DYE LASER MODEL WITH PRE-GAUSSIAN PUMP FLUCTUATIONS

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by

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Abstract. We propose a method to calculate explicitely the stationary probability of nonlinear systems subjected to a pre-gaussian fluctuation. Using this method we consider the local behaviour of the stationary probability near its singular points in the case of dye laser system.

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1. Introduction.

Recently the problem of dye laser statistics in presence of pump fluctuations is intensively considered [1,2] because of some striking discrepancies between the traditional laser theory (without pump fluctuations) and the experimental observations. Usually lasers used in dye laser pumping are thought to generate fields with Gaussian statistics (chaotic light). However the model with chaotic noise pump fluctuations can not be solved analytically and several authors have developed approximate theories to analyze it. As it has been emphasized in [2] , use of such theories in comparing with experiments could lead to incorrect values of the parameters as it is not known a priori whether the experimental conditions are such that the parameters are in the domain of applicability of these theories. To avoid this the solution of dye laser equation has been simulated numerically in [2].

In this paper we consider the dye-laser model with pre-gaussian (PG) pump fluctuations. The concept of a PG stochastic process has been introduced recently in order to model noises in strong laser-atom interactions [3-5]. It has been emphasized in [3-6] that the power of the PG process formalism comes from the exact solvability of the stochastic equations of motion involved in the problem. In the other hand PG fluctuations approximate very well the Ornstein-Uhlenbeck process (chaotic fluctuation) with just a few telegraphs involved in the calculations. Thus the commonly accepted gaussian character of pumping does not diminish the advantage of the PG model.

The problem with one telegraph has been analyzed by Kitahara et al in [7]. They have calculated analytically the stationary probability of the system and determined the "phase diagram" for the various noise-induced transitions. Now we solve the problem with PG fluctuations composed of two telegraphs using the results given in [8] . The stationary probability is calculated and its local behaviour near singular points is investigated, what provides some quantitative information about phase diagrams.

2. PG fluctuations in dye laser model.

We start from a general nonlinear differential stochastic equation

$$\frac{dx}{dt} = f(x) + z(t)g(x) \qquad (2.1)$$

where f(x) and g(x) are two arbitrary functions of the investigated dynamical variable x. The random process z(t) is the two telegraph PG noise:

$$z(t) = x_1(t) + x_2(t) \qquad (2.2)$$

$$< x_i(t) > = 0, \quad < x_i(t)x_j(s) > = \delta_{ij}\Gamma L/2T_c \exp\left[-\frac{|t-s|}{T_c}\right] \qquad (2.3)$$

The nonlinear equation (2.1) leads to a linear problem for the following quantity

$$\rho(\xi,t) = \delta(\xi - x(t)) \qquad (2.4).$$

Then from equation (2.1) and the definition (2.4) we obtain the Liouville equation (cf.[8])

$$\frac{d\rho}{dt} = M_0\rho + iz(t)M\rho \qquad (2.5)$$

where $M_0 = -\frac{\partial}{\partial x} f(x)$ and $M = i\frac{\partial}{\partial x}g(x)$. The distribution function

$$P(t) = < \rho(t) >$$

is a solution of the following integro-differential equation:

$$\frac{dP}{dt}(t) = M_0 P(t) - \int_0^t ds K(t-s)P(s) \qquad (2.6)$$

with the laplace transformed kernel ($t \geq 0$)

$$\tilde{K}(z) = \int_0^\infty e^{-zt}K(t)dt = $$
$$= M \frac{\Gamma_{L/T_c}}{z+1/T_c - M_o + M \frac{\Gamma L/T_c}{z+2/T_c - M_o} M} M \tag{2.7}$$

The steady state distribution function is then a solution of the following equation:

$$(M_o - \tilde{K}(0))P_{st} = 0 \tag{2.8}$$

The equation (2.8) leads to the following ordinary differential equation of second order

$$\left(2g - \frac{f^2}{ag}\right)\frac{d^2y}{dx^2} + \left(\frac{2g}{fT_c} - \frac{3f}{gaT_c} - \frac{1}{a}\frac{d}{dx}(f^2/g) + 2\frac{dg}{dx}\right)\frac{dy}{dx} +$$
$$+ \left(-\frac{2}{agT_c^2} - \frac{1}{aT_c}\frac{d(f/g)}{dx}\right)y = 0 \tag{2.9}$$

where $a = \Gamma_L/T_c$ and $y(x) = \int^x \frac{fP_{st}}{g}$.

It is worth noting that the integro-differential equation in the stationary regime leads to a ordinary differential equation of second order for which various powerful methods of solving have been developed. In many interesting problems of quantum optics f and g are just rational functions, hence equation (2.9) reduces to the Fux type equation [10,11].

Now as an example we consider the dye-laser equation

$$\frac{dx}{dt} = (\lambda - x)x + xz(t) \tag{2.10}$$

In this case $g(x) = x$, $f(x) = \lambda x - x^2$. After simple calculations we obtain the following equation

$$F(x)y'' + G(x)y' + H(x)y = 0 \tag{2.11}$$

where

$$F(x) = T_c^2 x^2 (2a - (\lambda - x)^2)(\lambda - x),$$

$$G(x) = xT_c(2a - 3(\lambda - x)^2 + T_c(\lambda - x)(2a - \lambda^2 + 4\lambda x - 3x^2)$$

$$H(x) = (\lambda - x)(xT_c - 2).$$

This equation is a subject of our analysis in the next sections. It is worth to note that the method introduced in [8] is quite general and easyly gives us the results of Klyatskin [9] and Kitahara [7], namely for one telegraph we have the kernel

$$\tilde{K}(z) = -\frac{\partial}{\partial x} g \frac{\Gamma_{L/T_c}}{z + \frac{1}{T_c} + \frac{\partial}{\partial x} f} \frac{\partial}{\partial x} g \tag{2.12}$$

Then from (2.8) we obtain the following equation for the stationary distribution function

$$f(x) P_{st}(x) = g(x)\left(\frac{1}{T_c} + \frac{d}{dx} f(x)\right)^{-1} \frac{d}{dx} g(x) P_{st}(x) \tag{2.13}$$

But this is exactly the equation obtained in [9] , which has a general solution [7,9]

$$P_{st}(x) = \frac{C|g(x)|}{g^2(x) - f^2(x)} \exp\left(\frac{1}{T_c} \int \frac{dx f(x)}{g^2(x) - f^2(x)} \right) \tag{2.14}$$

3. The stationary probability density.

At first we have to solve the following equation

$$F(x)y'' + G(x)y' + H(x)y = 0 \tag{3.1}$$

We see that (3.1) is a second order nonlinear differential equation, so we can use the standard methods of reduction (cf.[10]) and write down its general solution. After some calculations we obtain

$$P_{st}(x) = \frac{y'(x)}{\lambda - x} = \frac{C}{\lambda - x} \cos\theta\, e^{-A(x)} \exp\left[\int_\alpha^x (e^{-A(s)} - a_2 e^{A(s)}) \sin\theta \cos\theta \, ds \right], \tag{3.2}$$

where $a_1(x) = \frac{G(x)}{F(x)}$, $a_2(x) = \frac{H(x)}{F(x)}$, $A(x) = \int^x a_1$ and θ is a solution to the first order equation

$$\theta' = e^{-A} \cos^2\theta + a_2 e^{A} \sin^2\theta \tag{3.3}$$

Now by substitution the new function $u(x) = tg\,\theta$ we reduce (3.3) to a Riccati equation

$$u' = a_2 e^{A} u^2 + e^{-A} \qquad (3.4)$$

This equation can be solved by the well known approximate methods (recurrential approximate series) or by numerical analysis.

Let us now concentrate on the local behaviour of P_{st} near its singular points. One can easily check that if $0, \lambda, \lambda \pm \sqrt{2a}$ are different numbers then (3.1) is a Fuchs type equation [10] with the regular singular points (in a complex domain [11]). It can be written in the form:

$$P^2 y'' + PQy' + Ry = 0 \qquad (3.5)$$

where

$$P = x(2a-(\lambda-x)^2)(\lambda-x)T_c = T_c(x-a_1)(x-a_2)\ldots(x-a_m),$$

$$Q = 2a-3(\lambda-x)^2+T_c(\lambda-x)(2a-\lambda^2+4\lambda x-3x^2),$$

$$R = (xT_c-2)(\lambda-x)^2(2a-(\lambda-x)^2)$$

and

$$\deg Q = 3 \le m-1,$$

$$\deg R = 5 \le 2(m-1) \text{ for } m = 4.$$

Hence we can apply the well known methods for such equations in complex domain and obtain some local expansions of a solution of (3.1) near singular points. On the basis of elementary analysis of potential functions (see Figs. 1 and 2) for dye laser equation (2.10) ($\dot{x} = -\mathrm{grad}V$) at three values of the two-telegraph stochastic process $(-2a,0,2a)$ we can conclude that the support of P_{st} is following

$$U = [\,\max(0,\lambda-\sqrt{2a})\;,\;\lambda+2a\,]\;.$$

In the stationary limit the distribution function is vanishing out of the attraction interval U. It is implied by elementary statistical considerations. The support of $P_{st}(x)$ is indicated on the x-axis by the bold face line. In contrary of the one-telegraph problem in our case the point $\lambda \in U$ is a singular point for P_{st}.

In the neighbourhood of the singular point, say ξ, the equation (3.1) can be rewritten in the form

$$(x-\xi)^2 y'' + (x-\xi)p(x)y' + q(x)y = 0 \tag{3.6}$$

here $\xi = 0,\ \lambda-\sqrt{2a},\ \lambda,\ \lambda+\sqrt{2a}$. From the general theory of Fuchs type equations we conclude that for the real equation (3.6), i.e. p, q real analytic and ξ real, we have the following fundamental systems of real solutions:

a)
$$\begin{aligned} y_1(x) &= |x-\xi|^{\rho_1} \sum_{k=0}^{\infty} w_k (x-\xi)^k \\ y_2(x) &= |x-\xi|^{\rho_2} \sum_{k=0}^{\infty} w_k^* (x-\xi)^k \end{aligned} \tag{3.7}$$

$w_0,\ w_0^* = 1,$

if ρ_1, ρ_2 are real and $\rho_1-\rho_2$ is noninteger.

b)
$$\begin{aligned} y_1(x) &= |x-\xi|^{\rho_1} \sum_{k=0}^{\infty} w_k (x-\xi)^k, \\ y_2(x) &= \alpha y_1(x) \ln|x-\xi| + |x-\xi|^{\rho_2} \sum_{k=0}^{\infty} w_k^* (x-\xi)^k \end{aligned} \tag{3.8}$$

if $\rho_1-\rho_2$ is integer ≥ 0.

Coefficients w_k(w_k^*-respectively) satisfy the following reccurent system of equations:

$$\begin{aligned} &w_0 f(\rho) = 0, \\ &w_1 f(\rho+1) + w_0 f_1(\rho) = 0, \\ &\dots\dots\dots\dots\dots\dots \\ &w_k f(\rho+k) + w_{k-1} f_1(\rho+k-1) + \dots + w_0 f_k(\rho) = 0 \end{aligned} \tag{3.9}$$

where

$f(\rho) = \rho(\rho-1) + p_0\rho + q_0$, $f_n(\rho) = \rho p_n + q_n$, p_n and q_n are coefficients of series power expansion for p(x), q(x):

$$p(x) = \sum_{k=0}^{\infty} p_k(x-\xi)^k \ , \ q(x) = \sum_{k=0}^{\infty} q_k(x-\xi)^k \qquad (3.10)$$

ρ_1, ρ_2 are solutions of the characteristic equation

$$f(\rho) = 0 \qquad (3.11).$$

Using the above results we have respectivly for the singular points:

$\xi = \lambda$,

$$\rho_1 = 1 + \frac{1}{T_c\lambda} \ , \quad \rho_2 = 0,$$

$\xi = \lambda + \sqrt{2a}$,

$$\rho_1 = \frac{1}{T_c(\lambda+\sqrt{2a})} \ , \quad \rho_2 = 0,$$

$\xi = 0$, (if $\lambda - \sqrt{2a} \leq 0$).

$$\rho_1 = -\frac{1}{2}\left[\frac{2a-3\lambda^2}{\lambda T_c(2a-\lambda^2)} - \left(\frac{4a^2-28\lambda^2 a+17\lambda^4}{\lambda^2 T_c^2(2a-\lambda^2)^2}\right)^{1/2}\right],$$

$$\rho_2 = -\frac{1}{2}\left[\frac{2a-3\lambda^2}{\lambda T_c(2a-\lambda^2)} + \left(\frac{4a^2-28\lambda^2 a+17\lambda^4}{\lambda^2 T_c^2(2a-\lambda^2)^2}\right)^{1/2}\right].$$

$\xi = \lambda - \sqrt{2a}$, (if $\lambda - \sqrt{2a} > 0$),

$$\rho_1 = \frac{1}{T_c(\lambda-\sqrt{2a})} \ , \quad \rho_2 = 0.$$

Differentiating the solutions (3.7), (3.8) with the calculated above exponents ρ_1, ρ_2, and using the definition

$$P_{st}(x) = \frac{y'(x)}{\lambda - x}$$

we obtain the local expansions for P_{st} near singular points.

4. Final remarks.

In this paper we have proposed an approach to arbitrary nonlinear systems subjected to a PG fluctuation with two telegraphs. As an example the exact formula for P_{st} and its local expansins have been determined in the case of dye laser model. The behaviour of mean value of dye laser intensity and high correlations will be considered and compared with the results obtained by other approaches, in a forthcoming paper. Our paper is an extension of results obtained by Kitahara et al [7] for the case of one telegraph. We hope that our equation (2.1) is useful for several problems in quantum optics where f, g are simply specified and two-telegraph PG process approximates very well the chaotic gaussian process.

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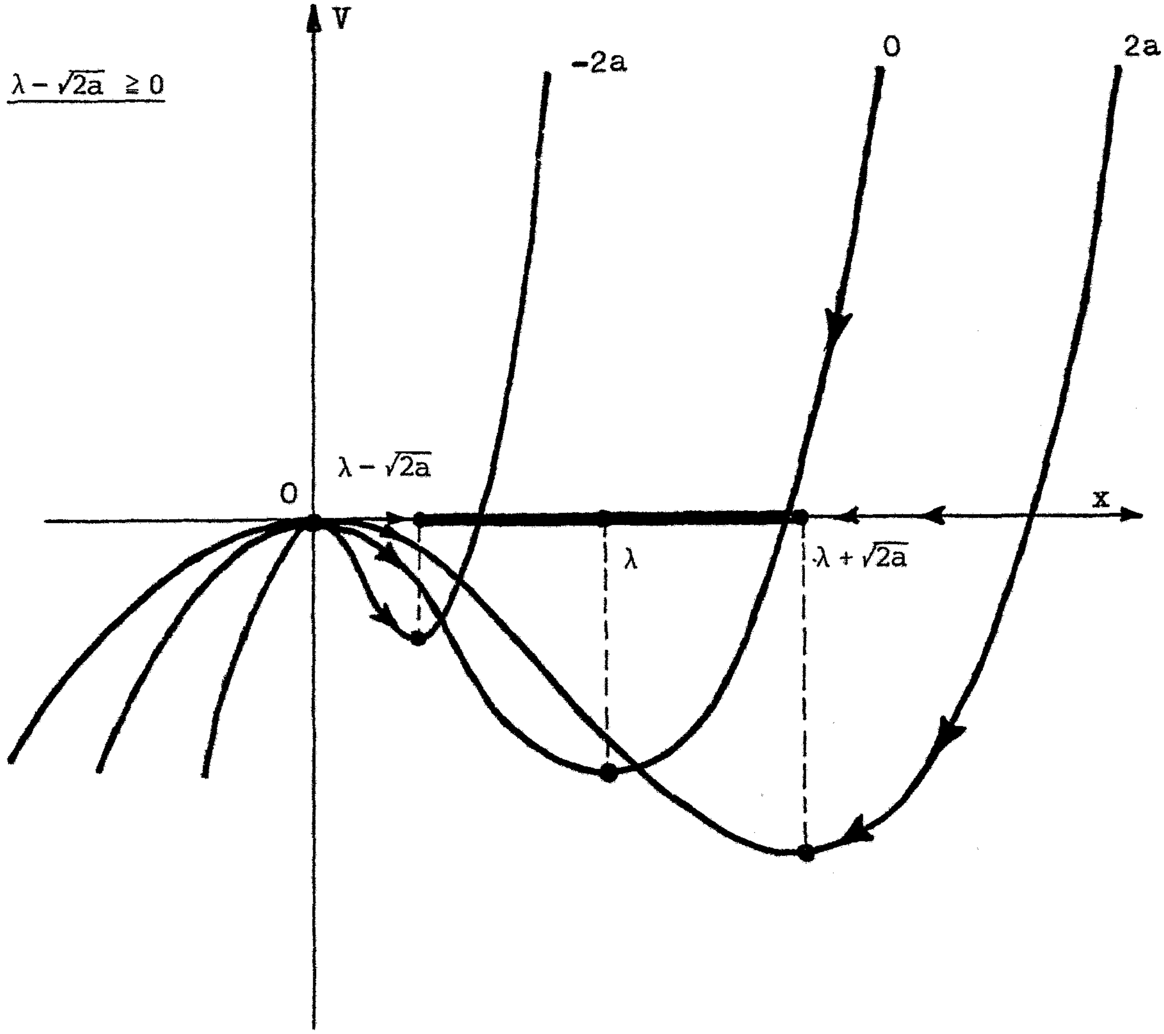

$\lambda - \sqrt{2a} \geqq 0$
V
-2a
0
2a
0
$\lambda - \sqrt{2a}$
x
λ
$\lambda + \sqrt{2a}$

Fig. 1.

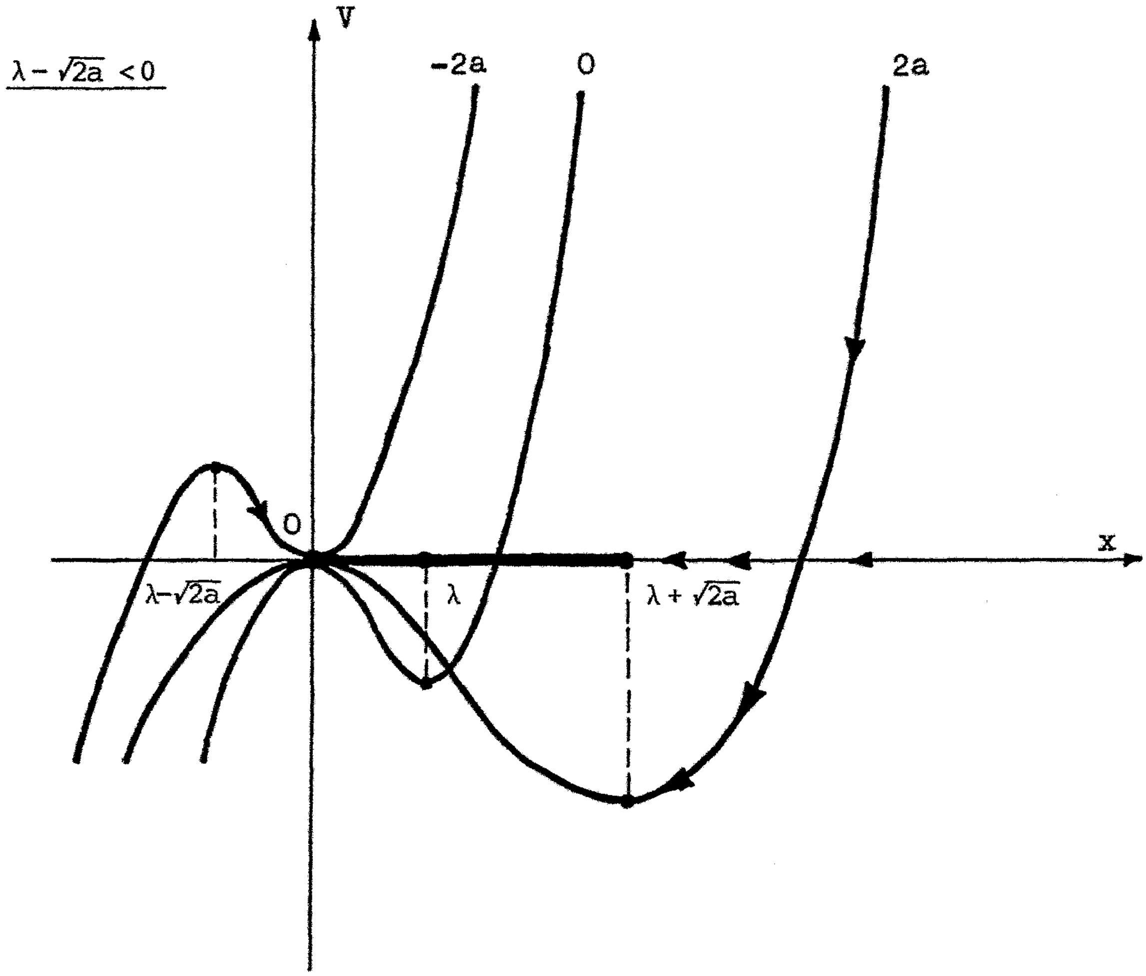

Fig. 2.