BOUNDS AND $\mathbb{Q}$-GORENSTEIN SMOOTHINGS OF SMOOTHABLE STABLE LOG SURFACES

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Dedicated to Professor Herb Clemens on the occasion of his sixtieth birthday

Abstract. This paper considers bounds and $\mathbb{Q}$-Gorenstein smoothings of smoothable stable log surfaces. The positivity of the second Chern class of smoothable stable log surfaces and its relation of bounds are presented. By studying the deformation theory of smoothable stable log surfaces, we provide the infinitesimal smoothable deformations explicitly under the assumption of permissble singularities. The existence of a $\mathbb{Q}$-Gorestein smoothing of a normal projective surface with a permissible singularity is interpreted by the relation between the tangent bundle of its minimal resolution and the normal bundle of the exceptional divisor.

1. Introduction

It has been known for a long time that the set of all curves of a fixed genus g naturally corresponds to the points on a certain variety with dimension $3g-3$, which is denoted by M_g. The moduli space M_g is not projective, and there has been considerable work in finding a natural compactification. An answer was provided by Deligne and Mumford in [DM] by introducing stable curves. A stable curve of genus g is defined as a 1-dimensional projective connected curve C which has only ordinary nodes as singularities such that ω_C is ample. This compactification is called Deligne-Mumford compactification of M_g, denoted by $\overline{M_g}$. The projectivity of $\overline{M_g}$ over any field is accomplished by using the geometric invariant theory and the theory of moduli.

Via the geometric invariant theory, Gieseker [Gi] succeeded in proving the existence of a coarse moduli space $M_{K^2,\chi}$ for canonically embedded smooth projective surfaces of general type with fixed numerical invariants K^2 and χ. The next natural problem is how to construct a geometric compactification of moduli of surfaces via similar way as $\overline{M_g}$: With the proof of the boundedness conjecture in dimension two, Alexeev [A] finished the construction of projective coarse moduli space of surfaces of general type with fixed K^2 that was started in [KSB]. The compactified moduli space should include (possibly reducible) surfaces with ordinary double curves and certain other mild singularities. These surfaces are called smoothable stable surfaces. This notion of smoothable stable surface can be generalized to smoothable stable log surface similar as the generalization of stable curve to stable

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pointed curve. The notion of nodes on a curve can be changed to semi log canonical singularities in higher dimensions.

Definition. (cf. [K et], [KM]) Let X be a normal surface and B an effective $\mathbb{Q}$-divisor such that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $f\colon Y \to X$ be a proper birational morphism. We write

$$K_Y \equiv f^*(K_X + B) + \sum a_i E_i.$$

Define discrep $(X, B) = \inf_{E_i}\{a_i | E_i$ is exceptional, and $Center_X(E_i) \neq \emptyset\}$. We say that (X, B), or $K_X + B$ is

$$\left.\begin{array}{l}\text{terminal}\\ \text{canonical}\\ \text{purely log terminal}\\ \text{log canonical}\end{array}\right\} \quad \text{if} \quad \text{discrep}(X,B) \quad \left\{\begin{array}{l} > 0,\\ \geq 0,\\ > -1,\\ \geq -1.\end{array}\right.$$

Moreover, (X, B) is *divisorial log terminal* (dlt) if there exists a log resolution such that the exceptional locus consists of divisors with all $a_i > -1$.

And we say that a log surface (X, B) has semi log canonical singularities if
(1) X satisfies Serre's condition S_2,
(2) X has normal crossing singularities in codimension one,
(3) $K_X + B$ is $\mathbb{Q}$-Cartier, and for any birational morphism $f\colon Y \to X$ from a normal $\mathbb{Q}$-Gorenstein surface Y we have

$$K_Y \equiv f^*(K_X + B) + \sum a_i E_i$$

where all $a_i \geq -1$.

Definition. A stable log surface is the pair (X, B) where

(1) X is a connected projective surface and B a reduced Weil divisor on X,
(2) (condition on singularities) the pair (X, B) has semi log canonical singularities,
(3) (numerical condition) $K_X + B$ is ample.

It is well known that a stable log surface (X, B) has finite automorphism groups. With the proof of boundedness conjecture, Alexeev [A] finished the construction of projective coarse moduli space of surfaces of general type with fixed K^2. And he proved that the existence of coarse moduli space for stable log surfaces by the log minimal model program (log MMP) of threefolds. Every stable curve is smoothable. But there is a small submoduli space of smoothable stable log surfaces inside of a moduli space of stable log surfaces. Smoothability means that (X, B) admits a deformation $(\mathcal{X}, \mathcal{B})$ over a small disk Δ in $\mathbb{C}$ where a generic fiber X_t has at most DuVal singularities. Furthermore $K_{\mathcal{X}} + \mathcal{B}$ is $\mathbb{Q}$-Cartier. Smoothable stable surfaces $(B = \emptyset)$ are added to the boundary points in the compactified the moduli space of smooth surfaces of general type.

One can replace a general relatively log minimal model $(\mathcal{X}, \mathcal{B}) \to \Delta$ with another relatively log minimal model, which has better singularities via base change and simultaneous resolution [Ka]. We introduce the notion of permissible singularities due to Kawamata [Ka].

Definition. Let $\pi\colon (\mathcal{X},\mathcal{B}) \to \Delta$ be a deformation of a smoothable stable log surface (X,B). We say that π is a *permissible degeneration* if the following conditions are satisfied: The analytic germ of each point $p \in (X,B)$ is isomorphic to one of the following germs, where $\pi = t$:
(1.3.1) $(xyz = t)$,
(1.3.2) $(xy = t) \subset A$, $A = \frac{1}{r}(a, r-a, 1, 0)$ for $(a,r) = 1$,
(3.3.3) $(xy = z^r + t^n) \subset A$ for some n.

In a permissible degeneration of (X,B), the central fiber has smooth double curves, and each component of (X,B) has cyclic quotient singularities of the following two types: $\frac{1}{r}(1,a)$ on the double curves and B, and $\frac{1}{r^2}(a, r-a)$ outside the double curves and B. And $(\mathcal{X},\mathcal{B})$ admits a semistable model by the succession of weighted blowing up of singular points. This construction is explicitly given in [C]. We say that a pair (X,B) is a log surface with *permissible singularities* if it can be the central fiber of a permissible degeneration. And we say that a pair (X,B) has a *permissible singularity at* p if local germ at p has a permissible degeneration.

The paper goes as follows:

Section 2 presents the positivity of the second Chern class of smoothable stable log surfaces and its relation of boundedness. In particular, it holds the following:

Theorem. Let (X,B) be a smoothable stable log surface and let $(\mathcal{X},\mathcal{B})$ be a deformation of (X,B). Then the number of components in (X,B) is bounded by the second Chern class of a generic fiber $c_2(X_t, B_t) = e_{top}(X_t) - e_{top}(B_t)$.

In Section 3, we study the deformation theory of smoothable stable log surfaces. Our main concern is the case of smoothable stable log surfaces with permissible singularities. We obtain the following for the infinitesimal smoothable deformations:

Theorem. Let (X,B) be a smoothable stable log surface with permissible singularities. Assume that B is Cartier. Then

$$\tilde{\mathcal{T}}^1_X = \oplus_p \mathbb{C}_p \oplus \mathcal{O}_D(\sum q)$$

where p is a singular point of X outside double curves D and q a singular point of X on D.

Let $\overline{P}_d$ denote the compactification of a coarse moduli space for smooth plane curves of degree $d \geq 4$ in the moduli space of stable log surfaces. Then it holds the following as a corollary:

Corollary. Let (X,B) be in $\overline{P}_d$ with permissible singularities. Assume that B is Cartier. Then

$$\ker[H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))] = 0.$$

In [L3], we intend to generalize the result without assuming that (X,B) has permissible singularities and B is Cartier. Corollary can be generalized to $(X,B) \in \overline{P}_d$ with semi-log terminal singularities. The details will be given in [L3].

In Section 4, we study a $\mathbb{Q}$-Gorestein smoothing of a normal projective surface with a permissible singularity. The existence of smoothing can be interpreted by the relation between the tangent bundle of its minimal resolution and the normal bundle of the exceptional divisor. It holds the following:

Theorem. *Let U be a normal projective surface with a permissible singularity. Let $f\colon V \to U$ be the minimal resolution of U and E the exceptional divisor. Assume that the map $\mathrm{T}^1_V \to H^1(E, \mathcal{O}_E(E))$ is surjective and that $\mathrm{T}^2_V = 0$. Then U has a $\mathbb{Q}$-Gorenstein smoothing.*

We work throughout over the complex number field $\mathbb{C}$. The notation here follows Hartshorne's Algebraic Geometry.

2. Numerical bounds for the number of components of smoothable stable log surfaces

If X is a smooth minimal projective surface then Bogomolov-Miyaoka-Yau inequality implies that $c_2(X) \geq \frac{1}{3}K_X^2$. This inequality can be generalized to stable log surfaces. Let (Y, D) be a dlt pair with D a reduced Weil divisor. The second Chern class of a dlt pair can be defined as orbifold Euler number (cf. [Mi], [W3]). Define $\mathrm{Sing}(Y, D)$ to be the set of singular points of Y outside D. Then

$$c_2(Y, D) = e_{top}(Y) - e_{top}(D) - \sum_{p \in \mathrm{Sing}(Y,D)} (1 - \frac{1}{r(p)})$$

where $r(p)$ is the local orbifold fundamental group. And $c_1(Y, D) = K_Y + D$. This definition can be generalized to a log canonical pair by defining $r(p) = \infty$ if p is a log canonical singularity which is not purely log terminal.

Let (X, B) be a stable log surface. Then each component of the normalization of (X, B) has log canonical singularities. Assume that X is normal. Then by the well-known theorem of dlt model of a log canonical pair of surfaces, the proof can be obtained by the classification of log canonical surfaces, there exists a projective birational morphism $f\colon Y \to X$ with (Y, D) a dlt pair such that
(1) $K_Y + D = f^*(K_X + B)$
(2) D is a reduced Weil divisor
(3) $c_2(X, B) = c_2(Y, D)$.
By the result for a dlt pair (cf. [Mi], [W3]), it holds the following:

Theorem 2.1. (cf. [Me]) Let (X, B) be a log canonical pair with B a reduced Weil divisor such that $K_X + B$ is nef. Then $c_2(X, B) \geq \frac{1}{3}(K_X + B)^2$. In particular $c_2(X, B) > 0$ if $K_X + B$ is ample.

By this log version of Bogomolov-Miyaoka-Yau inequality, we obtain a numerical bounds for the number of components of smoothable stable log surfaces, which is a generalization of the result in [L1].

Theorem 2.2. Let (X, B) be a smoothable stable log surface and let $(\mathcal{X}, \mathcal{B})$ be a deformation of (X, B). Then the number of components in (X, B) is bounded by the second Chern class of a generic fiber $c_2(X_t, B_t)$.

Proof. Since the proof is obtained by the same method in [L1], we provide a sketch of the proof. Let (X, B) be a smoothable stable log surface. Then by 3-dimensional log MMP there exists a semi-dlt model (Y, D) (the pair of a component Y_i and $D_i + F_i$ is a dlt pair where $D_i = D|_{Y_i}$, F_i is a double curve in Y_i) such that

(1) D is a reduced Weil divisor
(2) $K_Y + D$ is nef.

We change this log minimal model to another log minimal model (relatively minimal permissible model, cf. [Ka], [L1]) which admits a semistable model (in the sense of semistable reduction theorem). By this process, we can compare between the second Chern class of the central fiber and the second Chern class of a general fiber:

$$c_2(X_t, B_t) = c_2(Y_t, B_t) \geq \sum e_{top}(Y_i) - e_{top}(D_i + F_i).$$

On the other hand, by Theorem 2.1,

$$\begin{aligned} e_{top}(Y_i) - e_{top}(D_i + F_i) &= c_2(Y_i, D_i + F_i) + \sum_{p \in \operatorname{Sing}(Y_i, D_i + F_i)} (1 - \frac{1}{r(p)}) \\ &\geq \frac{1}{3}(K_{Y_i} + D_i + F_i)^2 + \sum_{p \in \operatorname{Sing}(Y_i, D_i + F_i)} (1 - \frac{1}{r(p)}). \end{aligned}$$

In particular, $e_{top}(Y_i) - e_{top}(D_i + F_i) > 0$ if Y_i is not contracted to a point or a curve by the log MMP. Therefore we obtain the proof. $\square$

Example 2.3. Let $X = C \times C$ where C is a union of two smooth elliptic curves E_1, E_2 with $E_1 \cap E_2 = \{p\}$. Then X is a smoothable stable surface with $12\chi(\mathcal{O}_X) - K_X^2 = 4$. Therefore the inequality in Theorem 2.2 (the case of $B = \emptyset$) is sharp.

Let $Z = E \times E$ such that E is a smooth elliptic curve and p_1, p_2 are two natural projections. Let $B = p_1^*(p) + p_2^*(q)$ where p, q are points in E. Then (Z, B) is a smoothable stable log surface with $c_2(Z, B) = 1$. The inequality in the case of $B \neq \emptyset$ is also sharp.

3. Deformation theory

Let (X, B) be a semi log canonical surface with B a reduced Weil divisor on X. By the formalism of deformation of pairs [R] we have:

$0 \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_B \oplus \mathrm{T}^0_X \to \mathrm{Hom}_X(\Omega_X, \mathcal{O}_B) \to \mathrm{T}^1_{X,B} \to \mathrm{T}^1_B \oplus \mathrm{T}^1_X$
$\to \mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_B) \to \mathrm{T}^2_{X,B} \to \mathrm{T}^2_X$.

The short exact sequence $0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to \mathcal{O}_B \to 0$, where $\mathcal{O}_X(-B)$ denotes the ideal sheaf of B, induces the long exact sequence:

$0 \to \mathrm{T}^0_X(-B) \to \mathrm{T}^0_X \to \mathrm{Hom}_X(\Omega_X, \mathcal{O}_B) \to \mathrm{T}^1_X(-B) \to \mathrm{T}^1_X \to \mathrm{Ext}^1_X(\Omega_X, \mathcal{O}_B)$
$\to \mathrm{T}^2_X(-B) \to \mathrm{T}^2_X$ where $\mathrm{T}^i_X = \mathrm{Ext}^i_X(\Omega^1_X, \mathcal{O}_X)$, $\mathrm{T}^i_X(-B) = \mathrm{Ext}^i_X(\Omega^1_X, \mathcal{O}_X(-B))$ for $i = 0, 1, 2$. By these two long exact sequences, we obtain a long exact sequence for deformation of pairs, which is a generalization of the long exact sequence in a smooth case (cf. [H]):

$$\begin{aligned} &0 \to \mathrm{T}^0_X(-B) \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_B \\ &\to \mathrm{T}^1_X(-B) \to \mathrm{T}^1_{X,B} \xrightarrow{t} \mathrm{T}^1_B \\ &\to \mathrm{T}^2_X(-B) \to \mathrm{T}^2_{X,B}. \end{aligned} \tag{3.1}$$

Assume that the index of $K_X + B$ is the same as the index of K_X for any singular points in (X, B). In particular, this is true if B is Cartier, or if (X, B) has permissible singularities. Consider the sheaves $\mathcal{T}^i_X = \mathcal{E}xt^i_X(\Omega^1_X, \mathcal{O}_X)$ for $i = 0, 1, 2$. Define the subsheaves $\tilde{\mathcal{T}}^i_X$ in $\mathcal{T}^i_X$ which represent the infinitesimal smoothable deformations: Let $U \subset X$ be an analytic neighborhood with an index one cover U'.

For the case of the field $\mathbb{C}$, this index one cover is unique up to isomorphism. Define $\tilde{\mathcal{T}}^i_X(U)$ as the invariant part of $\mathcal{T}^i_{U'}$. By this definition, it is obvious that $\tilde{\mathcal{T}}^i_X \subset \mathcal{T}^i_X$ and $\tilde{\mathcal{T}}^0_X = \mathcal{T}^0_X$. By the assumption, U' is also an index one cover of $K_X + B$. So we define $\tilde{\mathcal{T}}^i_X(-B)$ and $\tilde{\mathcal{T}}^i_{X,B}$ in the same way.

Define $\tilde{\mathrm{T}}^0_X = \mathrm{T}^0_X$, and $\tilde{\mathrm{T}}^1_X$ as the elements of T^1_X mapped to $H^0(\tilde{\mathcal{T}}^1_X)$. And the obstruction map can be computed by $\tilde{\mathrm{T}}^1_X \subset \mathrm{T}^1_X \xrightarrow{ob} \mathrm{T}^2_X$. Define $\tilde{\mathrm{T}}^0_{X,B} = \mathrm{T}^0_{X,B}$, and $\tilde{\mathrm{T}}^1_{X,B}$ as the elements of $\mathrm{T}^1_{X,B}$ mapped to $\tilde{\mathrm{T}}^1_X$. Also we define $\mathrm{T}^0_X(-B)$, $\tilde{\mathrm{T}}^1_X(-B)$ in the same way. Then we have a long exact sequence for a smoothable semi log canonical surface (X,B) with B a reduced Weil divisor on X:

$$0 \to \mathrm{T}^0_X(-B) \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_B \to \tilde{\mathrm{T}}^1_X(-B) \to \tilde{\mathrm{T}}^1_{X,B} \xrightarrow{t} \mathrm{T}^1_B\,. \tag{3.2}$$

If (X,B) is a smoothable stable log surface then it holds the long exact sequence $0 \to \tilde{\mathrm{T}}^1_X(-B) \to \tilde{\mathrm{T}}^1_{X,B} \to \mathrm{T}^1_B$ by the stability condition. Also $\tilde{\mathrm{T}}^1_X$ and $\tilde{\mathrm{T}}^1_X(-B)$ can be computed by the local-global spectral sequence ([Go], §7.3):
$0 \to H^1(\mathcal{T}^0_X) \to \tilde{\mathrm{T}}^1_X \to \ker[H^0(\tilde{\mathcal{T}}^1_X) \to H^2(\mathcal{T}^0_X)] \to 0$
$0 \to H^1(\mathcal{T}^0_X(-B)) \to \tilde{\mathrm{T}}^1_X(-B) \to \ker[H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))] \to 0.$
Therefore we slightly generalize the result in [H].

Lemma 3.1. *Let (X,B) be a smoothable semi log canonical surface with B a reduced Weil divisor on X such that the index of $K_X + B$ is the same as the index of K_X for any singular points in (X,B). Assume that $H^1(\mathcal{T}^0_X(-B)) = 0$ and that $H^0(\tilde{\mathcal{T}}^1_X(-B))$ injects into $H^2(\mathcal{T}^0_X(-B))$. Then $\tilde{\mathrm{T}}^1_{X,B}$ injects into T^1_B.*

Assume that B is Cartier. Then since the short exact sequence $0 \to \mathcal{O}_X(-B) \to \mathcal{O}_X \to \mathcal{O}_B \to 0$ gives a resolution of $\mathcal{O}_B$ by invertible sheaves, we have a long exact sequence for deformation of pairs

$$\begin{aligned} 0 &\to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_X \to H^0(B,\mathcal{O}_B(B)) \\ &\to \mathrm{T}^1_{X,B} \to \mathrm{T}^1_X \xrightarrow{n} H^1(B,\mathcal{O}_B(B)) \\ &\to \mathrm{T}^2_{X,B} \to \mathrm{T}^2_X\,. \end{aligned} \tag{3.3}$$

And we have a long exact sequence for smoothable deformation of pairs:

$$0 \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_X \to H^0(B,\mathcal{O}_B(B)) \to \tilde{\mathrm{T}}^1_{X,B} \to \tilde{\mathrm{T}}^1_X \xrightarrow{n} H^1(B,\mathcal{O}_B(B)). \tag{3.4}$$

Lemma 3.2. *Let (X,B) be a smoothable semi log canonical surface with B a reduced Weil divisor on X. Assume that B is Cartier. Then there are two long exact sequences for deformation of pairs, which are generalization of the long exact sequences in a smooth case:*
(3.2.1) $0 \to \mathrm{T}^0_X(-B) \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_B \to \tilde{\mathrm{T}}^1_X(-B) \to \tilde{\mathrm{T}}^1_{X,B} \to \mathrm{T}^1_B$
(3.2.2) $0 \to \mathrm{T}^0_{X,B} \to \mathrm{T}^0_X \to H^0(B,\mathcal{O}_B(B)) \to \tilde{\mathrm{T}}^1_{X,B} \to \tilde{\mathrm{T}}^1_X \to H^1(B,\mathcal{O}_B(B)).$
And $\tilde{\mathrm{T}}^1_X$ and $\tilde{\mathrm{T}}^1_X(-B)$ can be computed by the local-global spectral sequence:
(3.2.3) $0 \to H^1(\mathcal{T}^0_X) \to \tilde{\mathrm{T}}^1_X \to \ker[H^0(\tilde{\mathcal{T}}^1_X) \to H^2(\mathcal{T}^0_X)] \to 0$
(3.2.4) $0 \to H^1(\mathcal{T}^0_X(-B)) \to \tilde{\mathrm{T}}^1_X(-B) \to \ker[H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))] \to 0.$

Let (X,B) be a smoothable stable log surface with permissible singularities. Assume that B is Cartier. From the definition of permissible singularities, this assumption is the same as the smoothness of X on B.

Theorem 3.3. Let (X, B) be a smoothable stable log surface with permissible singularities. Assume that B is Cartier. Then it holds the following:

$$\tilde{\mathcal{T}}^1_X = \oplus_p \mathbb{C}_p \oplus \mathcal{O}_D(\sum q)$$

where p is a singular point of X outside double curves D, and q is a singular point of X on D.

Proof. Note that the singular points of (X, B) are isolated except double curves. Let (X, p) be the germ of singular points outside of double curves. Then by the result in [KSB], $\tilde{\mathcal{T}}^1_{X,p} = \mathbb{C}_p$.

Consider a semistable resolution of a permissible deformation $(\mathcal{X}, \mathcal{B})$ of (X, B) by the succession of weighted blowing up of singular points of $(\mathcal{X}, \mathcal{B})$. Since this semistable model is a d-semistable in the sense of Friedman [F] and each singularity is a rational singular point, we have $\mathcal{O}_D \subset \tilde{\mathcal{T}}^1_X$. Therefore it is clearly $\tilde{\mathcal{T}}^1_X = \oplus_p \mathbb{C}_p \oplus \mathcal{O}_D(\sum n_q q)$. Then we can prove that each $n_q = 1$ similarly as the argument in [H]: Since a permissible degeneration is smooth at the point analytically isomorphic to $(xyz = 0)$, we may assume that two components meets at q. It gives a proof of $\tilde{\mathcal{T}}^1_X = \oplus_p \mathbb{C}_p \oplus \mathcal{O}_D(\sum q)$. $\square$

Let P_d be a coarse moduli space for smooth plane curves of degree $d \geq 4$. For each $C \in P_d$, $(\mathbb{P}^2, C)$ is a stable log surface. Let $\overline{P}_d$ denote the closure of P_d in the moduli space of smoothable stable log surfaces. We have a morphism $j\colon P_d \to M_{g(d)}$ where $M_{g(d)}$ is the moduli space of curves of genus $g(d) = \frac{1}{2}(d-1)(d-2)$. The morphism j extends to $\bar{j}\colon \overline{P}_d \to \overline{M}_{g(d)}$ because B has only nodal singularities for $(X, B) \in \overline{P}_d$ and $K_X + B|_B = K_B$ is ample. Since j is isomorphic onto its image, $\bar{j}$ is birational onto its image.

Question. Is $\bar{j}\colon \overline{P}_d \to \overline{M}_{g(d)}$ isomorphic onto its image? Does $\bar{j}$ ever have positive dimension fibers?

In [H], he prove that $\bar{j}\colon \overline{P}_4 \to \overline{M}_3$ is an isomorphism. But in general $\bar{j}$ is not isomorphic. By Lemmas and Theorem 3.3, it holds the following:

Corollary 3.4. Let (X, B) be in $\overline{P}_d$ with permissible singularities. Assume that B is Cartier. Then

$$\ker[H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))] = 0.$$

Proof. If there is a nontrivial element in the kernel of the map $H^0(\tilde{\mathcal{T}}^1_X(-B)) \to H^2(\mathcal{T}^0_X(-B))$, then it gives a nontrivial element in the kernel of the map $H^0(\tilde{\mathcal{T}}^1_X) \to H^2(\mathcal{T}^0_X)$ by a commutative diagram. Embedded resolution of a plane curve singularity produces only a tree of $\mathbb{P}^1$. And since the log MMP program preserves a tree of $\mathbb{P}^1$, it holds the vanishing of $H^1(\tilde{\mathcal{T}}^1_X)$ by Theorem 3.3. Therefore we have a deformation of smoothings $(\mathcal{X}, \mathcal{B})$ of (X, B) without changing B. This gives a contradiction because the map j is isomorphic onto its image. $\square$

4. $\mathbb{Q}$-Gorenstein smoothings of a permissible singularity

Let U be a normal projective surface with a rational singularity p. Let $f: V \to U$ be the minimal resolution of U and E the exceptional divisor. Since $f_* \mathcal{T}_V = \mathcal{T}^0_U$

[BW] we have a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H^1(f_*\mathcal{T}_V) & \longrightarrow & \mathrm{T}^1_V & \longrightarrow & H^0(R^1f_*\mathcal{T}_V) & \longrightarrow & H^2(f_*\mathcal{T}_V) \\
 & & {\scriptstyle isom}\big\downarrow & & {\scriptstyle D_f}\big\downarrow & & & & {\scriptstyle isom}\big\downarrow \\
0 & \longrightarrow & H^1(\mathcal{T}^0_U) & \longrightarrow & \mathrm{T}^1_U & \longrightarrow & H^0(\mathcal{T}^1_U) & \longrightarrow & H^2(\mathcal{T}^0_U).
\end{array}
\tag{4.1}
$$

where the first raw is induced by the spectral sequence $H^i(R^jf_*\mathcal{T}_V)$ and the bottom raw is induced by the local-global spectral sequence [Go]. In the commutative diagram, the map D_f is derived by the natural map $\mathrm{T}^1_V = \mathrm{Ext}^1_V(\Omega^1_V, \mathcal{O}_V) \to \mathrm{Ext}^1_V(f^*\Omega^1_U, \mathcal{O}_V)$, and by the isomorphism $\mathrm{Ext}^1_V(f^*\Omega^1_U, \mathcal{O}_V) = \mathrm{T}^1_U$ proved in [W2] for a rational singularity. Consider the local cohomology [Gr]:

$$H^1_E(\mathcal{T}_V) \cong \lim_{\rightarrow} \mathrm{Ext}^1(\mathcal{O}_{mE}, \mathcal{T}_V) = \lim_{\rightarrow} H^0(\mathcal{T}_V(mE) \otimes \mathcal{O}_{mE}).$$

Then the depth argument in [BW] gives an exact sequence for a rational singularity:

$$0 \to H^1_E(\mathcal{T}_V) \to \mathrm{T}^1_V \xrightarrow{D_f} \mathrm{T}^1_U\,.$$

Therefore $H^1_E(\mathcal{T}_V)$ represents the first order deformations which blow down to the trivial deformations. Let E_i be an irreducible component of E. Then we have an exact sequence [W1] [W2]:

$$0 \to S \to \mathcal{T}_V \to \oplus_i \mathcal{O}_{E_i}(E_i) \to 0$$

where S is defined by the kernel of the map $\mathcal{T}_V \to \oplus_i \mathcal{O}_{E_i}(E_i)$, and it is locally free module of rank 2 on V. The cohomology $H^1(S)$ represents the first order deformations of V for which none of the E_i disappear. The injectivity of the map $H^1(S) \to H^0(\mathcal{T}^1_U)$, induced by the composition of the maps $H^1(S) \to \mathrm{T}^1_V \xrightarrow{D_f} \mathrm{T}^1_U$, is given in [W1] [W2] for a rational singularity p.

Let p be a permissible singularity. Then $\tilde{\mathcal{T}}^1_U = \mathbb{C}_p$ by Theorem 3.3. One of interesting question is whether the cohomology $H^0(\tilde{\mathcal{T}}^1_U)$, which represents the $\mathbb{Q}$-Gorenstein smoothing, is represented by the elements in T^1_V or not. By the above argument, we should look at the elements lifted by $\mathrm{T}^1_V \to \oplus_i H^1(E_i, \mathcal{O}_{E_i}(E_i))$.

Lemma 4.1. *Let U be a normal projective surface with a rational singularity p. Let $f\colon V \to U$ be the minimal resolution of U and E the exceptional divisor. And let E_i be an irreducible component of E. Let U_p be an affine neighborhood of p and let $V_E = f^{-1}(U_p)$. Then it holds the following:*
(1) The map $\mathrm{T}^1_{V_E} \to \oplus_i H^1(E_i, \mathcal{O}_{E_i}(E_i))$ is surjective.
(2) If p is a RDP, then $H^1_E(T_V) \cong \mathrm{T}^1_{V_E} \cong \oplus_i H^1(E_i, \mathcal{O}_{E_i}(E_i))$. In particular, the restriction map $\mathrm{T}^1_V \to \mathrm{T}^1_{V_E}$ is surjective.

The proof of Lemma 4.1 is in [BW].

Lemma 4.2. *Let $\bar{V}$ be a smooth projective surface with $\mathrm{T}^0_{\bar{V}} = 0$ and let $\bar{B}$ be a connected curve in $\bar{V}$ with at most nodal singularities. Let $f\colon V \to \bar{V}$ be a blowing up at a smooth point of $\bar{B}$ in $\bar{V}$. Let B' be the total transform of $\bar{B}$ and B the proper transform. Assume that the map $\mathrm{T}^1_{\bar{V}} \to H^1(\bar{B}, \mathcal{O}_{\bar{B}}(\bar{B}))$ is surjective.*

Then the natural maps $\mathrm{T}^1_V \to H^1(B, \mathcal{O}_B(B))$ *and* $\mathrm{T}^1_V \to H^1(B', \mathcal{O}_{B'}(B'))$ *are also surjective.*

Proof. Let E be the exceptional -1-curve, and $B' = B + E$. Let q be the intersection point between B and E. Since we have the exact sequence $0 \to \mathcal{O}_B(B) \to \mathcal{O}_{B'}(B') \to \mathcal{O}_E \to 0$, it is enough to check the surjection of $\mathrm{T}^1_V \to H^1(B, \mathcal{O}_B(B))$.

We have the following exact sequences and commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{T}_{V,B} & \longrightarrow & f^*\mathcal{T}_{\bar{V},\bar{B}} & \longrightarrow & \mathcal{O}_E & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{T}_V & \longrightarrow & f^*\mathcal{T}_{\bar{V}} & \longrightarrow & \mathcal{O}_E(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_B(B) & \longrightarrow & f^*\mathcal{O}_{\bar{B}}(\bar{B}) & \longrightarrow & \mathcal{O}_q & \longrightarrow & 0.
\end{array}$$

Then since the map $\mathrm{T}^1_{\bar{V}} \to H^1(\bar{B}, \mathcal{O}_{\bar{B}}(\bar{B}))$ is surjective and $H^0(\mathcal{O}_E(1))$ goes injectively into T^1_V, the map $n\colon \mathrm{T}^1_V \to H^1(B, \mathcal{O}_B(B))$ is surjective: In particular, an element $\theta \in \mathrm{T}^1_V$, corresponding to the direction of the first order deformation induced by the blowing up at points whose direction is left out $\bar{B}$, goes onto the image of the map $H^0(\mathcal{O}_q) \to H^1(B, \mathcal{O}_B(B))$. $\square$

The next lemma holds by Akizuki-Nakano vanishing theorem [AN]:

Lemma 4.3. *Let U be a normal projective surface with a singularity. Let V be a resolution of U and E the exceptional divisor. Assume that $-(K_V + E)$ is ample. Then the natural map $\mathrm{T}^1_V \to H^1(E, \mathcal{O}_E(E))$ is surjective.*

Theorem 4.4. *Let U be a normal projective surface with a permissible singularity p. Let $f\colon V \to U$ be the minimal resolution of U and E the exceptional divisor. Assume that the map $\mathrm{T}^1_V \to H^1(E, \mathcal{O}_E(E))$ is surjective and that $\mathrm{T}^2_V = 0$. Then U has a $\mathbb{Q}$-Gorenstein smoothing.*

Proof. Step 1: If we assume that U has a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{U} \to \Delta$ with the central fiber U, then by the succession of weighted blowing up of the singular point in $\mathcal{U}$, $\mathcal{U}$ admits a semi-stable model $\mathcal{X} \to \mathcal{U} \to \Delta$ without changing a base and a general fiber. Let X be the central fiber of $\mathcal{X}$. It is obvious that the existence of a $\mathbb{Q}$-Gorenstein smoothing of U is the same as the existence of a smoothing of X. X is a simple normal crossing surface and $\mathcal{T}^1_X = \mathcal{O}_D$ where $D = Sing\, X$. Then $X = \sum_{k=0}^n W_k$ where W_0 is a resolution of U, W_k is a rational surface for $k = 1, \ldots, n$, and each double curve D_{W_i} is a tree of $\mathbb{P}^1$ for $k = 0, \ldots, n$.

Step 2: We have the vanishing of $\mathrm{T}^2_{W_k}$ for $k = 0, \ldots, n$:
By the assumption of the vanishing T^2_V, and since the vanishing of T^2 is preserved by blowing up of a smooth point, we have the vanishing of $\mathrm{T}^2_{W_0}$. Also $\mathrm{T}^2_{W_k} = 0$ because W_k is a rational surface for $k = 1, \ldots, n$.

Step 3: We have the vanishing of $\mathrm{T}^2_{W_k, D_{W_k}}$ for $k = 0, \ldots, n$:
By the assumption and Lemma 4.2, it holds the surjection of the map $\mathrm{T}^1_{W_0} \to H^1(D_{W_0}, \mathcal{O}_{D_{W_0}}(D_{W_0}))$. For $k = 1, \ldots, n$, by the succession of blowing down of -1-curves we consider the relatively smooth log minimal model $(\bar{W}_k, \bar{D}_{W_k})$ of (W_k, D_{W_k}) where $\bar{D}_{W_k}$ is the image of D_{W_k}. Then $\bar{W}_k$ is a rational surface and

$-(K_{\bar{W}_k}+D_{\bar{W}_k})$ is ample. Furthermore by tracing back the morphism, we conclude $\bar{D}_{W_k}$ has at worst nodes and D_{W_k} is obtained by the succession of blowing up of the smooth points of $\bar{D}_{W_k}$. Then by Lemmas 4.2, 4.3, and by the tracking back the above morphism, it holds the surjection of the map $\mathrm{T}^1_{W_k} \to H^1(D_{W_k}, \mathcal{O}_{D_{W_k}}(D_{W_k}))$ for $k = 1, \ldots, n$. By Step 2 and by the long exact sequence (3.3), we have $\mathrm{T}^2_{W_k, D_{W_k}} = 0$ for $k = 0, \ldots, n$.

Step 4: We have the vanishing of T^2_X:
By Step 3 and by the vanishing of $H^1(\mathcal{T}^0_{D_{W_k}})$ for $k = 0, \ldots, n$, we have the vanishing of $H^2(\mathcal{T}^0_X)$. And it holds the vanishing of $H^1(\mathcal{T}^1_X) = H^1(\mathcal{O}_D)$ because D is a tree of $\mathbb{P}^1$. $\square$

Example 4.5. Let W be a Godeaux surface (a minimal surface of general type with $p_g = q = 0, K^2 = 1$) with an involution. Assume that the fixed divisor is a -3-curve D ($D = \mathbb{P}^1$ with $D^2 = -3$). The possible fixed locus of a Godeaux surface with an involution is classified in [KL]. Assume that $T^2_W = 0$ and surjectivity of the map $T^1_W \to H^1(D, \mathcal{O}_D(D))$. There is an example satisfying this assumption [L2]. Let V be a blowing up at a point of D in W. Let $\bar{D}$ be the proper transform of D in V. And let U be a normal surface obtained by the contraction of $\bar{D}$ ($\bar{D}^2 = -4$). The smoothablity of U is given by Theorem 4.4. But U is not a stable surface because $K_U\bar{E} < 0$ where $\bar{E}$ is the image of -1-curve. Let $\mathcal{X} \to \Delta$ be a smoothings of U. Then the canonical model of $\mathcal{X} \to \Delta$ is obtained by flip. In fact, the canonical model is a deformation of W disappearing -3-curve in W.

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