setminus \{o\} \to S^1$ which is an ε-perturbation of the standard plane dipole with $\varepsilon < \frac{\pi}{4}$ but is not a local diffeomorphism of any plane dipole.

Let $\eta: \mathbb{C}^* \to \mathbb{C}^*$, $\eta(z) = z/|\,z\,|^2$, be the inversion in the unit circle. Note that $\eta^{-1} = \eta$. The standard plane dipole foliation is mapped by η onto the foliation of $\mathbb{R}^2$ by horizontal lines with the orientation given by the constant vector field $(-1, 0)$. Let $f: \mathbb{R} \to \mathbb{R}$ be a C^1 function with $|f'(x)| \le \tan\varepsilon$, where $\varepsilon \in \left[0, \frac{\pi}{4}\right)$. Let $Y(x, y) = (-1, -f'(x))$ and let $X = \eta_*(Y)/|\eta_*(Y)|$. Since η is a conformal map

$$\langle X, D\rangle = \left\langle \frac{Y}{|Y|}, (-1,0)\right\rangle \ge \frac{1}{\sqrt{1+\tan^2\varepsilon}} = \cos\varepsilon$$

so that X is an ε-perturbation of the standard plane dipole. Choosing f such that f' does not have a limit as $x \to \pm\infty$ (for example $f(x) = \tan\varepsilon \sin x$) we obtain a vector field X which does not have a limit as it approaches o along any leaf. This shows that X cannot be diffeomorphic to a plane dipole (since the plane dipole D_α tends to $e^{i\alpha}$ as it approaches o along any of its leaves and any of its local diffeomorphs will have the same behavior).

Example 4.2. We will now construct a diffeomorph E of the standard plane dipole which does not satisfy the conditions of the Theorem, i.e., which is not an ε-perturbation of any plane dipole ($\varepsilon < \frac{\pi}{4}$). The simple elliptic version of the standard plane dipole given below is such an example and the diffeomorphism can be explicitly written down.

Let $E(x,y) = (x^2 - k^2y^2, 2xy)$, $k > 0$. The foliation $\mathcal{E}$ generated by E consists of the family of ellipses $\frac{x^2}{k^2h^2} + \frac{(y-h)^2}{h^2} = 1$, $h \neq 0$ together with the x-axis except o. Note that $\mathcal{E}$ is the image of the standard plane dipole foliation $\mathcal{D}$ under the linear transformation $T(x,y) = (kx, y)$.

Let $\phi\colon \mathbb{R}^2 \setminus \{o\} \to (-\pi, \pi]$ be the oriented angle from D to E. We will show that if $k \geq 3$ then $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \subseteq \phi\big(\Delta \setminus \{o\}\big)$ for any disk Δ centered at o. Notice that, since both E and D are homogeneous maps , we have $\phi\big(\Delta \setminus \{o\}\big) = \phi(S^1)$ for any disk Δ centered at o.

To simplify the notation we set

$$N(x,y) = (x^2+y^2)|E(x,y)| = (x^2+y^2)\sqrt{(x^2-k^2y^2)^2 + 4x^2y^2}$$

Then ($\sin\phi = \langle E/|E|, iD\rangle$)

$$\cos\big(\phi(x,y)\big) = \frac{x^4 + (3-k^2)x^2y^2 + k^2y^4}{N(x,y)} \quad \text{and} \quad \sin\big(\phi(x,y)\big) = \frac{2(k^2-1)xy^3}{N(x,y)}.$$

These formulas show that $0 \leq \phi(e^{i\theta}) < \pi$ for $0 \leq \theta \leq \frac{\pi}{2}$ and $\phi(e^{i(\pi-\theta)}) = -\phi(e^{i\theta}) = -\phi(-e^{i\theta})$. We conclude that $\phi(S^1)$ is a symmetric interval $[-\alpha, \alpha]$ for some $\alpha = \alpha(k) \in (0, \pi)$ and $[0, \alpha] = \phi(Q)$, where $Q = \{e^{i\theta} \,|\, 0 \leq \theta \leq \frac{\pi}{2}\}$.

With $t = \tan^2\theta$ we have

$$c(\theta) \colon= \cos\big(\phi(e^{i\theta})\big) = \frac{k^2t^2 + (3-k^2)t + 1}{(t+1)\sqrt{(k^2t-1)^2 + 4t}} =\colon f(t),$$

and a straightforward computation shows that

$$f'(t) = \frac{2(k^2-1)^2t^2(k^2t-3)}{(t+1)^2((k^2t-1)^2+4t)^{3/2}}.$$

Let $\theta_0 = \arctan(\sqrt{3}/k)$ and choose $k \geq 1$. We obtain that $c(\theta)$ is decreasing for $\theta \in [0, \theta_0]$ and increasing for $\theta \in [\theta_0, \frac{\pi}{2}]$. Thus $\phi(Q) = [0, \alpha]$, where $\alpha = \arccos\big(c(\theta_0)\big)$. The numerator of f takes nonpositive values if $k^2 \geq 3$, and with this choice for k we make sure that $\alpha \geq \frac{\pi}{2}$ (see Figure 4).

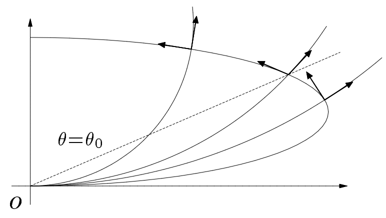

FIGURE 4.

Finally, note that $\sup \phi = \alpha$ and $\inf \phi = -\alpha$ and so if $\tilde{\phi}$ is the oriented angle from another plane dipole to E then either $\sup \tilde{\phi} > \alpha$ or $\inf \tilde{\phi} < -\alpha$. This shows that E cannot be an ε-perturbation of any plane dipole if $\varepsilon < \frac{\pi}{4}$.

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