Analytic Torsion Forms for Noncompact Fiber Bundles

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Abstract

We develop an L^2-analytic torsion form on the base space of a smooth fiber bundle with the covering space of a compact manifold as its typical fiber. Under certain natural hypotheses this torsion form is closed and represents a characteristic cohomology class. We also analyze our construction in some interesting cases.

1 Introduction

In [BL] Bismut and Lott introduced an analytic torsion form for a smooth fiber bundle with closed Riemannian manifold as fiber. Such an analytic torsion form arises naturally from a transgression formula for the Euler form and hence plays a crucial role in their proof of a C^∞-form version of the Riemann-Roch-Grothendieck theorem for flat bundles and submersions. From our point of view the most interesting property of this torsion form is that it globalizes Ray-Singer analytic torsion [RS]. Just as the index of a family of elliptic operators globalizes via Atiyah-Singer to a cohomology class on the parameter space, parameterized family of torsions yields in most interesting cases a cohomology class on the parameter space.

Given the success of the initiative of Bismut-Lott it is natural to extend their work to cover L^2-analytic torsion. The hope is that this will link up cohomology with geometric data, and

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even produces new geometrically defined cohomology classes in many important situations. For example, for a finitely generated discrete group Γ, its classifying space $B(\Gamma)$ often appears as the base space of smooth fiber bundles. However, in most interesting and natural cases the fibers are noncompact and thus it is necessary to study the L^2-situation.

In this paper we are primarily concerned to develop the foundational material which allows us to extend the Bismut-Lott construction. The technical machinery needed, beyond the Bismut superconnection formalism utilized by Bismut-Lott, involves traces in von Neumann algebras and families of the Novikov-Shubin invariants. Some crucial estimates are much more difficult in the noncompact case. This paper is hardly the last word on this, and many very interesting foundational questions remain. As far as calculations go, even in the presumably simpler situation analyzed by Bismut-Lott, the surface has only been scratched, and real breakthroughs are needed to turn these constructions into practical tools.

¿From the point of view of topology, a purely topological or combinatorial description of the cohomology classes derived from analytic techniques, remains a basic challenge. In fact, prior to the work of Bismut-Lott, Igusa-Klein [IK] defined torsion type classes in the parameterized situation, using more topological methods, in particular the Waldhausen K-theory of spaces machinery, along with Igusa's parameterized Morse theory. Even earlier Bismut, Gillet and Soulé [BGS] had defined analogous classes in the holomorphic situation via holomorphic analysis. Very recently, Dwyer-Weiss-Williams [DWW] have abstracted and generalized the C^∞-Riemann-Roch-Grothendieck theorem and indicated how to derive characteristic classes by utilizing and generalizing Waldhausen's techniques. Although all this work is very interesting and suggestive, precise links between the analytic and topological approaches have yet to be established. Sample calculations in the Igusa-Klein theory and the Bismut-Lott theory, which remain extremely difficult and restricted, provide evidence that the defined classes are closely interwined.

The aim of the present paper is to develop an L^2-analytic torsion form for a fiber bundle M with the covering space of a closed Riemannian manifold as typical fiber. This L^2-analytic torsion form is a natural extension of the L^2-analytic torsion invariant for covering spaces to fiber bundles. It has many properties of Bismut-Lott's torsion form. As in the case of the L^2-analytic torsion invariant, the L^2-analytic torsion form is defined only for fiber bundles satisfying a positivity condition on the family Novikov-Shubin invariants. Because the fibers are noncompact, delicate and complex problems arise in establishing the existence of such forms, which do not occur in the compact case. Sections 2 and 3 deal with such matters.

This paper is arranged as follows. In Section 2 we will discuss spectral analysis on the noncompact fiber bundle M, in particular the smooth dependence of the family of spectral projections associated with the fiberwise Laplacian with respect to the variables of the base space of M. The difficulty is caused by the continuous spectrum of the Laplacian on the noncompact fibers of M. We will consider in Section 3 the index formula for the fiberwise Euler-de Rham operator $d_M^* + d_M$ on the fiber bundle M and use quite delicate analysis to prove Lemma 3.3, the most technical lemma of this paper. We will then obtain an L^2-form version of the Riemann-Roch- Grothendieck theorem under a decay condition. Such a decay condition will carefully be studied in Section 4 and will give rise to the so-called family Novikov-Shubin invariants. These invariants share the properties of the usual Novikov-Shubin invariants. With positivity of the family Novikov- Shubin invariants, we also define in Section 3 an L^2-analytic torsion form $\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)$ of the fiber bundle M and discuss its properties in Section 5. Finally in Section 6, we will address the computation problem of the L^2-analytic torsion form for fiber bundles with fibers homogeneous spaces. The application of this L^2-torsion form will be discussed elsewhere (e.g., see [Go 2]). We also include an appendix to discuss the existence and uniqueness of local Γ-invariance structures on the fiber bundle M.

This draft is a strengthened version of a manuscript circulated two years ago. We now remove a crucial condition in the previous version of Lemma 3.3. This enables us to overcome an error in the earily proof of Lemma 3.3 which was pointed out to us by Lott. The present version of Lemma 3.3 is significantly stronger and its proof turns out to be rather formidable and demanding than we had originally expected. The fact that two authors resided respectively for the year 96-97 in Bonn and Chicago has much to do with the late appearance of this manuscript. In the interin, Heitsch and Lazaroff observed that ideas utilized in this paper could be applied to develop analogous invariants of foliated manifolds. They have carried through this program in [HL]. We wish to thank both John Lott for his expert attention, which had a major impact on this work, and James Heitsch for his useful comments. The first author also thanks the Max-Planck-Institut für Mathematik, Bonn, Germany, for hospitality and support.

2 Spectral Analysis on Noncompact Fiber Bundles

Let B be a not necessarily closed Riemannian manifold and $\tilde{Z} \to M \xrightarrow{\pi} B$ a smooth fiber bundle over B with fiber $\tilde{Z}$ a regular covering space of a closed, connected Riemannian manifold Z, with covering group $\Gamma = \pi_1(Z)/\pi_1(\tilde{Z})$. Thus M has a covering by open sets of the form $\{U_i\} \times \tilde{Z}$, where $\{U_i\}$ is an open covering of B, and π restricted to each $\{U_i\} \times \tilde{Z}$ is just projection into the first factor. Such an open subset of M will be called a bundle chart for the fiber bundle M. These bundle charts yield a family of "local actions " of Γ on M. We do **not** assume they fit together to yield a global action. Denote as usual by $T\tilde{Z}$ the vertical tangent bundle of M, that is the tangent vectors of M tangent to the fibers. Assume that there is a connection of the fiber bundle M so that the tangent bundle TM of M splits as $TM \simeq T\tilde{Z} \oplus \pi^*(TB)$. We will assume also that we are given the following data, $\{g^{\tilde{Z}}, E, \nabla^E, g^E\}$, associated with this fiber bundle, where $g^{\tilde{Z}}$ is a Riemannian metric on the vertical tangent bundle $T\tilde{Z}$, E is a complex flat bundle over M with flat connection ∇^E and (not necessarily flat) Hermitian metric g^E. The data is said to be locally Γ-invariant if the following condition holds. There exists a family of bundle charts $\{U_i\} \times \tilde{Z}$, called a family associated to the data, such that the diffeomorphisms $\varphi_i : M|_{U_i} \to U_i \times \tilde{Z}$ and $\psi_i : E|_{U_i} \to U_i \times E|_{\tilde{Z}}$, and the corresponding transition function $\varphi_{ij} : U_i \cap U_j \to Diff(\tilde{Z})$, the diffeomorphism group of $\tilde{Z}$, satisfy the following:

1) $\{g^{\tilde{Z}}, E, \nabla^E, g^E\}|_{U_i \times \tilde{Z}}$ is the pullback under the projection $U_i \times \tilde{Z} \to U_i \times Z$ of corresponding data, $\{g^Z, E_i, \nabla^{E_i}, g^{E_i}\}$, on $U_i \times Z$.

2) For each $x \in U_i \cap U_j$, there is an isometry $\beta_{ij}(x)$ of $(\tilde{Z}, g^{\tilde{Z}}|_{x \times \tilde{Z}}) \hookrightarrow (\tilde{Z}, g^{\tilde{Z}}|_{U_j \times \tilde{Z}})$ such that $\beta_{ij}(x)$ is covered by an isometry of $E|_{x \times \tilde{Z}}$, and $\beta_{ij}(x)\varphi_{ij}(x)$ takes a fundamental domain of Γ into a fundamental domain of Γ.

A local Γ-invariance structure on the bundle $\pi' : E \to M$ is determined by specifying a local Γ-invariant data and a maximal family of bundle charts associated to the data.

We will refer to the above 2) as the strong local Γ-invariance condition. We note that a local Γ-invariance structure exists providing $\pi' : E \to M$ covers a fiber bundle with fiber Z, i.e., if the map π factors as $\pi''\rho$, where π'' is a bundle map $\bar{M} \to B$ with fiber Z, and $\rho : M \to \bar{M}$ is a not necessarily regular covering which restricts over each fiber Z to the projection $p : \tilde{Z} \to Z$. In fact, any data on $\bar{M} \xrightarrow{\pi''} B$ induces a locally Γ-invariant structure on the pullback data

$M \xrightarrow{\pi} B$. The most natural example of a fiber bundle covering a fiber bundle with compact fiber is where a discrete group Γ acts freely and smoothly on a simply connected smooth manifold W with compact quotient. Then $W \times_\Gamma W \to W/\Gamma$ is a bundle of this type. Note that the map $W \times_\Gamma W \to W/\Gamma \times W/\Gamma$ is a regular covering only if Γ is Abelian.

It is natural and convenient to define two locally Γ-invariant structures on $\pi : M \to B$ as equivalent if there exists data with a locally Γ-invariant structure on $\tilde{\pi} : M \times [0,1] \to B$ which restricts to the two given ones on $M \times \{i\} \to B$, $i = 0, 1$. If π factors as above as $\pi''\rho$, then for any two data on $\pi'' : \bar{M} \to B$ with the same flat bundle, the induced locally Γ-invariant structures on the pullback data are equivalent. Thus given a factorization and a flat bundle on $\bar{M}$, we have a unique equivalence class of locally Γ-invariant structures on π.

Given $\pi : M \to B$, $p : \tilde{Z} \to Z$ and $E \to M$, there may not exist data with a locally Γ-invariant structure, and for given data there may exist more than one locally Γ-invariant structures up to equivalence. The cohomology classes which we define, utilizing specific locally Γ-invariant structures will turn out to depend only on the equivalence class of such structures, and thus the problem of classifying such equivalence classes is rather fundamental from our point of view. We analyze the question of the existence and uniqueness of locally Γ-invariant structures in more detail in Appendix.

A differential operator on forms on M with values in E, D, will be called fiberwise Γ-invariant (with respect to a given locally Γ-invariant structure) if it is locally the lifting of an operator on $U_i \times Z$ with respect to any family of bundle charts associated to the locally Γ-invariant structure. From now on, $\{U_i \times Z\}$ will always denote an associated family.

Let $\wedge^*(T^*M_x, E|_{M_x})$ be the $E|_{M_x}$-valued smooth forms on M_x, where M_x is the fiber of M over $x \in B$. We have a smooth infinite dimensional vector bundle $\mathcal{E}$ over B whose fiber at $x \in B$ is $\wedge^*(T^*M_x, E|_{M_x})$. We also denote $\mathcal{E}$ by $\wedge^*(T^*\tilde{Z}, E|_{\tilde{Z}})$. If $\Omega(M, E)$ (resp., $\Omega^v(M, E)$) denotes as usual all smooth forms on M with values in E whose restrictions to M_x are in $\wedge^*(T^*M_x, E|_{M_x})$ (resp., which are annihilated by interior multiplication with $X \in \pi^*(TB)$), then

$$\Omega^v(M, E) \simeq C^\infty(B, \mathcal{E}),$$

and

$$\Omega(M, E) \simeq \Omega(B)\hat{\otimes}\Omega^v(M, E) \simeq \Omega(B)\hat{\otimes}C^\infty(B, \mathcal{E}) \simeq \Omega(B, \mathcal{E}).$$

Observe that the exterior differential d_M on M with coefficients in E satisfies

$$d_M((\pi^*\phi)\xi) = \pi^*(d_B\phi) \wedge \xi + (\pi^*\phi)d_M(\xi), \quad \phi \in C^\infty(B), \xi \in \Omega(M, E).$$

Let $\{\tilde{e}_i\}$ and $\{f_\alpha\}$ be local orthonormal bases of the vertical tangent bundle $T\tilde{Z}$ and TB, respectively, with the corresponding dual bases $\{\tilde{e}^i\}$ and $\{f^\alpha\}$. We also use the same symbols $\{f_\alpha\}$ and $\{f^\alpha\}$ to denote the liftings of $\{f_\alpha\}$ and $\{f^\alpha\}$ to $\pi^*(TB)$ and $\pi^*(T^*B)$, respectively. Denote by $\tilde{m}^i$ (resp., m^α) the exterior multiplication by $\tilde{e}^i$ (resp., f^α), and by $\tilde{\iota}^i$ (resp., ι^α) the interior multiplication by $\tilde{e}_i$ (resp., f_α). Let $c(X) = (X\wedge) - i_X$ and $c_v(X) = (X\wedge) + i_X$ for X a tangent vector or 1-form, where i_X is the usual interior multiplication by X. In particular, let $\tilde{c}^i = c(\tilde{e}_i), c^\alpha = c(f_\alpha), \tilde{c}^i_v = c_v(\tilde{e}_i), c^\alpha_v = c_v(f_\alpha)$. There are the following identities:

$$m^a\iota^b + \iota^b m^a = \delta_{ab},$$

$$c(X^a)c(X^b) + c(X^b)c(X^a) = -2\delta_{ab},$$

$$c_v(X^a)c_v(X^b) + c_v(X^b)c_v(X^a) = 2\delta_{ab},$$

and

$$c(X^a)c_v(X^b) + c_v(X^b)c(X^a) = 0,$$

where $X^a = \tilde{e}^i$ or f^α, and m^a and ι^b mean $\tilde{m}^i, m^\alpha, \tilde{\iota}^i$ or ι^α, respectively. Let g^B be a Riemannian metric on B and $g^M = g^{\tilde{Z}} \oplus \pi^* g^B$ be the corresponding Riemannian metric on M. Let ∇^M be the Levi-Civita connection associated with g^M. For the projection $P^{\tilde{Z}} : TM = T\tilde{Z} \oplus T^H M \to T\tilde{Z}$, we take $\nabla^{\tilde{Z}} = P^{\tilde{Z}}\nabla^M$. Let $w_{abc} = X^a(\nabla^M_{X_c} X_b)$, where $X_b = \tilde{e}_i$ or f_α, and we used the Einstein summation. These $w'_{abc}s$ satisfy

$$w_{\alpha kj} = w_{\alpha jk} = -w_{j\alpha k} = -w_{k\alpha j},$$

and

$$w_{\beta\alpha j} = -w_{\alpha\beta j} = -w_{\alpha j\beta} = w_{\beta j\alpha} = -w_{j\beta\alpha}.$$

Let $\nabla^{\tilde{Z}\otimes E}$ be the tensor product of the connections $\nabla^{\tilde{Z}}$ and ∇^E, $\nabla^{\tilde{Z}\otimes E} = \nabla^{\tilde{Z}} \otimes 1 + 1 \otimes \nabla^E$. Acting on $C^\infty(B, \mathcal{E})$, d_M is locally given by

$$d_M \;=\; \tilde{m}^j \nabla^{M\otimes E}_{\tilde{e}_j} + m^\alpha \nabla^{M\otimes E}_{f_\alpha}$$

$$
\begin{aligned}
&= \tilde m^j(\nabla^{\tilde Z\otimes E}_{\tilde e_j} + w_{\alpha kj} m^\alpha \tilde\iota^k) + m^\alpha(\nabla^{\tilde Z\otimes E}_{f_\alpha} + w_{\beta k\alpha} m^\beta \tilde\iota^k)\\
&= \tilde m^j\nabla^{\tilde Z\otimes E}_{\tilde e_j} + (m^\alpha\nabla^{\tilde Z\otimes E}_{f_\alpha} + \tilde m^j w_{\alpha kj} m^\alpha \tilde\iota^k) + m^\alpha w_{\beta k\alpha} m^\beta \tilde\iota^k\\
&= d_{\tilde Z} + \nabla^{\mathcal{E}} + i_{\tilde T}, \qquad (2.1)
\end{aligned}
$$

where $d_{\tilde Z} \in C^\infty(B, Hom(\mathcal{E}^*, \mathcal{E}^{*+1}))$ is the exterior differential along fibers, $\nabla^{\mathcal{E}}$ is the connection on $\mathcal{E}$ given by

$$\nabla^{\mathcal{E}}_U \phi = L_{U^H}\phi.$$

Here $\phi \in C^\infty(B, \mathcal{E})$, U is a vector field on B with $U^H \in C^\infty(M, T^H M)$ a horizontal lift of U, $\pi^* U^H = U$, and L_{U^H} is the Lie differentiation on $\Omega^v(M,E)$, $i_{\tilde T} \in \Omega^2(B, Hom(\mathcal{E}^*, \mathcal{E}^{*-1}))$ is defined as the interior multiplication by $\tilde T(U_1, U_2) = -P^{\tilde Z}[U_1^H, U_2^H] \in C^\infty(M, T\tilde Z)$ for vector fields U_i on B. We should mention that the operators d_M, $d_{\tilde Z}, L_{U^H}$ and $i_{\tilde T}$ are defined on L^2-smooth forms whose images under these operators over $x \in B$ lie in the L^2-completion $\wedge^*_2(T^*M_x, E|_{M_x})$ of $\wedge^*(T^*M_x, E|_{M_x})$. Formula (2.1) gives local representations of $d_{\tilde Z}, \nabla^{\mathcal{E}}$ and $i_{\tilde T}$. d_M is a " flat super connection" on $\mathcal{E}$ of degree 2. More precisely, $d_{\tilde Z}, \nabla^{\mathcal{E}}$ and $i_{\tilde T}$ are the degrees $0,1$ and 2-components of the super flat connection d_M, respectively, and satisfy

$$
\begin{aligned}
(d_{\tilde Z})^2 = 0, (i_{\tilde T})^2 = 0,\\
d_{\tilde Z}\nabla^{\mathcal{E}} + \nabla^{\mathcal{E}} d_{\tilde Z} = 0,\\
(\nabla^{\mathcal{E}})^2 + d_{\tilde Z} i_{\tilde T} + i_{\tilde T} d_{\tilde Z} = 0,
\end{aligned}
$$

and

$$\nabla^{\mathcal{E}} i_{\tilde T} + i_{\tilde T}\nabla^{\mathcal{E}} = 0.$$

Using the Hermitian metric g^E on E, we can define a Hermitian metric $g^{\mathcal{E}}$ on $\mathcal{E}$ such that

$$(< w_1, w_2 >_{g^{\mathcal{E}}})(x) = \int_{M_x} < w_1(x) \wedge * w_2(x) >_{g^{\mathcal{E}}} .$$

The fiberwise adjoint $(d_M)^*$ of d_M as a super connection is given by

$$d^*_M = d^*_{\tilde Z} + (\nabla^{\mathcal{E}})^* - \tilde T\wedge,$$

where by (2.1),

$$
\begin{aligned}
d^*_{\tilde Z} &= -\tilde\iota^j(\nabla^{\tilde Z\otimes E}_{\tilde e_j} + \psi_j),\\
(\nabla^{\mathcal{E}})^* &= m^\alpha(\nabla^{\tilde Z\otimes E}_{f_\alpha} - w_{\alpha jk}\tilde\iota^j \tilde m^k + \psi_\alpha),
\end{aligned}
$$

and $\psi = (g^E)^{-1}(\nabla^E g^E) \in \Omega^1(M, End(E))$ with $\psi_j = (g^E)^{-1}(\nabla^E_{\tilde{e}_j} g^E)$ and $\psi_\alpha = (g^E)^{-1}(\nabla^E_{f_\alpha} g^E)$.

Let N be the number operator on $C^\infty(B, \mathcal{E})$ given by $N(\xi) = l\xi$, where l is the degree of a form $\xi \in C^\infty(B, \mathcal{E})$. We use d_M, d_M^* and N to define two basic families of operators $\tilde{C}_t$ and $\tilde{D}_t$,

$$\tilde{C}_t = \frac{1}{2}(t^{-N/2} d_M^* t^{N/2} + t^{N/2} d_M t^{-N/2})$$

and

$$\tilde{D}_t = \frac{1}{2}(t^{-N/2} d_M^* t^{N/2} - t^{N/2} d_M t^{-N/2}), \quad t \in (0, \infty).$$

Let $\tilde{V} = d_{\tilde{Z}}^* - d_{\tilde{Z}}$ and $w(\mathcal{E}, g^{\mathcal{E}}) = (\nabla^{\mathcal{E}})^* - \nabla^{\mathcal{E}} = m^\alpha (w_{\alpha jk} \tilde{c}^j \tilde{c}_v^k + \psi_\alpha)$. Then

$$\tilde{D}_t = \frac{\sqrt{t}}{2} \tilde{V} + \frac{1}{2} w(\mathcal{E}, g^{\mathcal{E}}) - \frac{c_v(\tilde{T})}{2\sqrt{t}}. \tag{2.2}$$

To define an L^2-analytic torsion form, we need a super Γ-trace tr_Γ^s on each fiber of M. This is exactly where the strong local Γ-invariance enters in. A fiberwise Γ-operator A on $\Omega(B, \mathcal{E})$ is called a fiberwise Γ-Hilbert-Schmidt operator if its restriction A_{U_i} to $\Omega(U_i, \mathcal{E}_{U_i})$ is Γ-Hilbert-Schmidt in the usual sense, namely, the kernel $A_{U_i}(y_1, y_2)$ of A_{U_i} satisfies

$$\int_{(\tilde{Z} \times \tilde{Z})/\Gamma} \|A_{U_i}(y_1, y_2)\|^2 < \infty.$$

A is of fiberwise Γ-trace class operator if it is a product of two fiberwise Γ-Hilbert-Schmidt operators. For a fiberwise Γ-trace class operator A on $\Omega(B, \mathcal{E})$ with smooth kernel $A(y_1, y_2)$, the Γ-trace $tr_\Gamma(A)$ is equal to

$$tr_\Gamma(A)(x) = \int_{\mathcal{F}_x} tr(A(y, y)) = \sum_i (Au_i, u_i), \quad x \in B,$$

where $\mathcal{F}_x$ is a fundamental domain of Γ in M_x and $\{u_i\}$ is an orthonormal basis of the Hilbert space completion of $\wedge^*(T^*\mathcal{F}_x, E|_{\mathcal{F}_x})$ (see [At] for more details). This Γ-trace does not depend on the choice of the associated bundle chart $M|_{U_i} \overset{\varphi_{U_i}}{\simeq} U_i \times \tilde{Z}$. Indeed, $tr_\Gamma(A) = tr(\chi_1 A \chi_2)$ for compactly supported bounded measurable functions χ_1 and χ_2 on $\tilde{Z}$ such that $\sum_{\nu \in \Gamma} \nu(\chi_1 \chi_2) = 1$. Then on $U_i \cap U_j \neq \emptyset$, $i \neq j$, the isomorphism $\varphi_{ij} \equiv \varphi_{U_i} \varphi_{U_j}^{-1}$ will not change the Γ-trace. To prove this, let φ_{ij}^* be the operator on $\Omega(U_i \cap U_j, \mathcal{E}_{U_i \cap U_j})$ induced by φ_{ij}. Then the two restrictions A_{U_i} and A_{U_j} of A corresponding to these two isomorphisms φ_{U_i} and φ_{U_j} have the relation $A_{U_j} = \varphi_{ij}^* A_{U_i} (\varphi_{ij}^*)^{-1}$. We have

$$tr_\Gamma(A_{U_j}) = tr(\chi_1 A_{U_j} \chi_2) = tr(\tilde{\chi}_1 A_{U_i} \tilde{\chi}_2) = tr_\Gamma(A_{U_i}),$$

where χ_i are compactly supported bounded measurable functions on $U_i \cap U_j \times \tilde{Z}$ and $\tilde{\chi}_l = \chi_l(\varphi_{ij})$. The strong local Γ-invariance assumption guarantees that $\sum_{\nu\in\Gamma} \nu(\tilde{\chi}_1\tilde{\chi}_2) = 1$. We endow a $\mathbb{Z}_2$-grading on $\mathcal{E}$ by the degree of forms, $\mathcal{E} = \mathcal{E}^{ev} \oplus \mathcal{E}^{odd}$. Hence, we have a $\mathbb{Z}_2$-grading on $C^\infty(B, \mathcal{E})$. If A is a $\mathbb{Z}_2$-graded fiberwise Γ-trace class operator, $A = (A_{ij})_{i,j=1}^2$, then the super Γ-trace is $tr_\Gamma^s(A) = tr_\Gamma(A_{11}) - tr_\Gamma(A_{22})$.

We will use the functional calculus of $\tilde{D}_t$ as a fundamental tool to develop the L^2-analytic torsion form. Some preparations on spectral analysis of $\tilde{D}_t$ are thus necessary, since we are dealing with a noncompact fiber bundle. The occurrence of continuous spectrum of $\tilde{D}_t$ on each fiber makes the discussion harder. Some results in the compact case may not be extended to the situation here. For instance, we do not know at present whether the spectral projections E_λ of $(i\frac{\tilde{V}}{2})^2$ depend smoothly on the variables of the base space B $(\lambda > 0)$.

We begin with a subalgebra of Γ-trace class operators on a single fiber $\tilde{Z}$. Let $\tilde{F}$ be a Γ-vector bundle over $\tilde{Z}$ which is the lifting of a vector bundle F over Z. Let $S(\tilde{F})$ be the space consisting of the following bounded operators A on the space $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ of L^2-$\tilde{F}$-valued forms on $\tilde{Z}$:

(1) A has an $End(\tilde{F})$-valued smooth kernel $k_A(\tilde{x}, \tilde{y})$ and is Γ-invariant.

(2) There exists a constant $C > 0$ such that

$$
\begin{aligned}
&(a) \qquad \int_{\tilde{Z}} ||k_A(\tilde{x}, \tilde{y})||_{\tilde{F}}^2 d\tilde{y} < C, \quad \forall \tilde{x} \in \tilde{Z}, \\
&(b) \qquad \int_{\tilde{Z}} ||k_A(\tilde{x}, \tilde{y})||_{\tilde{F}}^2 d\tilde{x} < C, \quad \forall \tilde{y} \in \tilde{Z},
\end{aligned}
$$

where $||k_A(\tilde{x}, \tilde{y})||_{\tilde{F}}$ is the norm of operators from $\tilde{F}_{\tilde{y}}$ to $\tilde{F}_{\tilde{x}}$.

(3) For each Γ-invariant differential operator D on $\tilde{Z}$ of order k, $k > 0$, DA and DA^* satisfy (2).

Compared to a definition in [Ro], we have posed an extra condition (3) on the elements of $S(\tilde{F})$. Condition (3) enables us to simplify the proof of Lemma 3.3 in the next section. It is modeled on the heat kernel of the Laplacian on $\tilde{Z}$. Clearly, if $A \in S(\tilde{F})$, then the adjoint $A^* \in S(\tilde{F})$. By the Riesz representation theorem for functionals on a Hilbert space, Condition (2)(a) (resp. (b)) means that A (resp. A^*) is a bounded operator from $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ to the space $C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ of bounded continuous sections of $\wedge^*(T^*\tilde{Z}) \otimes \tilde{F}$. From this we get that $S(\tilde{F})$ is a $*$-algebra.

Let $\nabla^{\tilde{F}}$ be the Γ-invariant connection on $\tilde{F}$ which is the lifting of a connection ∇^F on F. Let $d_{\tilde{F}} = d\hat{\otimes}1 + 1\hat{\otimes}\nabla^{\tilde{F}}$ and $D_{\tilde{F}} = \frac{1}{2}(d_{\tilde{F}} + (d_{\tilde{F}})^*)$. We first prove that if $\{E_\lambda\}$ is the spectral projections of $(D_{\tilde{F}})^2$, $(D_{\tilde{F}})^2 = \int \lambda dE_\lambda$, then for $\lambda \geq 0$ and $t > 0$, $E_\lambda \in S(\tilde{F})$ and $e^{-t(D_{\tilde{F}})^2} \in S(\tilde{F})$. Indeed, we can prove the following more general result.

Lemma 2.1 *Let $\psi : \mathbb{R} \to \mathbb{C}$ be a rapidly decreasing function, i.e., $\forall k \geq 0$, there exists $C_k > 0$ such that*

$$|\psi(x)| \leq C_k(1 + |x|)^{-k}, \quad \forall x \in \mathbb{R}.$$

Then $\psi(D_{\tilde{F}}) \in S(\tilde{F})$.

Proof: The proof is based on the following noncompact version of the Sobolev embedding theorem (see [Ro], Prop. 13.7). Let $C^r(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ be the r-times continuously differentiable sections of $\wedge^*(T^*\tilde{Z})\otimes\tilde{F}$ with bounded first r-derivatives. $C^r(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ is a Banach space with the supremum norm of the first r-derivatives. Then for $n = \dim\tilde{Z}$, integer $p > n/2$, and $r \geq 0$, there exists a constant $C > 0$ such that

$$\|\xi\|_{C^r} \leq C(\|\xi\|_{L^2} + \|D_{\tilde{F}}\xi\|_{L^2} + \cdots + \|(D_{\tilde{F}})^{p+r}\xi\|_{L^2}), \quad \forall\xi \in Dom((D_{\tilde{F}})^{p+r}),$$

where $\|\cdot\|_{C^r}$ and $\|\cdot\|_{L^2}$ mean the norms of $C^r(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ and $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$, respectively. See the proof of Proposition 13.7 in [Ro]. Since for each integer $l \geq 0$, $(D_{\tilde{F}})^l\psi(D_{\tilde{F}})$ is a bounded operator on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$, we obtain that for all integers $r, l \geq 0$, $(D_{\tilde{F}})^l\psi(D_{\tilde{F}}) : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F}) \to C^r(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ is bounded. Let $\psi = \psi^+ - \psi^-$ be written as the difference of two positive rapidly decreasing functions. In view of a general fact that if a bounded selfadjoint operator A on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ maps $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ continuously to $C^{r+1}(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$, then A^2 has a C^r-kernel, we get that $\psi^\pm(D_{\tilde{F}})$ and hence $\psi(D_{\tilde{F}})$ have smooth kernels. Furthermore, $\psi(D_{\tilde{F}})$ satisfies Condition (2). Let D be a Γ-invariant differential operator of order k, then $D : C^k(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ is bounded. Therefore, $D\psi(D_{\tilde{F}}) : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ is bounded. This implies that $D\psi(D_{\tilde{F}})$ satisfies Condition $(2)(a)$.

To check that $D\psi(D_{\tilde{F}})$ satisfies Condition $(2)(b)$, we want to prove that the adjoint $\bar{\psi}(D_{\tilde{F}})D^*$ of $D\psi(D_{\tilde{F}})$ on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ extends to a bounded operator from $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z})\otimes\tilde{F})$ to

$C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Since $(D_{\tilde{F}})^2$ is Γ-elliptic on $\tilde{Z}$, there exists a Γ-parametrix $G_{\tilde{F}}$ of $(D_{\tilde{F}})^2$ such that $G_{\tilde{F}}(D_{\tilde{F}})^2 = I - A_{\tilde{F}}$ and $(D_{\tilde{F}})^2 G_{\tilde{F}} = I - B_{\tilde{F}}$ for some Γ-invariant bounded smoothing operators $A_{\tilde{F}}$ and $B_{\tilde{F}}$ on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Indeed, this parametrix can be obtained by lifting the corresponding parametrix of $(D_F)^2 = (\frac{1}{2}(d_F + d_F^*))^2$ on $L^2(Z, \wedge^*(T^*Z) \otimes F)$.

$$G_{\tilde{F}}^k (D_{\tilde{F}})^{2k} = G_{\tilde{F}}^{k-1}(I - A_{\tilde{F}})(D_{\tilde{F}})^{2(k-1)} = I - \sum_{j=0}^{k-1} G_{\tilde{F}}^j A_{\tilde{F}} D_{\tilde{F}}^{2j}. \quad (2.3)$$

$$(D_{\tilde{F}})^{2k} G_{\tilde{F}}^k = I - \sum_{j=0}^{k-1} D_{\tilde{F}}^{2j} B_{\tilde{F}} G_{\tilde{F}}^j. \quad (2.4)$$

Hence,

$$(D_{\tilde{F}})^l \bar{\psi}(D_{\tilde{F}}) D^* = (D_{\tilde{F}})^l \bar{\psi}(D_{\tilde{F}})((D_{\tilde{F}})^{2k} G_{\tilde{F}}^k + \sum_{j=0}^{k-1} D_{\tilde{F}}^{2j} B_{\tilde{F}} G_{\tilde{F}}^j) D^*.$$

The operators $G_{\tilde{F}}^k D^*$ and $B_{\tilde{F}} G_{\tilde{F}}^j D^*$ are Γ-invariant pseudodifferential operators of order< 0, and therefore they are bounded. Also $(D_{\tilde{F}})^l \bar{\psi}(D_{\tilde{F}}) D_{\tilde{F}}^{2j}$, $j = 0, 1, \cdots k$, are bounded. Thus for each $l > 0$, $(D_{\tilde{F}})^l \bar{\psi}(D_{\tilde{F}}) D^*$ is bounded on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. This implies by the Sobolev embedding theorem that $\psi(D_{\tilde{F}})^* D^* : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ is bounded.

The above argument also proves that $D\psi(D_{\tilde{F}})^*$ satisfies Condition (2). ∎

For $\varepsilon > 0$ let $E_\varepsilon^\perp = I - E_\varepsilon$ and G_ε be the Green operator of the operator $E_\varepsilon^\perp (D_{\tilde{F}})^2 E_\varepsilon^\perp$, i.e., G_ε is the inverse of $E_\varepsilon^\perp (D_{\tilde{F}})^2 E_\varepsilon^\perp$ on $Im(E_\varepsilon^\perp)$ and is zero on $Im(E_\varepsilon)$. We will need the following theorem.

Theorem 2.1 *Let $A \in S(\tilde{F})$. Then for each integer $l > 0$, $G_\varepsilon^l A \in S(\tilde{F})$.*

Proof. We divide the proof into three steps.

Step 1. We want to prove that for each integer $r \geq 0$ there exists an integer $k > 0$ such that $(G_\varepsilon)^k : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}) \to C^r(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ is bounded. We use the parametrix $G_{\tilde{F}}$ in the proof of Lemma 2.1. For a Γ-invariant differential operator D of order $\leq r$, we choose an integer $k > 1$ such that as a Γ-invariant pseudodifferential operator of order $\leq -2k + r$ $DG_{\tilde{F}}^k$ maps $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ into a Sobolev space on $\tilde{Z}$ contained in $C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Since the operator $A_{\tilde{F}}$ in (2.3) is a bounded Γ-invariant smoothing operator on $\tilde{Z}$, $DG_{\tilde{F}}^j A_{\tilde{F}}$, $j = 0, 1, \cdots, k-1$, also map $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ continuously to $C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. On the other hand, for $0 \leq j \leq k$, $D_{\tilde{F}}^{2j} G_{\varepsilon}^{k} = E_{\varepsilon}^{\perp} D_{\tilde{F}}^{2(-k+j)} E_{\varepsilon}^{\perp}$ is a bounded operator on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Combining these together, we conclude by (2.3) that

$$DG_{\varepsilon}^{k} = D(G_{\tilde{F}}^{k} D_{\tilde{F}}^{2k} + \sum_{j=0}^{k-1} G_{\tilde{F}}^{j} A_{\tilde{F}} D_{\tilde{F}}^{2j}) G_{\varepsilon}^{k} : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$$

is bounded.

Step 2. Let $C_2^{\infty}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ be the space of all L^2-$\tilde{F}$-valued smooth forms ξ on $\tilde{Z}$ such that for each Γ-invariant differential operator D of order k, $k \in \mathbb{N}$, $D\xi \in L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Then G_{ε} is a map on $C_2^{\infty}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. We first prove that for $\xi \in C_2^{\infty}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$, $G_{\varepsilon}\xi$ is smooth. Indeed, for each integer $r \geq 0$, choose an integer k big enough as in Step 1. Then since $G_{\varepsilon} D_{\tilde{F}}^{2k} = I$ on $Im(E_{\varepsilon}^{\perp})$,

$$G_{\varepsilon}\xi = G_{\varepsilon}^{k} D_{\tilde{F}}^{2k} G_{\varepsilon}\xi = G_{\varepsilon}^{k} G_{\varepsilon} D_{\tilde{F}}^{2k} \xi \in C^{r}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}).$$

To check that $G_{\varepsilon}\xi \in C_2^{\infty}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$, let D be a Γ-invariant differential operator of order k. By (2.3),

$$DG_{\varepsilon}\xi = DG_{\tilde{F}}^{k} G_{\varepsilon} D_{\tilde{F}}^{2k} \xi + \sum_{j=0}^{k-1} DG_{\tilde{F}}^{j} A_{\tilde{F}} G_{\varepsilon} D_{\tilde{F}}^{2j} \xi.$$

Since $DG_{\tilde{F}}^{k}$ and $DG_{\tilde{F}}^{j} A_{\tilde{F}}$ are Γ-invariant pseudodifferential operator of order < 0, they are bounded operators on $L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. Hence from $D_{\tilde{F}}^{2j}\xi \in L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$, we get the result.

Step 3. We prove that $G_{\varepsilon} A \in S(\tilde{F})$. Note that for a fixed variable x or y, the kernel $k_A(\tilde{x}, \tilde{y})$ of the operator A is in $C_2^{\infty}(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$. By Step 2, $G_{\varepsilon} A$ has a smooth kernel. To show that $G_{\varepsilon} A$ satisfies Conditions $(2) - (3)$ in the definition of $S(\tilde{F})$, we proceed exactly as we did in Step 2 and prove that for each Γ-invariant differential operator D of order k, $DG_{\varepsilon} A : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ is bounded. Clearly, $A^* G_{\varepsilon}^* D^*, DA^* G_{\varepsilon}^*$ and $G_{\varepsilon} A D^* : L^2(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F}) \to C(\tilde{Z}, \wedge^*(T^*\tilde{Z}) \otimes \tilde{F})$ are bounded. Hence, $G_{\varepsilon} A \in S(\tilde{F})$.

Repeatly using the above result, we conclude that for each integer $l \geq 1$, $G_{\varepsilon}^{l} A \in S(\tilde{F})$. ∎

We now extend the above results to the case of fiber bundles. We will continue to use the notation setting up at the beginning of this section. Since we are only interested in estimates local in B, we now assume $M \simeq B \times \tilde{Z}$. Let $C^{\infty}(B, \mathcal{D}(E))$ be the algebra of all smooth maps $D : B \to Diff^{\Gamma}(E)$ such that for $x \in B$, $D(x)$ is a Γ-invariant differential operator on M_x with coefficients depending smoothly on the variables of B. Let $\mathcal{N}^s = C^\infty(B, S(E))$ be the space of all smooth maps $A : B \to S(E)$ satisfying that for $x \in B$, $A(x) \in S(E|_{M_x})$ and that the norm $\|A(x)\|$ of $A(x)$ can be estimated uniformly on each compact subset of B. To clarify the meaning of $A \in \mathcal{N}^s$ to be smooth, let us set

$$M \times_\pi M = \{(x,y) \in M \times M : \pi(x) = \pi(y) \in B\},$$

where $\pi : M \to B$ is the bundle projection. $M \times_\pi M$ is a fiber bundle over B whose fiber over $x \in B$ is equal to $M_x \times M_x$. We have two obvious projections P_1 and P_2 from $M \times_\pi M$ to M. Given a bundle E over M as in the locally Γ-invariant data, we denote by $E \otimes_\pi E^*$ the bundle $P_1 E^* \otimes P_2^* E^*$ over $M \times_\pi M$, where E^* is the dual bundle of E. We say $A \in \mathcal{N}^s$ depends smoothly on the variables of B if its kernel $k_A(\tilde{x}, \tilde{y}) \in C^\infty(M \times_\pi M, E \otimes_\pi E^*)$. It may not automatically imply, however, that the norm $\|A(x)\|$ of $A(x)$ can be estimated uniformly on each compact subset of B, since the fiber M_x is noncompact in general. This differs from the compact situation. We may demand more by interpreting A locally to be a smooth family of Γ-trace class operators on a fixed Hilbert space which can be done via the local structure maps of the bundles M and E.

With the above notation, we first have the following basic fact.

Lemma 2.2 *For $t > 0$, $e^{-t(\frac{i\tilde{V}}{2})^2} \in \mathcal{N}^s$.*

Proof. We point out that $\tilde{V}_x = d^*_{\tilde{Z}} - d_{\tilde{Z}}$ depends on $x \in B$ smoothly via the 1-form part of the connection $\nabla^{E|_{M_x}}$ and the adjoint $d^*_{\tilde{Z}} = \pm\tau_x^{-1}(d\hat{\otimes}1 + 1\hat{\otimes}\nabla^{\bar{E}^*|_{M_x}})\tau_x$, where $\bar{E}^*$ is the antidual of the bundle E, and τ_x is the Hodge type star operator depending smoothly on the metrics on M_x and $E|_{M_x}$ (cf.[BL] (3.33)). By Lemma 2.1, it suffices to prove that $e^{-t(\frac{i\tilde{V}}{2})^2}$ depends smoothly on $x \in B$.

Using a contour integral representation of $e^{-t(\frac{i\tilde{V}}{2})^2}$ and as in ([BGV], p. 79), one can easily check that the derivative $\frac{d}{dx}e^{-t(\frac{i\tilde{V}}{2})^2}$ with respect to $x \in B$ is a bounded operator. Furthermore, by the uniqueness of the solution of the heat equation, we get the following Duhamel formula:

$$\frac{d}{dx}e^{-t(\frac{i\tilde{V}}{2})^2} = -\int_0^t e^{-(t-s)(\frac{i\tilde{V}}{2})^2}\frac{d}{dx}(\frac{i\tilde{V}}{2})^2 e^{-s(\frac{i\tilde{V}}{2})^2}ds. \tag{2.5}$$

Since $\frac{d}{dx}(\frac{i\tilde{V}}{2})^2$ is just a sum of some products of $d_{\tilde{Z}}$, τ_x, $\frac{d}{dx}\tau_x$, and derivatives of 1-form parts of $\nabla^{E|_{M_x}}$ and $\nabla^{\bar{E}^*|_{M_x}}$, Lemma 2.1 tells us that the right hand side of (2.5) is in $S(E|_{M_x})$. We can repeatly use the Duhamel formula to deal with higher derivatives. Hence, $e^{-t(\frac{i\tilde{V}}{2})^2} \in \mathcal{N}^s$. ∎

However, we meet some difficulties in dealing with continuity or smooth property of the spectral family $\{E_\lambda\}$ of $(\frac{i\tilde{V}}{2})^2$ on the variables of B. There may not exist any gap in the spectrum of $(\frac{i\tilde{V}}{2})^2$ on a fiber. Therefore, we may not be able to explicitly express E_λ as we do in the compact case by the integral of $(z-(\frac{i\tilde{V}}{2})^2)^{-1}$ over a contour surrounding the corresponding isolated spectral subset. Thus we are forced to use our locally Γ-invariant bundle structure instead of general operator theory to prove the following fundamental result.

Theorem 2.2 *The orthogonal projection $P_{\tilde{V}}$ from $C^\infty(B,\mathcal{E})$ onto $Ker((\frac{i\tilde{V}}{2})^2)$ is in $\mathcal{N}^s$.*

Proof. By Lemma 2.1, we need only to prove that $P_{\tilde{V}}$ depends smoothly on the variables of B, since $\|P_{\tilde{V}}\|=1$.

Let $U\subset B$ be a local chart and fix a point $x_0\in U$ such that $M_U \overset{\varphi_U}{\simeq} U\times M_{x_0}$, $M_{x_0}\simeq \tilde{Z}$, and $E|_U \overset{\psi_U}{\simeq} U\times E|_{M_{x_0}}$ are Γ-invariant diffeomorphisms. For each $x\in B$, $\Phi_U^*(x): L^2(M_{x_0},\wedge^*(T^*M_{x_0})\otimes E|_{M_{x_0}})\to L^2(M_x,\wedge^*(T^*M_x)\otimes E|_{M_x})$ defined by

$$\Phi_U^*(x)\xi(y)=\psi_U^{-1}(x)(\xi(\varphi_U(x)y))$$

is an invertible bounded operator and depends on $x\in U$ smoothly. The smooth property means that Φ_U^* maps smooth sections in $C^\infty(U\times M_{x_0}, U\times E|_{M_{x_0}})$ to smooth sections in $C^\infty(M_U,E|_U)$. In view of the compatibility of the connection ∇^E and ψ_U, we have the following commutative diagram:

$$\begin{array}{ccc} L^2(M_{x_0},\wedge^*(T^*M_{x_0})\otimes E|_{M_{x_0}}) & \overset{d_{M_{x_0}}}{\rightarrow} & L^2(M_{x_0},\wedge^*(T^*M_{x_0})\otimes E|_{M_{x_0}}) \\ \downarrow \Phi_U^*(x) & & \downarrow \Phi_U^*(x) \\ L^2(M_x,\wedge^*(T^*M_x)\otimes E|_{M_x}) & \overset{d_{M_x}}{\rightarrow} & L^2(M_x,\wedge^*(T^*M_x)\otimes E|_{M_x}). \end{array} \tag{2.6}$$

Here the exterior differential $d_{M_{x_0}}=d\hat{\otimes}1+1\hat{\otimes}\nabla^{E|_{M_{x_0}}}$ is of course densely defined. From this we get $\Phi_U^*(x)(Ker(d_{M_{x_0}}))\subset Ker(d_{M_x})$ and $\Phi_U^*(x)(\overline{Im(d_{M_{x_0}})})\subset\overline{Im(d_{M_x})}$. By Hodge theory, we have

$$\begin{array}{c} L^2(M_{x_0},\wedge^*(T^*M_{x_0})\otimes E|_{M_{x_0}})=Ker(d_{M_{x_0}}+d^*_{M_{x_0}})\oplus\overline{Im(d_{M_{x_0}})}\oplus\overline{Im(d^*_{M_{x_0}})} \\ \downarrow \Phi_U^*(x)=\begin{bmatrix} \Phi_{11}^*(x) & 0 & \Phi_{13}^*(x) \\ \Phi_{21}^*(x) & \Phi_{22}^*(x) & \Phi_{23}^*(x) \\ 0 & 0 & \Phi_{33}^*(x) \end{bmatrix} \\ L^2(M_x,\wedge^*(T^*M_x)\otimes E|_{M_x})=Ker(d_{M_x}+d^*_{M_x})\oplus\overline{Im(d_{M_x})}\oplus\overline{Im(d^*_{M_x})}. \end{array}$$

It follows clearly that $\Phi_{22}^*(x)$ is an invertible operator depending on $x \in U$ smoothly. Let $P_{\overline{Im(d_{M_{x_0}})}}$ be the orthogonal projection from $L^2(M_{x_0}, \wedge^*(T^*M_{x_0}) \otimes E|_{M_{x_0}})$ onto $\overline{Im(d_{M_{x_0}})}$. Define $\tilde{P}_{\overline{Im(d_{M_x})}} = \Phi_{22}^*(x) P_{\overline{Im(d_{M_{x_0}})}} (\Phi_U^*(x))^{-1}$. $\tilde{P}_{\overline{Im(d_{M_x})}}$ varies smoothly on $x \in U$ and is an idempotent. It may not be, however, an orthogonal projection. Let

$$Q_{\overline{Im(d_{M_x})}} = I + (\tilde{P}_{\overline{Im(d_{M_x})}} - \tilde{P}^*_{\overline{Im(d_{M_x})}})(\tilde{P}^*_{\overline{Im(d_{M_x})}} - \tilde{P}_{\overline{Im(d_{M_x})}}).$$

$Q_{\overline{Im(d_{M_x})}}$ is a smooth family of invertible selfadjoint operators. $P_{\overline{Im(d_{M_x})}} = \tilde{P}_{\overline{Im(d_{M_x})}} \tilde{P}^*_{\overline{Im(d_{M_x})}} Q^{-1}_{\overline{Im(d_{M_x})}}$ is the orthogonal projection onto $\overline{Im(d_{M_x})}$ and is smooth with respect to $x \in U$.

Let τ_x be the Hodge type operator such that $d^*_{M_x} = \pm \tau_x^{-1} \hat{d}_{M_x} \tau_x$, where $\hat{d}_{M_x}$ is the diferentiation associated with the antidual bundle $\bar{E}^*$ of E. τ_x^{-1} maps $\overline{Im(\hat{d}_{M_x})}$ onto $\overline{Im(d^*_{M_x})}$. Let $\tilde{P}_{\overline{Im(d^*_{M_x})}} = \tau_x^{-1} \tilde{P}_{\overline{Im(\hat{d}_{M_x})}} \tau_x$. $\tilde{P}_{\overline{Im(d^*_{M_x})}}$ is an idempotent depending smoothly on $x \in U$, since $\tilde{P}_{\overline{Im(\hat{d}_{M_x})}}$ is smooth as we showed above (with E replaced by its antidual). Using the above argument, we get an orthogonal projection $P_{\overline{Im(d^*_{M_x})}}$ onto $\overline{Im(d^*_{M_x})}$ which depends on $x \in U$ smoothly. Therefore, we obtain that the orthogonal projection $P_{\tilde{V}} = I - (P_{\overline{Im(d_{M_x})}} + P_{\overline{Im(d^*_{M_x})}})$ is smooth with respect to $x \in U$. ∎

It is not clear in general, however, whether the projection E_λ, $\lambda > 0$, depends smoothly on the variables of B. Under a condition that $\{\varphi_U\}$ and $\{\psi_U\}$ in the locally Γ-invariant data be isometries, namely, for $x \in B$, $\varphi_U(x) : M_x \simeq M_{x_0}$ and $\psi_U(x) : E|_{M_x} \simeq E|_{M_{x_0}}$ are isometries, E_λ is a smooth family of projections.

The smooth property of elements in the algebra $\mathcal{N}^s$ is not convenient for us to verify. For the purpose of the next section, we introduce a weak notion as a tool to deal with the proof of Lemma 3.3. Let $\mathcal{N}$ be the space of maps A from B to $S(E)$ satisfying that for $x \in B$ $A(x) \in S(E|_{M_x})$ and the usual operator norm $\|A(x)\|$ of $A(x)$ can be estimated uniformly on each compact subset of B. Obviously, $\mathcal{N}$ is a $*$-algebra. Also $P_{\tilde{V}}$ and $e^{-t(\frac{i\tilde{V}}{2})^2}$ are in $\mathcal{N}$. The following lemma says that $\mathcal{N}$ is closed under left and right multiplications by G_ε ($\varepsilon > 0$).

Lemma 2.3 *Let $A \in \mathcal{N}$. Then $G_\varepsilon A$ and AG_ε are in $\mathcal{N}$.*

Proof. By Theorem 2.1, we know that both $G_\varepsilon A$ and AG_ε are the maps from B to $S(E)$. Since $\|G_\varepsilon\| \leq \frac{1}{\varepsilon}$, the norms of these two operators can uniformly be estimated on each compact subset of B. ∎

3 L^2-Transgression Formula and The R-R-G Theorem

With the preparation of Section 2, we are now able in this section to derive an L^2-version of the Riemann-Roch-Grothendieck theorem and the corresponding transgression formula.

Using the notation in Section 2, we denote by $R^{\tilde{Z}}$ the curvature of $\nabla^{\tilde{Z}}$ and

$$R^{\tilde{Z}}_{jk} = < \tilde{e}_j, R^{\tilde{Z}}\tilde{e}_k >_{g^{\tilde{Z}}} .$$

Let $\hat{R}^{\tilde{Z}} \in \Omega^2(M, End(\wedge(T^*\tilde{Z})))$ and $R \in \Omega^2(M, End(\wedge^*(T^*\tilde{Z}) \otimes E))$ be defined by

$$\hat{R}^{\tilde{Z}} = \frac{1}{4} R^{\tilde{Z}}_{jk}\tilde{c}^j_v\tilde{c}^k_v,$$

and

$$R = \hat{R}^{\tilde{Z}} \otimes I + I \otimes R^{E,u},$$

where $R^{E,u}$ is the curvature of $\nabla^{E,u} = \frac{1}{2}(\nabla^E + (\nabla^E)^*)$. Note that $\nabla^{E,u} = \nabla^E + \frac{1}{2}\psi$ and $R^{E,u} = -\frac{1}{4}\psi^2$. Let z be an auxiliary odd Grassmann variable which anticommutes with all other odd Grassmann variables and $z^2 = 0$. Define

$$\mathcal{D}_{\tilde{e}_j} = \nabla^{\tilde{Z}\otimes E,u}_{\tilde{e}_j} - \frac{1}{2\sqrt{t}} w_{\alpha jk} m^\alpha \tilde{c}^k - \frac{1}{4t} w_{\alpha\beta j} m^\alpha m^\beta - \frac{1}{2\sqrt{t}} z\tilde{c}^j_v$$

and

$$\mathcal{D}^2 = \mathcal{D}_{\tilde{e}_j}\mathcal{D}_{\tilde{e}_j} - \mathcal{D}_{\nabla^{\tilde{Z}}_{\tilde{e}_j}\tilde{e}_j}.$$

The following Lichnerowicz type identity is important.

Lemma 3.1 ([BL]) *For $t > 0$,*

$$\begin{aligned}
\tilde{C}^2_{4t} - z\tilde{D}_{4t} &= t(-\mathcal{D}^2 + \frac{R}{4}) + \frac{t}{2}\tilde{c}^i\tilde{c}^j R(\tilde{e}_i, \tilde{e}_j) + \sqrt{t}\tilde{c}^i m^\alpha R(\tilde{e}_i, f_\alpha) \\
&+ \frac{1}{2} m^\alpha m^\beta R(f_\alpha, f_\beta) + t(\frac{1}{4}\psi^2_j + \frac{1}{8}\tilde{c}^j_v\tilde{c}^k_v[\psi_j, \psi_k] - \frac{1}{2}\tilde{c}^j\tilde{c}^k_v(\nabla^{\tilde{Z}\otimes E,u}_{\tilde{e}_j}\psi_k)) \\
&- \frac{\sqrt{t}}{2} m^\alpha \tilde{c}^j_v(\nabla^{\tilde{Z}\otimes E,u}_{f_\alpha}\psi_j) - \frac{z\sqrt{t}}{2}\tilde{c}^j\psi_j - \frac{z}{2} m^\alpha\psi_\alpha \\
&- \frac{\sqrt{t}}{2}(\sqrt{t}\tilde{c}^j c_v(\nabla^{\tilde{Z}}_{\tilde{e}_j}\tilde{e}_k)\psi_k + m^\alpha c_v(\nabla^{\tilde{Z}}_{f_\alpha}\tilde{e}_k)\psi_k).
\end{aligned}$$

See Theorem 3.11 [BL] for the proof. Here the last two terms in the formula come from the left hand side of (3.62) in [BL] and vanish for an appropriate basis of TM.

Let $O(T\tilde{Z})$ be the orientation line bundle of $T\tilde{Z}$. Recall that the Euler class $e(T\tilde{Z})$ of $\tilde{Z}$ in $H^{\dim(\tilde{Z})}(M, O(T\tilde{Z}))$ can be represented by an $O(T\tilde{Z})$-valued closed form $e(T\tilde{Z}, \nabla^{\tilde{Z}})$ on M,

$$e(T\tilde{Z}, \nabla^{\tilde{Z}}) = \begin{cases} Pf(\frac{R^{\tilde{Z}}}{2\pi}) \equiv T(\exp \frac{-R^{\tilde{Z}}}{2\pi}), & \dim(\tilde{Z}) \ \ even, \\ \qquad 0, & \dim(\tilde{Z}) \ \ odd, \end{cases} \tag{3.1}$$

where T is the Berezin integral $T : \wedge(T^*\tilde{Z}) \to \mathbb{R}$. When $\tilde{Z}$ is oriented, T is given by $T(\tilde{e}^I) = 1$ for $|I| = \dim(\tilde{Z})$, and 0 otherwise. T extends easily to a form-valued linear map.

Let $\varphi : \Omega(B) \to \Omega(B)$ be the linear map defined by

$$\varphi(\phi) = (2\pi i)^{-\frac{k}{2}}\phi, \quad \phi \in \Omega^k(B).$$

Set $f(z) = z\exp(z^2)$. $f(z)$ is a holomorphic real odd function in $\mathbb{C}$, i.e., $f(-z) = -f(z)$ and $f(\bar{z}) = \overline{f(z)}$. We wish to apply the function f to the operator $\tilde{D}_t$. To do this we regard $\tilde{D}_t$ as the sum of forms of degrees $0, 1$ and 2 on B with values in the operators $End(\mathcal{E})$. Each fiber $\mathcal{E}_x$ of $\mathcal{E}$ completes to a Hilbert space, and thus $\tilde{D}_t$ can be extended to an unbounded essentially selfadjoint operator on each fiber of $\mathcal{E}$. We can apply the functional calculus fiberwise to define $f(\tilde{D}_t)$. Now standard calculations of heat kernel on covering spaces show that $f(\tilde{D}_t)$ is itself given by a form on B with values in the fiberwise Γ-trace class operators on $\mathcal{E}$ (see the proof of Lemma 2.2). Thus we define

$$f(\tilde{D}_t, g^{\mathcal{E}}) = (2\pi i)^{\frac{1}{2}}\varphi tr^s_\Gamma(f(\tilde{D}_t)) \in \Omega(B)$$

and

$$f(\nabla^E, g^E) = (2\pi i)^{\frac{1}{2}}\varphi tr^s(f(\frac{1}{2}((\nabla^E)^* - \nabla^E))).$$

As in [BL], one can see that $f(\tilde{D}_t, g^{\mathcal{E}})$ and $f(\nabla^E, g^E)$ are real, odd and closed forms on B. For convenience, we also denote $w(E, g^E) = (g^E)^{-1}(\nabla^E g^E)$. Recall that this is ψ preceding (2.2). Then $(\nabla^E)^* - \nabla^E = w(E, g^E)$ and

$$f(\nabla^E, g^E) = (2\pi i)^{\frac{1}{2}}\varphi tr^s(f(\frac{1}{2}w(E, g^E))).$$

We now work out the limit of $f(\tilde{D}_t, g^{\mathcal{E}})$ as $t \to \infty$ and $t \to 0$, respectively. We say a family of smooth forms $\{w_t\}_{t\in[0,\infty)}$ on B has asymptotic w_∞ as $t \to \infty$ (resp., w_0 as $t \to 0$) if for each compact set $K \subset B$, there are constants C_K and $\alpha_K > 0$ such that the supreme of the norm of $w_t - w_\infty$ (resp., $w_t - w_0$) on K is bounded above by $\frac{C_K}{t^{\alpha_K/2}}$ (resp., $C_K t^{\alpha_K/2}$). We denote it by $w_t \to_{t\to\infty} w_\infty + O(t^{-\alpha_K/2})$ (resp., $w_t \to_{t\to 0} w_0 + O(t^{\alpha_K/2})$).

Lemma 3.2 *The de Rham cohomology class of the closed form $f(\tilde{D}_t, g^{\mathcal{E}})$ is independent of the parameter t, and depends only on the equivalence class of the local Γ-invariance structure. Furthermore, as $t \to 0$,*

$$f(\tilde{D}_t, g^{\mathcal{E}}) = \begin{cases} \int_{\mathcal{F}} e(T\tilde{Z}, \nabla^{\tilde{Z}}) f(\nabla^E, g^E) + O(t), & if \quad \dim(\tilde{Z}) \ \ even, \\ O(\sqrt{t}), & if \quad \dim(\tilde{Z}) \ \ odd, \end{cases} \tag{3.2}$$

where $\mathcal{F}$ is a fundamental domain of Γ in $\tilde{Z}$ with respect to some local bundle chart of an associated family.

Proof. The proof of the first assertion is exactly the same as that of Theorem 3.16 [BL]. To prove (3.2), we use the Getzler rescaling together with the heat kernel method for covering spaces. Since the proof is similar to that of the family index theorem [BGV], we only indicate some main points and refer the details to [BGV][BL].

Step 1. Using the odd Grassmann variable z defined as before, we can write each $w \in \Omega(B) \otimes \mathbb{C}[z]$ as

$$w = w_0 + zw_1, \quad w_i \in \Omega(B),$$

and set $w^z = w_1$. Since $d_M^2 = 0$ and $[\tilde{D}_t, \tilde{C}_t] = 0$, we have $\tilde{C}_t^2 = -\tilde{D}_t^2$ and

$$\begin{aligned} f(\tilde{D}_t, g^{\mathcal{E}}) &= (2\pi i)^{\frac{1}{2}} \varphi tr_\Gamma^s(\tilde{D}_t \exp(-\tilde{C}_t^2)) \\ &= (2\pi i)^{\frac{1}{2}} \varphi [tr_\Gamma^s(\exp(-\tilde{C}_t^2 + z\tilde{D}_t))]^z. \end{aligned}$$

If $\tilde{N}$ denotes the number operator on $\Omega(B) \otimes \mathbb{C}[z]$ given by $\tilde{N}(w) = deg(w)w$, where $deg(w)$ is the total degree of w, then

$$f(\tilde{D}_t, g^{\mathcal{E}}) = (2\pi i)^{\frac{1}{2}} \varphi [t^{-\tilde{N}/2} tr_\Gamma^s(\exp(t(-\tilde{C}_1^2 + z\tilde{D}_1)))]^z.$$

Step 2. To find the asymptotics of the kernel $K_t^\Gamma(y_1, y_2)$ of the operator $\exp(t(-\tilde{C}_1^2 + z\tilde{D}_1))$ on the diagonal as $t \to 0$, let us note that $K_t^\Gamma(y_1, y_2)$ is locally fiberwise Γ-equivariant, since

$-\tilde{C}_1^2 + z\tilde{D}_1$ has such a property. For each $x \in B$, let $(-C_1^2 + zD_1)_x$ be the pushdown of the operator $(-\tilde{C}_1^2 + z\tilde{D}_1)|_{M_x}$ to $M_x/\Gamma \simeq Z$ and $K_t^x(y_1, y_2)$ be its kernel. Then over each fiber,

$$K_t^x(y_1, y_2) = \sum_{g\in\Gamma} K_t^\Gamma(\tilde{y}_1, g\tilde{y}_2)(x), \qquad \pi(\tilde{y}_1) = y_1, \pi(\tilde{y}_2) = y_2.$$

Hence, $K_t^\Gamma(\tilde{y}_1, \tilde{y}_2)$ and $K_t(y_1, y_2)$ have the same asymptotics as $t \to 0$ for $y_1 = y_2$ [Lo]. This implies the asymptotic rate of the RHS of (3.2). The rest of proof follows exactly [BL]. Let the dimension of $\tilde{Z}$ be even. By Lemma 3.1, rescaling that $\partial_j \to s^{-\frac{1}{2}}\partial_j, \tilde{c}^j \to s^{-\frac{1}{2}}\tilde{m}^j - s^{\frac{1}{2}}\tilde{\iota}^j, \tilde{c}_v^j \to \tilde{c}_v^j$, $z \to s^{-\frac{1}{2}}z$ and $m^\alpha \to s^{-\frac{1}{2}}m^\alpha$, we get that as $s \to 0$,

$$s(\tilde{C}_4^2 - z\tilde{D}_4) \to (\partial_j - \frac{1}{4}R_{jk}^{\tilde{Z}}x_k - \frac{1}{2}z\tilde{c}_v^j)^2 + R - \frac{1}{2}z\psi.$$

This together with local index theory techniques implies

$$\begin{aligned}
\lim_{t\to 0} f(\tilde{D}_t, g^{\mathcal{E}}) &= \lim_{t\to 0}(2\pi i)^{\frac{1}{2}}\varphi[t^{-\tilde{N}/2}tr^s(\exp(t(-\tilde{C}_1^2 + z\tilde{D}_1))|_{\mathcal{F}})]^z \\
&= \int_{\mathcal{F}} e(T\tilde{Z}, \nabla^{\tilde{Z}})(2\pi i)^{\frac{1}{2}}\varphi tr^s(\exp(-R^{E,u} + \frac{z}{2}w(E, g^E))))^z \\
&= \lim_{t\to 0} \ \textit{of RHS of } (3.2).
\end{aligned}$$

If $\dim(\tilde{Z})$ is odd, using the fact that $tr^s(S) = (-2)^{\dim(\tilde{Z})}$ for $S = \tilde{c}^1\tilde{c}_v^1\cdots\tilde{c}^{\dim(\tilde{Z})}\tilde{c}_v^{\dim(\tilde{Z})}$ and $tr^s(S) = 0$ otherwise, we obtain the result by the above local index theory techniques. ∎

But the asymptotics of $f(\tilde{D}_t, g^{\mathcal{E}})$ as $t \to \infty$ is complicated. Unlike the situation of compact fibers, we will have to control the decay rate of $f(\tilde{C}_t, g^{\mathcal{E}})$ as $t \to \infty$. Let $H_2(\tilde{Z}, E|_{\tilde{Z}})$ be the vector bundle over B whose fiber over $x \in B$ is $H_2^*(M_x, E|_{M_x})$, the L^2-cohomology of the fiber M_x with coefficients in $E|_{M_x}$. By the L^2-version of Hodge theory, $H_2^*(M_x, E|_{M_x}) \simeq Ker(\tilde{V}_x)$. To define a connection on $H_2(\tilde{Z}, E|_{\tilde{Z}})$, let $P : Ker(d_{\tilde{Z}}) \to H_2(\tilde{Z}, E|_{\tilde{Z}})$ be the natural projection. Given a smooth section ξ of $H_2(\tilde{Z}, E|_{\tilde{Z}})$, let $\tilde{\xi} \in Ker(d_{\tilde{Z}})$ be such that $P(\tilde{\xi}) = \xi$. Then for a vector field U on B, we set

$$\nabla_U^{H_2}\xi = P(\nabla_U^{\mathcal{E}}\tilde{\xi}).$$

One can easily check that ∇^{H_2} is well-defined and is flat. In fact, using the smooth family of projections $P_{\tilde{V}}$ onto $Ker(\tilde{V})$, one may also define a connection on $H_2(\tilde{Z}, E|_{\tilde{Z}})$ to be $P_{\tilde{V}}\nabla^{\mathcal{E}}$. Let g^{H_2} be the metric on $H_2(\tilde{Z}, E|_{\tilde{Z}})$ inherited from $Ker(\tilde{V})$ and $f(\nabla^{H_2}, g^{H_2}) = (2\pi i)^{\frac{1}{2}}\varphi tr_\Gamma^s(f(\frac{1}{2}(\nabla^{H_2^*} - \nabla^{H_2})))$. The following lemma is important for the present paper and its proof is much more complicated than that in the compact case.

Lemma 3.3 *Assume that for each compact subset $K \subset B$ there exists a positive number $\beta_K > 0$ such that*

$$tr_\Gamma(e^{(P_{\tilde{V}}^{\perp}\frac{\sqrt{t}}{2}\tilde{V}P_{\tilde{V}}^{\perp})^2}) \to_{t\to\infty} O(t^{-\beta_K/2}) \tag{3.3}$$

on K, where $P_{\tilde{V}}^{\perp} = 1 - P_{\tilde{V}}$. Then for some $\beta'_K > 0$,

$$f(\tilde{D}_t, g^{\mathcal{E}}) \to_{t\to\infty} f(\nabla^{H_2}, g^{H_2}) + O(t^{-\beta'_K/2}) \ \text{ on } \ K. \tag{3.4}$$

Proof. Note that $\tilde{V}$ is essentially antiselfadjoint and $(\tilde{V})^2$ is a nonpositive operator. We divide the proof into five steps.

Step 1. Let $(i\frac{\tilde{V}}{2})^2 = \int \lambda dE_\lambda$ be the spectral representation of the operator $(i\frac{\tilde{V}}{2})^2$. In view of a relation between the decay rate of $tr_\Gamma(E_\lambda - E_0)$ and that of (3.3) [ES], we obtain

$$tr_\Gamma(E_\lambda - E_0) \to O(\lambda^{\beta_K/2}), \ \ \lambda \to 0, \tag{3.5}$$

on the compact set $K \subset B$. Note that for $x \in Im(E_\lambda)$, $< (i\frac{\tilde{V}}{2})^2 x, x > \leq \lambda \|x\|^2$. This implies that $E_0 \leq P_{\tilde{V}}$. But $tr_\Gamma(E_0) = tr_\Gamma(P_{\tilde{V}})$, and as a usual finite trace on the von Neumann algebra generated by Γ, tr_Γ is faithful, we have $E_0 = P_{\tilde{V}}$. Let $P_\varepsilon = E_\varepsilon - E_0$ and $E_\varepsilon^\perp = I - E_\varepsilon$, where $\varepsilon > 0$. Let

$$\begin{aligned} \bar{D}(\varepsilon) &= P_{\tilde{V}} i\frac{w}{2} P_{\tilde{V}} + P_{\tilde{V}}\frac{iw}{2}E_\varepsilon^\perp + E_\varepsilon^\perp \frac{iw}{2} P_{\tilde{V}} + E_\varepsilon^\perp i\frac{\tilde{V}+w}{2}E_\varepsilon^\perp + P_\varepsilon i\frac{\tilde{V}+w}{2}P_\varepsilon \\ &\stackrel{def}{=} \hat{D}(\varepsilon) + P_\varepsilon i \frac{\tilde{V}+w}{2}P_\varepsilon. \end{aligned} \tag{3.6}$$

Here $w = w(\mathcal{E}, g^{\mathcal{E}})$. Let $\mathcal{M}_j = \oplus_{l\geq j}\mathcal{B}(B, \wedge^l(T^*B)\otimes(D(\mathcal{E})+S(\mathcal{E})))$ and $\mathcal{N}_j = \oplus_{l\geq j}\mathcal{B}(B, \wedge^l(T^*B)\otimes S(\mathcal{E}))$, where $\mathcal{B}(B, \wedge^l(T^*B)\otimes(D(\mathcal{E})+S(\mathcal{E})))$ is the space of $D(\mathcal{E})+S(\mathcal{E})$-valued l-forms on B, and $\mathcal{B}(B, \wedge^l(T^*B)\otimes S(\mathcal{E}))$ is the space of $S(E)$-valued l-forms A on B such that the operator norm $\|A(x)\|$ of $A(x)$ can uniformly be estimated on each compact subset of B. We will only use $\mathcal{N}$ to estimate the difference of the smooth form $f(\tilde{D}_t, g^{\mathcal{E}}) - f(\nabla^{H_2}, g^{H_2})$ by the bound stated in (3.4). So we do not need $A \in \mathcal{N}$ to be smooth with respect to the variables of B, since we are only concerned with the uniform boundedness of the L^1-norm and the operator norm of the above smooth form on each compact subset of B. Let $(\mathcal{N}_j, k)$ be the ideal of those operators in $\mathcal{N}_j$ with k $\hat{G}_\varepsilon$'s. The $\hat{G}_\varepsilon$ is the Green operator of $(E_\varepsilon^\perp \frac{i}{2}\tilde{V}E_\varepsilon^\perp)^2$. It is defined to be the inverse of $(E_\varepsilon^\perp \frac{i}{2}\tilde{V}E_\varepsilon^\perp)^2$ on $Im(E_\varepsilon^\perp)$ and zero on $Im(P_{\tilde{V}}) \oplus Im(P_\varepsilon)$. We will associate $1/\sqrt{t}$ to one form on B and $t^{\frac{1}{8}}$ to one factor $\hat{G}_\varepsilon$, and hence $t^{(k-4j)/8}$ to $(\mathcal{N}_j, k)$. We say $(\mathcal{N}_j, k) \leq (\mathcal{N}_{j'}, k')$ if $t^{(k-4j)/8} \leq t^{(k'-4j')/8}$. By $(\mathcal{N}_j, k, \geq)$ we denote the vector space generated by the union of all $(\mathcal{N}_{j'}, k')$ greater than or equal to $(\mathcal{N}_j, k)$.

We want to prove that there exists a $T(\varepsilon) \in \mathcal{M}_0$ with $T(\varepsilon) - I \in (\mathcal{N}_1, 1, \geq)$ such that for $\varepsilon > 0$,

$$(\bar{D}(\varepsilon))^2 = T(\varepsilon)^{-1} \begin{bmatrix} X(\varepsilon) & 0 & 0 \\ 0 & Y(\varepsilon) & 0 \\ 0 & 0 & Z(\varepsilon) \end{bmatrix} T(\varepsilon), \tag{3.7}$$

where the decompostion is with respect to $I = E_0 \oplus P_\varepsilon \oplus E_\varepsilon^\perp$, and

$$\begin{aligned} X(\varepsilon) &= (P_{\tilde{V}} \frac{i}{2} w P_{\tilde{V}})^2 \;\; mod\; (\mathcal{N}_3, 2, \geq) \\ Y(\varepsilon) &= (E_\varepsilon^\perp (\frac{i}{2}\tilde{V} + \frac{i}{2} w) E_\varepsilon^\perp)^2 \;\; mod\; (\mathcal{N}_2, 0, \geq) \\ Z(\varepsilon) &= (P_\varepsilon i \frac{\tilde{V} + w}{2} P_\varepsilon)^2. \end{aligned}$$

To prove (3.7), we want first to get a corresponding result for $(\hat{D}(\varepsilon))^2$. Such a result was proved in ([BGV], Chapter 9, Lemma 9.4) for the compact case. The major difficulty in the noncompact case is that the Green operator G_r of $(\frac{i}{2}\tilde{V})^2$ might not be bounded. This forces us to consider the above type of operator $\bar{D}(\varepsilon)$ instead of $i\frac{\tilde{V}+w}{2}$. The main idea is to avoid using the Green operator on $Im(P_\varepsilon)$. Note that the Green operator $\hat{G}_\varepsilon$ is bounded,

$$\hat{G}_\varepsilon = \int_\varepsilon^\infty \frac{1}{\lambda} d(E_\lambda - E_\varepsilon).$$

Hence, its norm is bounded by $\frac{1}{\varepsilon}$. For each $\varepsilon > 0$, we can therefore use the argument in [BGV] on $(\hat{D}(\varepsilon))^2$. Since in Step 3 below we will take $\varepsilon = t^{-\frac{1}{8}}$ as $t \to \infty$, we have to carefully count the combination of forms on the base space B and $\hat{G}_\varepsilon$. A one-form on B will give us a factor $\frac{1}{\sqrt{t}}$ in the estimate, but the norm of $\hat{G}_\varepsilon$ will contribute the factor $t^{\frac{1}{8}}$. This means that the product of a one-form on B and $\hat{G}_\varepsilon$ will produce a factor $t^{-\frac{3}{8}}$. Thus, we should be able to get the estimate (3.4) as $t \to \infty$. The task now is to check how the forms on B and $\hat{G}_\varepsilon$ occur in the process of estimation. In view of this and use of the next step, we include a detailed proof here.

The proof is by induction on j in $\mathcal{N}_j$. For $j = 1$, take $T_1(\varepsilon) = I$.

$$(\hat{D}(\varepsilon))^2 \;\; = \;\; \begin{bmatrix} P_{\tilde{V}} \frac{i}{2} w P_{\tilde{V}} & P_{\tilde{V}} \frac{i}{2} w E_\varepsilon^\perp \\ E_\varepsilon^\perp \frac{i}{2} w P_{\tilde{V}} & E_\varepsilon^\perp (\frac{i}{2}\tilde{V} + \frac{i}{2} w) E_\varepsilon^\perp \end{bmatrix}^2$$

$$= \begin{bmatrix} X_1(\varepsilon) & Y_1(\varepsilon) \\ Z_1(\varepsilon) & W_1(\varepsilon) \end{bmatrix} \in \begin{bmatrix} \mathcal{M}_2 & \mathcal{N}_1 \\ \mathcal{N}_1 & \mathcal{M}_0 \end{bmatrix},$$

where

$$X_1(\varepsilon) = (P_{\tilde{V}}\frac{i}{2}wP_{\tilde{V}})^2 + P_{\tilde{V}}\frac{i}{2}wE_\varepsilon^\perp\frac{i}{2}wE_\varepsilon^\perp \in \mathcal{N}_2, \tag{3.8}$$

$$Y_1(\varepsilon) = P_{\tilde{V}}\frac{i}{2}w(P_{\tilde{V}} + E_\varepsilon^\perp)\frac{i}{2}wE_\varepsilon^\perp + P_{\tilde{V}}\frac{i}{2}wE_\varepsilon^\perp\frac{i}{2}\tilde{V}E_\varepsilon^\perp \in \mathcal{N}_1, \tag{3.9}$$

$$Z_1(\varepsilon) = E_\varepsilon^\perp\frac{i}{2}w(P_{\tilde{V}} + E_\varepsilon^\perp)\frac{i}{2}wP_{\tilde{V}} + E_\varepsilon^\perp\frac{i}{2}\tilde{V}\frac{i}{2}wP_{\tilde{V}} \in \mathcal{N}_1, \tag{3.10}$$

$$W_1(\varepsilon) = E_\varepsilon^\perp\frac{i}{2}wP_{\tilde{V}}\frac{i}{2}wE_\varepsilon^\perp + (E_\varepsilon^\perp i\frac{\tilde{V}+w}{2}E_\varepsilon^\perp)^2 = (E_\varepsilon^\perp i\frac{\tilde{V}+w}{2}E_\varepsilon^\perp)^2 \ \ mod\ (\mathcal{N}_2, 0), \tag{3.11}$$

and

$$X_1(\varepsilon) - Y_1(\varepsilon)\hat{G}_\varepsilon Z_1(\varepsilon) = (P_{\tilde{V}}\frac{i}{2}wP_{\tilde{V}})^2 \ \ mod\ (\mathcal{N}_3, 1).$$

Here we have used the fact that by Lemma 2.4, for $A \in \mathcal{N}$,AP_ε, $\hat{G}_\varepsilon A$, $E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp A \in \mathcal{N}$. We have also utilized the fact that $E_\varepsilon^\perp\frac{i\tilde{V}}{2}E_\varepsilon^\perp\frac{iw}{2}P_{\tilde{V}}$ and $P_{\tilde{V}}\frac{iw}{2}E_\varepsilon^\perp\frac{i\tilde{V}}{2}E_\varepsilon^\perp$ are in $\mathcal{N}_1$. The uniform estimates of the norms of these operators on a compact subset K can easily be obtained.

The case $j = 2$ is crucial for the induction. Take

$$T_2(\varepsilon) = \begin{bmatrix} 1 & -Y_1(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_1(\varepsilon) & 1 \end{bmatrix} \in \begin{bmatrix} \mathcal{M}_0 & \mathcal{N}_1 \\ \mathcal{N}_1 & \mathcal{M}_0 \end{bmatrix}. \tag{3.12}$$

Since $Y_1(\varepsilon)\hat{G}_\varepsilon$ and $\hat{G}_\varepsilon Z_1(\varepsilon)$ are forms on B, the following series is finite:

$$\begin{aligned} T_2^{-1}(\varepsilon) &= \sum_{k=0}^{\infty}(-1)^k \begin{bmatrix} 0 & -Y_1(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_1(\varepsilon) & 0 \end{bmatrix}^k \\ &= \begin{bmatrix} 1 & Y_1(\varepsilon)\hat{G}_\varepsilon \\ -\hat{G}_\varepsilon Z_1(\varepsilon) & 1 \end{bmatrix} \ \ mod\ (\mathcal{N}_2, 2, \geq). \end{aligned}$$

Hence, $T_2(\varepsilon) - I$ is in $(\mathcal{N}_1, 1)$, and $T_2^{-1}(\varepsilon) - I$ is in $(\mathcal{N}_1, 1, \geq)$. We have

$$T_2(\varepsilon)(\hat{D}(\varepsilon))^2 T_2^{-1}(\varepsilon) = \begin{bmatrix} X_2(\varepsilon) & Y_2(\varepsilon) \\ Z_2(\varepsilon) & W_2(\varepsilon) \end{bmatrix},$$

where

$$\begin{aligned}
X_2(\varepsilon) &= X_1(\varepsilon) - Y_1(\varepsilon)\hat{G}_\varepsilon Z_1(\varepsilon) \;\; mod\; (\mathcal{N}_3, 2, \geq) \\
&= (P_{\tilde{V}} \frac{iw}{2} P_{\tilde{V}})^2 \;\; mod\; (\mathcal{N}_3, 2, \geq), \\
Y_2(\varepsilon) &= Y_1(\varepsilon) - Y_1(\varepsilon)\hat{G}_\varepsilon W_1(\varepsilon) \;\; mod\; (\mathcal{N}_2, 1, \geq) \\
&= 0 \;\; mod\; (\mathcal{N}_2, 1, \geq), \\
Z_2(\varepsilon) &= Z_1(\varepsilon) - W_1(\varepsilon)\hat{G}_\varepsilon Z_1(\varepsilon) \;\; mod\; (\mathcal{N}_2, 1, \geq) \\
&= 0 \;\; mod\; (\mathcal{N}_2, 1, \geq), \\
W_2(\varepsilon) &= Z_1(\varepsilon)Y_1(\varepsilon)\hat{G}_\varepsilon + \hat{G}_\varepsilon Z_1(\varepsilon)Y_1(\varepsilon) + W_1(\varepsilon) \;\; mod\; (\mathcal{N}_3, 2, \geq) \\
&= (E_\varepsilon^\perp i \frac{\tilde{V} + w}{2} E_\varepsilon^\perp)^2 \;\; mod\; (\mathcal{N}_2, 0, \geq).
\end{aligned}$$

Clearly, $Y_2(\varepsilon)\hat{G}_\varepsilon, \hat{G}_\varepsilon Z_2(\varepsilon), Y_2(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp, E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_2(\varepsilon) \in (\mathcal{N}_2, 2, \geq)$, and $Y_2(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp$, $E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_2(\varepsilon) \in (\mathcal{N}_3, 2, \geq)$.

Suppose that there exists $T_j(\varepsilon) \in (\mathcal{N}_j, j, \geq)$ such that $I - T_j(\varepsilon) \in \mathcal{N}_j$ and

$$\begin{aligned}
T_j(\varepsilon)T_{j-1}(\varepsilon)\cdots T_2(\varepsilon)(\hat{D}(\varepsilon))^2 T_2^{-1}(\varepsilon)T_3^{-1}(\varepsilon)\cdots T_j^{-1}(\varepsilon) &= \begin{bmatrix} X_j(\varepsilon) & Y_j(\varepsilon) \\ Z_j(\varepsilon) & W_j(\varepsilon) \end{bmatrix} \\
&\in \begin{bmatrix} (\mathcal{N}_2, 2, \geq) & (\mathcal{N}_j, j-1, \geq) \\ (\mathcal{N}_j, j-1, \geq) & \mathcal{M}_0 \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
X_j(\varepsilon) &= (P_{\tilde{V}} \frac{iw}{2} P_{\tilde{V}})^2 \;\; mod\; (\mathcal{N}_3, 2, \geq), \\
W_j(\varepsilon) &= (E_\varepsilon^\perp i \frac{\tilde{V} + w}{2} E_\varepsilon^\perp)^2 \;\; mod\; (\mathcal{N}_2, 0, \geq),
\end{aligned}$$

$Y_j(\varepsilon)\hat{G}_\varepsilon, \hat{G}_\varepsilon Z_j(\varepsilon), Y_j(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp, E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_j(\varepsilon) \in (\mathcal{N}_j, j, \geq)$, and $Y_j(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp$, $E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_j(\varepsilon) \in (\mathcal{N}_{j+1}, j, \geq)$. Take

$$T_{j+1}(\varepsilon) = \begin{bmatrix} 1 & -Y_j(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_j(\varepsilon) & 1 \end{bmatrix} \in \begin{bmatrix} \mathcal{M}_0 & (\mathcal{N}_j, j, \geq) \\ (\mathcal{N}_j, j, \geq) & \mathcal{M}_0 \end{bmatrix}. \tag{3.13}$$

Then

$$T_{j+1}^{-1}(\varepsilon) = \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix} 0 & -Y_j(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_j(\varepsilon) & 0 \end{bmatrix}^k$$

$$= \begin{bmatrix} 1 & Y_j(\varepsilon)\hat{G}_\varepsilon \\ -\hat{G}_\varepsilon Z_j(\varepsilon) & 1 \end{bmatrix} \ mod\ (\mathcal{N}_{2j}, 2j, \geq).$$

We have that $T_{j+1}(\varepsilon) - I$ and $T_{j+1}(\varepsilon)^{-1} - I$ are in $(\mathcal{N}_{j+1}, j+1, \geq)$, and

$$\begin{bmatrix} 1 & -Y_j(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_j(\varepsilon) & 1 \end{bmatrix} \begin{bmatrix} X_j(\varepsilon) & Y_j(\varepsilon) \\ Z_j(\varepsilon) & W_j(\varepsilon) \end{bmatrix} \begin{bmatrix} 1 & -Y_j(\varepsilon)\hat{G}_\varepsilon \\ \hat{G}_\varepsilon Z_j(\varepsilon) & 1 \end{bmatrix}^{-1} = \begin{bmatrix} X_{j+1}(\varepsilon) & Y_{j+1}(\varepsilon) \\ Z_{j+1}(\varepsilon) & W_{j+1}(\varepsilon) \end{bmatrix},$$

where

$$\begin{aligned} X_{j+1}(\varepsilon) &= X_j(\varepsilon) - Y_j(\varepsilon)\hat{G}_\varepsilon Z_j(\varepsilon) - (Y_j(\varepsilon) - Y_j(\varepsilon)\hat{G}_\varepsilon W_j(\varepsilon))\hat{G}_\varepsilon Z_j(\varepsilon) \ \ mod\ (\mathcal{N}_{2j}, 2j-1, \geq) \\ &= (P_{\tilde{V}} \frac{iw}{2} P_{\tilde{V}})^2 \ \ mod\ (\mathcal{N}_3, 2, \geq), \\ Y_{j+1}(\varepsilon) &= (X_j(\varepsilon) - Y_j(\varepsilon)\hat{G}_\varepsilon Z_j(\varepsilon))Y_j(\varepsilon)\hat{G}_\varepsilon + Y_j(\varepsilon) - Y_j(\varepsilon)\hat{G}_\varepsilon W_j(\varepsilon) \ \ mod\ (\mathcal{N}_{2j}, 2j-1, \geq) \\ &= 0 \ \ mod\ (\mathcal{N}_{j+1}, j, \geq), \\ Z_{j+1}(\varepsilon) &= \hat{G}_\varepsilon Z_j(\varepsilon)X_j(\varepsilon) + Z_j(\varepsilon) - (\hat{G}_\varepsilon Z_j(\varepsilon)Y_j(\varepsilon) + W_j(\varepsilon))\hat{G}_\varepsilon Z_j(\varepsilon) \ \ mod\ (\mathcal{N}_{2j}, 2j-1, \geq) \\ &= 0 \ \ mod\ (\mathcal{N}_{j+1}, j, \geq), \\ W_{j+1}(\varepsilon) &= (\hat{G}_\varepsilon Z_j(\varepsilon)X_j(\varepsilon) + Z_j(\varepsilon))Y_j(\varepsilon)\hat{G}_\varepsilon + \hat{G}_\varepsilon Z_j(\varepsilon)Y_j(\varepsilon) + W_j(\varepsilon) \ \ mod\ (\mathcal{N}_{2j}, 2j-1, \geq) \\ &= (E_\varepsilon^\perp i \frac{\tilde{V} + w}{2} E_\varepsilon^\perp)^2 \ \ mod\ (\mathcal{N}_2, 0, \geq). \end{aligned}$$

Obviously, $Y_{j+1}(\varepsilon)\hat{G}_\varepsilon, \hat{G}_\varepsilon Z_{j+1}(\varepsilon), Y_{j+1}(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp, E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_{j+1}(\varepsilon) \in (\mathcal{N}_{j+1}, j+1, \geq)$, and $Y_{j+1}(\varepsilon)\hat{G}_\varepsilon E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp, E_\varepsilon^\perp i\frac{\tilde{V}}{2}E_\varepsilon^\perp i\frac{w}{2}E_\varepsilon^\perp \hat{G}_\varepsilon Z_{j+1}(\varepsilon) \in (\mathcal{N}_{j+2}, j+1, \geq)$. Therefore, we can continue the induction on j to get the assertion. The required $T(\varepsilon)$ is given by $T(\varepsilon) = \hat{T}(\varepsilon) \oplus I$, where $\hat{T}(\varepsilon) = T_{\dim(B)}(\varepsilon)T_{\dim(B)-1}(\varepsilon) \cdots T_2(\varepsilon)$. This implies that $\hat{T}(\varepsilon) - I \in (\mathcal{N}_1, 1, \geq)$, and hence proves (3.7).

Step 2. With the $\hat{T}(\varepsilon)$ in Step 1,

$$\hat{D}(\varepsilon) = (\hat{T}(\varepsilon))^{-1} \begin{bmatrix} P_{\tilde{V}}\frac{i}{2}wP_{\tilde{V}} & 0 \\ 0 & E_\varepsilon^\perp(i\frac{\tilde{V}+w}{2})E_\varepsilon^\perp \end{bmatrix} \hat{T}(\varepsilon) \ \ mod\ \begin{bmatrix} (\mathcal{N}_2, 2, \geq) & (\mathcal{N}_2, 2, \geq) \\ (\mathcal{N}_2, 2, \geq) & (\mathcal{N}_2, 2, \geq) \end{bmatrix}. \quad (3.14)$$

We prove this by induction. For $j = 1, T_1(\varepsilon) = I$. For $T_2(\varepsilon)$ in (3.12), one can easily compute

$$T_2(\varepsilon)\hat{D}(\varepsilon)T_2^{-1}(\varepsilon) = \begin{bmatrix} P_{\tilde{V}}\frac{i}{2}wP_{\tilde{V}} & 0 \\ 0 & E_\varepsilon^\perp i\frac{\tilde{V}+w}{2}E_\varepsilon^\perp \end{bmatrix} \ \ mod\ \begin{bmatrix} (\mathcal{N}_2, 2, \geq) & (\mathcal{N}_2, 2, \geq) \\ (\mathcal{N}_2, 2, \geq) & (\mathcal{N}_2, 2, \geq) \end{bmatrix}.$$

Suppose that for $T_j(\varepsilon)$ in Step 1, $j>2$,

$$T_j(\varepsilon)T_{j-1}(\varepsilon)\cdots T_2(\varepsilon)\hat{D}(\varepsilon)T_2^{-1}(\varepsilon)\cdots T_j^{-1}(\varepsilon)=\begin{bmatrix} P_{\tilde V}\frac{i}{2}wP_{\tilde V} & 0\\ 0 & E_\varepsilon^\perp i\frac{\tilde V+w}{2}E_\varepsilon^\perp\end{bmatrix}\ mod\ (\mathcal{N}_2,2,\geq).$$

Then for $T_{j+1}(\varepsilon)$ in (3.13), $T_{j+1}(\varepsilon)=I\ \ mod\ \ (\mathcal{N}_j,j,\geq)$, we clearly have the required form:

$$T_{j+1}(\varepsilon)T_j(\varepsilon)\cdots T_2(\varepsilon)\hat{D}(\varepsilon)T_2^{-1}(\varepsilon)\cdots T_{j+1}^{-1}(\varepsilon)=\begin{bmatrix} P_{\tilde V}\frac{i}{2}wP_{\tilde V} & 0\\ 0 & E_\varepsilon^\perp i\frac{\tilde V+w}{2}E_\varepsilon^\perp\end{bmatrix}\ mod\ \ (\mathcal{N}_2,2,\geq).$$

This concludes Step 2.

Step 3. Let u_t be the map of $\Omega(B,D(\mathcal{E})+S(\mathcal{E}))$ given by $u_tf_j=t^{-j/2}f_j$ for $f_j\in\Omega^j(B,D(\mathcal{E})+S(\mathcal{E}))$. Then

$$\begin{aligned}\bar{D}_t(\varepsilon)\ &\stackrel{def}{\equiv}\ i(P_\varepsilon\frac{\sqrt{t}\tilde V+w}{2}P_\varepsilon+E_\varepsilon^\perp\frac{\sqrt{t}\tilde V+w}{2}E_\varepsilon^\perp+P_{\tilde V}\frac{w}{2}P_{\tilde V}+P_{\tilde V}\frac{w}{2}E_\varepsilon^\perp+E_\varepsilon^\perp\frac{w}{2}P_{\tilde V}) \\ &=\ \sqrt{t}u_t\bar{D}(\varepsilon)u_t^{-1},\end{aligned}\tag{3.15}$$

and

$$(\bar{D}_t(\varepsilon))^2=tu_t(\bar{D}(\varepsilon))^2u_t^{-1}=tu_t(\hat{D}(\varepsilon))^2u_t^{-1}+(P_\varepsilon i\frac{\sqrt{t}\tilde V+w}{2}P_\varepsilon)^2.$$

Take $\varepsilon=t^{-\frac18}$. $\|\hat{G}_\varepsilon\|\leq\frac{1}{\varepsilon}=t^{\frac18}$ and $\|E_\varepsilon^\perp i\frac{\tilde V}{2}E_\varepsilon^\perp\hat{G}_\varepsilon\|\leq t^{\frac{1}{16}}$. This implies that for those operators A constructed in Steps 1 and 2, $A\in(\mathcal{N}_j,k,\geq)$, we have

$$\|u_tAu_t^{-1}\|\leq const.t^{-\frac{j}{2}}t^{\frac{k}{8}},\quad t\to\infty.$$

where the constant is independent of t. Hence, by (3.7) and (3.14), we get

$$\begin{aligned}\sqrt{t}u_t\hat{D}(\varepsilon)u_t^{-1}\ &=\ u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}\begin{bmatrix} P_{\tilde V}\frac{i}{2}wP_{\tilde V} & 0\\ 0 & E_\varepsilon^\perp i\frac{\sqrt{t}\tilde V+w}{2}E_\varepsilon^\perp\end{bmatrix}u_t(\hat{T}(\varepsilon))u_t^{-1} \\ &+\ u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}O(\sqrt{t}t^{-1}t^{\frac14})u_t(\hat{T}(\varepsilon))u_t^{-1},\end{aligned}\tag{3.16}$$

$$\begin{aligned}tu_t(\hat{D}(\varepsilon))^2u_t^{-1}\ &=\ u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}(\begin{bmatrix} (P_{\tilde V}\frac{i}{2}wP_{\tilde V})^2 & 0\\ 0 & (E_\varepsilon^\perp i\frac{\sqrt{t}\tilde V+w}{2}E_\varepsilon^\perp)^2\end{bmatrix} \\ &+\ \begin{bmatrix} O(tt^{-\frac32}t^{\frac14}) & 0\\ 0 & O(tt^{-1})\end{bmatrix})u_t(\hat{T}(\varepsilon))u_t^{-1}.\end{aligned}\tag{3.17}$$

Here $O(t^{-\alpha})$ means an operator in $\mathcal{N}$ with norm dominated by $t^{-\alpha}$ for sufficiently large t.

Let $\|A\|_1$ be the L^1-norm of a Γ-trace class operator A. It can be defined as an integral of Γ-singular numbers of the operator [Fa]. The L^1-norm of a positive Γ-trace class operator is equal to its Γ-trace. Using spectral analysis and the fact that $e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2}$ is a positive operator, we have

$$\begin{aligned} |tr_\Gamma(e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2})| &= \|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2}\|_1 \leq \|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2/2}\|\|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2/2}\|_1 \\ &= \|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2/2}\| tr_\Gamma(e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2/2}) \\ &\leq e^{-t\frac{\varepsilon}{2}}O(t^{-\beta_K/2}) = e^{-\frac{t^{7/8}}{2}}O(t^{-\beta_K/2}), \quad t\to\infty. \end{aligned} \tag{3.18}$$

To estimate the L^1-norm $\|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2}\|_1$, we use the following Volterra formula ([BGV], p. 78):

$$e^{-(A+C)} = e^{-A} + \sum_{k=1}^{\infty}(-1)^k\int_{\Delta_k} e^{-\sigma_0 A}Ce^{-\sigma_1 A}\cdots Ce^{-\sigma_k A}d\sigma_1\cdots d\sigma_k, \tag{3.19}$$

where A and C are selfadjoint operators, and $\Delta_k = \{(t_0,t_1,\cdots,t_k) : t_i\geq 0, \sum t_i = 1\}$ is the standard simplex. We may write $(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2 = A+C$, where $A = (E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}}{2}E_\varepsilon^\perp)^2$. C is an operator valued form on M of degree ≥ 1. The right hand side of (3.19) is hence a finite sum. The sub-exponential type decay in (3.18) implies that for $\varepsilon = t^{-\frac{1}{8}}$ and some $\alpha > 0$,

$$\max_{x\in K}\{\|e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2}\|_1\} \quad\leq\quad const.e^{-\alpha t^{7/8}/2}t^{\frac{\dim(B)}{2}}, \quad t\to\infty. \tag{3.20}$$

The same estimate is obviously true for $tr_\Gamma^s(e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2})$ and for $tr_\Gamma^s(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2})$, since the norm of the operator $E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2/2}$ can be estimated by a constant independent of t. To estimate the super Γ-trace of $E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-tu_tY(\varepsilon)u_t^{-1}}$ for $Y(\varepsilon)$ in (3.7), we use

$$tu_tY(\varepsilon)u_t^{-1} = (E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2 + O(1), \quad t\to\infty, \tag{3.21}$$

Take $A = (E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp)^2/2$ and C to be half $O(1)$ in (3.21). C is a bounded operator valued form of degree ≥ 1 on M. The right hand side of (3.19) for just chosen operators is also a finite sum. Note that there is at leat one $\sigma_j \geq \frac{1}{k+1}$ for the integral over Δ_k. Hence, by the sub-exponential type decay in (3.20), (3.19) implies that the super Γ-trace of the operator

$$E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-tu_tY(\varepsilon)u_t^{-1}} = E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-tu_tY(\varepsilon)u_t^{-1}/2}e^{-tu_tY(\varepsilon)u_t^{-1}/2}$$

can be dominated by $e^{-t^{7/8}\alpha_1}$ for some constant $\alpha_1 > 0$, $t \to \infty$.

Similarly, using (3.16) – (3.18), we get that for $\varepsilon = t^{-\frac{1}{8}}$ and $t \to \infty$,

$$tr_\Gamma^s(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-tu_tX(\varepsilon)u_t^{-1}}) \to tr_\Gamma^s(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}})^2}) + O(t^{-\beta_1/2}) \tag{3.22}$$

for some $\beta_1 > 0$. Therefore, combining (3.16) – (3.17) and using the fact that $u_t(\hat{T}(\varepsilon))^{\pm 1}u_t^{-1} = 1 + O(t^{-\frac{3}{8}})$, we obtain the following main result of Step 3:

$$\begin{aligned}
& tr_\Gamma^s(\sqrt{t}u_t\hat{D}(\varepsilon)u_t^{-1}e^{-tu_t(\hat{D}(\varepsilon))^2u_t^{-1}}) \\
= \ & tr_\Gamma^s(u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-tu_tX(\varepsilon)u_t^{-1}}u_t(\hat{T}(\varepsilon))u_t^{-1}) \\
+ \ & tr_\Gamma^s(u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-tu_tY(\varepsilon)u_t^{-1}}u_t(\hat{T}(\varepsilon))u_t^{-1}) \\
+ \ & tr_\Gamma^s(u_t(\hat{T}(\varepsilon))^{-1}u_t^{-1}O(t^{-\frac{1}{4}})e^{-tu_t(X(\varepsilon)+Y(\varepsilon))u_t^{-1}}u_t(\hat{T}(\varepsilon))u_t^{-1}) \\
= \ & tr_\Gamma^s(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-tu_tX(\varepsilon)u_t^{-1}}) + tr_\Gamma^s(E_\varepsilon^\perp i\frac{\sqrt{t}\tilde{V}+w}{2}E_\varepsilon^\perp e^{-tu_tY(\varepsilon)u_t^{-1}}) \\
+ \ & tr_\Gamma^s(O(t^{-\frac{1}{4}})e^{-tu_t(X(\varepsilon)+Y(\varepsilon))u_t^{-1}}) + O(t^{-\beta_2'/2}) && (3.23) \\
= \ & tr_\Gamma^s(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}})^2}) + O(t^{-\beta_2/2}), \ \ \varepsilon = t^{-1/8}, \ \ t \to \infty, && (3.24)
\end{aligned}$$

for some positive constants β_2' and β_2. Here the estimate of the last two terms of (3.23) is exactly the same as those for (3.22).

We should point out that by (3.16) – (3.17), a similar result to (3.24) also holds for the super trace $tr_\Gamma^s(t^{\frac{l}{2}}u_t(\hat{D}(\varepsilon))^lu_t^{-1}e^{-tu_t(\hat{D}(\varepsilon))^2u_t^{-1}})$, where $l \geq 1$ is an integer. The proof of this is the same as above.

Step 4. We now prove that for $\varepsilon = t^{-\frac{1}{8}}$ and some $\beta_3 > 0$,

$$\begin{aligned}
tr_\Gamma^s(\bar{D}_t(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2}) &= tr_\Gamma^s(\hat{D}_t(\varepsilon)e^{-(\hat{D}_t(\varepsilon))^2}) + tr_\Gamma^s(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon e^{-(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2}) \\
&= tr_\Gamma^s(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}}e^{-(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}})^2}) + O(t^{-\beta_3/2}), \ \ t \to \infty. && (3.25)
\end{aligned}$$

By (3.24), it is enough to check the following:

$$\begin{aligned}
\max_{x\in K}\{|tr_\Gamma^s(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon e^{-(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2})|\} &= \max_{x\in K}\{|tr_\Gamma(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon e^{-(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2}\nu P_\varepsilon)|\} \\
&\leq \max_{x\in K}\{\|(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon e^{-(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2}\nu P_\varepsilon\|_1\} \\
&\leq const.tr_\Gamma(P_\varepsilon) \\
&= O(t^{-\beta_K/16}), \ \ \varepsilon = t^{-1/8}, \ \ t \to \infty. && (3.26)
\end{aligned}$$

Here ν is the grading operator. Indeed, $P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon e^{-t(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2}$ is a bounded operator with norm dominated by $\frac{1}{\sqrt{2e}}$, and $\|P_\varepsilon\|_1 = tr_\Gamma(P_\varepsilon)$. We knew already from (3.5) that $tr_\Gamma(P_\varepsilon)$ has the required estimate. Hence (3.26) holds.

Step 5. Finally, we want to prove (3.4). Let

$$\bar{D}_t = i\frac{\sqrt{t}\tilde{V}+w}{2}$$

and

$$A(\varepsilon) = P_\varepsilon\frac{iw}{2}P_{\tilde{V}} + P_{\tilde{V}}\frac{iw}{2}P_\varepsilon + P_\varepsilon\frac{iw}{2}E_\varepsilon^\perp + E_\varepsilon^\perp\frac{iw}{2}P_\varepsilon.$$

$\bar{D}_t = \bar{D}_t(\varepsilon) + A(\varepsilon)$. Let $\bar{D}_t(z) = \bar{D}_t(\varepsilon) + zA(\varepsilon), 0 \le z \le 1$. Using the Duhamel formula (see (2.5)),

$$\frac{d}{dz}e^{-(\bar{D}_t(z))^2} = -\int_0^1 e^{-(1-s)(\bar{D}_t(z))^2}\frac{d}{dz}((\bar{D}_t(z))^2)e^{-s(\bar{D}_t(z))^2}ds,$$

we get

$$\begin{aligned} & tr_\Gamma^s(\bar{D}_te^{-(\bar{D}_t)^2} - \bar{D}_t(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2}) = tr_\Gamma^s(\bar{D}_t(e^{-(\bar{D}_t)^2} - e^{-(\bar{D}_t(\varepsilon))^2})) + tr_\Gamma^s(A(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2}) \\ = & -\int_0^1\int_0^1 tr_\Gamma^s(\bar{D}_te^{-(1-s)(\bar{D}_t(z))^2}(\bar{D}_t(z)A(\varepsilon) + A(\varepsilon)\bar{D}_t(z))e^{-s(\bar{D}_t(z))^2})dsdz \\ + & \ tr_\Gamma^s(A(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2}). \end{aligned} \tag{3.27}$$

Note that both $A(\varepsilon)$ and $e^{-(\bar{D}_t(\varepsilon))^2}$ are bounded operators with norms dominated by a constant independent of t. The crucial fact is that $A(\varepsilon)$ involves the projection P_ε which enables us to estimate (3.27).

$$\begin{aligned} \max_{x\in K}\{|tr_\Gamma^s(A(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2})|\} &= \max_{x\in K}\{|tr_\Gamma(P_\varepsilon\frac{iw}{2}P_{\tilde{V}}e^{-(\bar{D}_t(\varepsilon))^2}\nu P_\varepsilon) \\ &+ tr_\Gamma(P_\varepsilon e^{-(\bar{D}_t(\varepsilon))^2}\nu P_{\tilde{V}}\frac{iw}{2}P_\varepsilon) + tr_\Gamma(P_\varepsilon\frac{iw}{2}E_\varepsilon^\perp e^{-(\bar{D}_t(\varepsilon))^2}\nu P_\varepsilon) \\ &+ tr_\Gamma(P_\varepsilon e^{-(\bar{D}_t(\varepsilon))^2}\nu E_\varepsilon^\perp\frac{iw}{2}P_\varepsilon)|\} \le const.tr_\Gamma(P_\varepsilon) \\ &= O(t^{-\beta_K/16}), \ \ \varepsilon = t^{-1/8}, \ \ t \to \infty. \end{aligned} \tag{3.28}$$

Since $\tilde{V}$ anticommutes with the grading operator ν on the fiber, we obtain that the first term of (3.27) can be estimated by

$$|\int_0^1\int_0^1 tr_\Gamma(\nu\bar{D}_te^{-(1-s)(\bar{D}_t(z))^2}(\bar{D}_t(z)A(\varepsilon) + A(\varepsilon)\bar{D}_t(z))e^{-s(\bar{D}_t(z))^2})dsdz|$$

$$
\begin{aligned}
\leq & \int_0^1\int_0^1 |tr_\Gamma(e^{-s(\bar{D}_t(z))^2}(-i\frac{\sqrt{t}\tilde{V}}{2}\nu+\nu i\frac{w}{2})e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)A(\varepsilon))|dsdz \\
+ & \int_0^1\int_0^1 |tr_\Gamma(\nu\bar{D}_t e^{-(1-s)(\bar{D}_t(z))^2}A(\varepsilon)\bar{D}_t(z)e^{-s(\bar{D}_t(z))^2})|dsdz \\
\leq & \int_0^1\int_0^1 |tr_\Gamma(e^{-s(\bar{D}_t(z))^2}\bar{D}_t(z)\nu e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)A(\varepsilon))|dsdz \\
+ & \int_0^1\int_0^1 |tr_\Gamma(e^{-s(\bar{D}_t(z))^2}[(P_{\tilde{V}}i\frac{w}{2}P_{\tilde{V}}+E_\varepsilon^\perp i\frac{w}{2}P_{\tilde{V}}+P_{\tilde{V}}i\frac{w}{2}E_\varepsilon^\perp+zA(\varepsilon))\nu+\nu i\frac{w}{2}] \\
\cdot & \ e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)A(\varepsilon))|dsdz \\
+ & \int_0^1\int_0^1 |tr_\Gamma(\nu\bar{D}_t(z)e^{-(1-s)(\bar{D}_t(z))^2}A(\varepsilon)\bar{D}_t(z)e^{-s(\bar{D}_t(z))^2})|dsdz \\
+ & \int_0^1\int_0^1 |tr_\Gamma(\nu(1-z)A(\varepsilon)e^{-(1-s)(\bar{D}_t(z))^2}A(\varepsilon)\bar{D}_t(z)e^{-s(\bar{D}_t(z))^2})|dsdz.
\end{aligned}
\tag{3.29}
$$

In (3.29) and below, we omit the symbol $\max_{x\in K}$. To estimate the first term of (3.29), note that

$$
\|\bar{D}_t(z)e^{-s(\bar{D}_t(z))^2}\| \leq \frac{1}{\sqrt{2es}}. \tag{3.30}
$$

The first term of (3.29) is equal to

$$
\begin{aligned}
& \int_0^1\int_0^1 |tr_\Gamma(P_\varepsilon i\frac{w}{2}P_{\tilde{V}}e^{-s(\bar{D}_t(z))^2}\bar{D}_t(z)\nu e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)P_\varepsilon) \\
+ & \ tr_\Gamma(P_\varepsilon e^{-s(\bar{D}_t(z))^2}\bar{D}_t(z)\nu e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)P_{\tilde{V}}i\frac{w}{2}P_\varepsilon) \\
+ & \ tr_\Gamma(P_\varepsilon i\frac{w}{2}E_\varepsilon^\perp e^{-s(\bar{D}_t(z))^2}\bar{D}_t(z)\nu e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)P_\varepsilon) \\
+ & \ tr_\Gamma(P_\varepsilon e^{-s(\bar{D}_t(z))^2}\bar{D}_t(z)\nu e^{-(1-s)(\bar{D}_t(z))^2}\bar{D}_t(z)E_\varepsilon^\perp i\frac{w}{2}P_\varepsilon)|dsdz \\
\leq & \ const.\int_0^1\int_0^1 \frac{1}{\sqrt{2es}}\frac{1}{\sqrt{2e(1-s)}}tr_\Gamma(P_\varepsilon)dsdz \\
\leq & \ const.O(t^{-\beta_K/16}), \ \ t\to\infty.
\end{aligned}
$$

We point out that this estimate of the first term of (3.29) can be obtained using the supertrace and the fact that $(\bar{D}_t(z))^2$ is an even operator. Similarly, we can estimate the rest of terms in (3.29) by $O(t^{-\beta_K/16})$. Therefore,

$$
|tr_\Gamma^s(\bar{D}_t(e^{-(\bar{D}_t)^2}-e^{-(\bar{D}_t(\varepsilon))^2}))| \leq const.O(t^{-\beta_K/16}), \ \ t\to\infty.
$$

This together with (3.27) – (3.28) proves that

$$
|tr_\Gamma^s(\bar{D}_te^{-(\bar{D}_t)^2}-\bar{D}_t(\varepsilon)e^{-(\bar{D}_t(\varepsilon))^2})| \leq const.O(t^{-\beta_K/16}), \ \ t\to\infty. \tag{3.31}
$$

Combining (3.25) and (3.31), we have proved that for some constant $\beta_4 > 0$,

$$tr_\Gamma^s(\bar{D}_t e^{-(\bar{D}_t)^2}) \to tr_\Gamma^s(P_{\tilde{V}} i\frac{w}{2} P_{\tilde{V}} e^{-(P_{\tilde{V}} i\frac{w}{2} P_{\tilde{V}})^2}) + O(t^{-\beta_4/2}), \ \ t \to \infty. \tag{3.32}$$

Using (3.19), (3.32), and the Volterra series, one can easily deal with the term $\frac{c_v(\tilde{T})}{2\sqrt{t}}$ in $\tilde{D}_t$. The proof of Lemma 3.3 is completed. ∎

A similar result to (3.32) is also true for the super trace $tr_\Gamma^s((\bar{D}_t)^2 e^{-(\bar{D}_t)^2})$. Combining Lemmas 3.2 and 3.3 and noting that the cohomology class of $f(\tilde{D}_t, g^{\mathcal{E}})$ is independent of t, we get the following L^2-form version of the Riemann-Roch-Grothendieck theorem for fiber bundles with covering spaces as typical fibers.

Theorem 3.1 *With the condition of Lemma* 3.3*, then in* $H^*(B, \mathbb{R})$*,*

$$f(\nabla^{H_2}, g^{H_2}) = \begin{cases} \int_{\mathcal{F}} e(T\tilde{Z}, \nabla^{\tilde{Z}}) f(\nabla^E, g^E), & if \ \ \dim(\tilde{Z}) \ \ even \\ 0, & if \ \ \dim(\tilde{Z}) \ \ odd. \end{cases}$$

Let us now find a transgression formula relating $f(\nabla^{H_2}, g^{H_2})$ and $\int_{\mathcal{F}} e(T\tilde{Z}, \nabla^{\tilde{Z}}) f(\nabla^E, g^E)$. Let $f'(z)$ be the first derivative of $f(z)$. Take

$$f'(\tilde{D}_t, g^{\mathcal{E}}) = \varphi tr_\Gamma^s(\frac{N}{2} f'(\tilde{D}_t)).$$

Then $f'(\tilde{D}_t, g^{\mathcal{E}})$ is an even real form on B. Since $\frac{\partial}{\partial t}\tilde{D}_t = [\tilde{C}_t, \frac{N}{2t}]$ and $[\tilde{C}_t, \tilde{D}_t] = 0$, we have

$$\begin{aligned} \frac{\partial}{\partial t} f(\tilde{D}_t, g^{\mathcal{E}}) &= (2\pi i)^{\frac{1}{2}} \varphi tr_\Gamma^s(\frac{\partial}{\partial t}\tilde{D}_t f'(\tilde{D}_t)) \\ &= (2\pi i)^{\frac{1}{2}} \varphi tr_\Gamma^s[\tilde{C}_t, \frac{N}{2t} f'(\tilde{D}_t)] \\ &= d(\varphi tr_\Gamma^s(\frac{N}{2t} f'(\tilde{D}_t))) = \frac{1}{t} d f'(\tilde{D}_t, g^{\mathcal{E}}). \end{aligned} \tag{3.33}$$

See [BGV]. Let $\chi_2(\tilde{Z}, E) = \sum_{i=0}^{\dim(\tilde{Z})} (-1)^i i tr_\Gamma(H_2^i(\tilde{Z}, E|_{\tilde{Z}}))$. Denote by $\chi_2(\tilde{Z})$ the L^2-Euler characteristic of $\tilde{Z}$ which is equal to $\chi(Z)$, the Euler characteristic of Z, by the Γ-index theorem [At]. The following lemma is an analogue of Lemmas 3.2 and 3.3 for $f'(\tilde{D}_t, g^{\mathcal{E}})$.

Lemma 3.4 *(i) As* $t \to 0$*,*

$$f'(\tilde{D}_t, g^{\mathcal{E}}) \to \begin{cases} \frac{1}{4}\dim(\tilde{Z}) rk(E) \chi_2(\tilde{Z}) + O(t), & if \ \ \dim(\tilde{Z}) \ \ even, \\ O(\sqrt{t}), & if \ \ \dim(\tilde{Z}) \ \ odd. \end{cases}$$

(ii) With the condition of Lemma 3.3, then for each compact subset $K \subset B$, there exists a constant $\beta'_K > 0$ such that

$$f'(\tilde{D}_t, g^{\mathcal{E}}) \to_{t\to\infty} \frac{1}{2}\chi_2(\tilde{Z}, E) + O(t^{-\beta'_K/2}) \ \text{ on } \ K.$$

Proof. By Lemmas 3.2 and 3.3, the lemma follows from the proof of Theorem 3.21 [BL] which holds in our situation.

One may also derive the second part of the lemma from the proof of Lemma 3.3. Indeed, by assumption,

$$tr^s_\Gamma(f'(P^\perp_{\tilde{V}}\frac{\sqrt{t}}{2}\tilde{V}P^\perp_{\tilde{V}})) \to_{t\to\infty} O(t^{-\beta_K/2}).$$

Steps $1-4$ of the proof of Lemma 3.3 work for the present case. We get that for some $\beta_5 > 0$,

$$\begin{aligned} tr^s_\Gamma((\bar{D}_t(\varepsilon))^2 e^{-(\bar{D}_t(\varepsilon))^2}) &= tr^s_\Gamma((\hat{D}_t(\varepsilon))^2 e^{-(\hat{D}_t(\varepsilon))^2}) + tr^s_\Gamma((P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2 e^{-(P_\varepsilon i\frac{\sqrt{t}\tilde{V}+w}{2}P_\varepsilon)^2}) \\ &= tr^s_\Gamma((P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}})^2 e^{-(P_{\tilde{V}}\frac{iw}{2}P_{\tilde{V}})^2}) + O(t^{-\beta_5/2}), \ \ t\to\infty, \ \ \varepsilon = t^{-\frac{1}{8}}. \end{aligned}$$

Furthermore,

$$\begin{aligned} & tr^s_\Gamma((\bar{D}_t)^2 e^{-(\bar{D}_t)^2} - (\bar{D}_t(\varepsilon))^2 e^{-(\bar{D}_t(\varepsilon))^2}) = tr^s_\Gamma((\bar{D}_t(z))^2(e^{-(\bar{D}_t)^2} - e^{-(\bar{D}_t(\varepsilon))^2})) \\ + \ & tr^s_\Gamma(((1-z)(\bar{D}_t(z)A(\varepsilon) + A(\varepsilon)\bar{D}_t(z)) + (1-z)^2A^2(\varepsilon))e^{-(\bar{D}_t)^2}) \\ - \ & tr^s_\Gamma(-(\bar{D}_t(z)A(\varepsilon) + A(\varepsilon)\bar{D}_t(z)) + A^2(\varepsilon))e^{-(\bar{D}_t(\varepsilon))^2}). \end{aligned}$$

Using $A(\varepsilon)$ as in the proof of Lemma 3.3 and $\bar{D}_t(z) = \bar{D}_t(\varepsilon) + A(\varepsilon) = \bar{D}_t + (z-1)A(\varepsilon)$, we can easily estimate the last two terms by $t^{-\beta_6/2}$ for some $\beta_6 > 0$ as $t \to \infty$. We can write the first term as

$$\begin{aligned} tr^s_\Gamma((\bar{D}_t(z))^2(e^{-(\bar{D}_t)^2} - e^{-(\bar{D}_t(\varepsilon))^2})) &= -\int_0^1\int_0^1 tr^s_\Gamma((\bar{D}_t(z))^2 e^{-(1-s)(\bar{D}_t(z))^2} \\ &\cdot \ \ (\bar{D}_t(z)A(\varepsilon) + A(\varepsilon)\bar{D}_t(z))e^{-s(\bar{D}_t(z))^2})dsdz. \end{aligned}$$

The norm of $(\bar{D}_t(z))^2 e^{-s(\bar{D}_t(z))^2}$ is dominanted by $\frac{1}{s}$. We apply this estimate to those s away from zero, and (3.30) to those small s. Therefore, we obtain that for some $\beta'_K > 0$,

$$f'(\tilde{D}_t, g^{\mathcal{E}}) \to_{t\to\infty} f'(\nabla^{H_2}, g^{H_2}) + O(t^{-\beta'_K/2})$$

on K. By the proof of Proposition 1.3 [BL], $f'(\nabla^{H_2}, g^{H_2}) = \frac{1}{2}\chi_2(\tilde{Z}, E)$. ∎

Using Lemma 3.4, we can define a form on B as follows.

Definition 3.1 *Assume that the condition of Lemma* 3.3 *is satisfied. Then the L^2-analytic torsion form $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ of the bundle M associated with E is defined as*

$$\begin{aligned}\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E) &= \int_0^\infty [f'(\tilde{D}_t, g^{\mathcal{E}}) - \frac{1}{2}\chi_2(\tilde{Z}, E) - (\frac{1}{4}\dim(\tilde{Z})rk(E)\chi_2(\tilde{Z}) - \\ &- \frac{1}{2}\chi_2(\tilde{Z}, E))f'(\frac{i\sqrt{t}}{2})]\frac{dt}{t}.\end{aligned}$$

Lemma 3.4 ensures that the above integral is convergent. We have immediately the following transgression formula.

Theorem 3.2 *With the condition of Lemma* 3.3, *$\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ has a weak exterior derivative and*

$$-d\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E) = \int_{\mathcal{F}} e(T\tilde{Z}, \nabla^{\tilde{Z}})f(\nabla^E, g^E) - f(\nabla^{H_2}, g^{H_2}). \tag{3.34}$$

Proof. Indeed, by Lemmas 3.2 and 3.3 together with (3.33), we get (3.34). ∎

This L^2-analytic torsion form is the main construction in this paper. We clearly have by the proof of Lemma 3.4 that $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is a continuous form on B. Moreover, since the right hand side of (3.34) is a smooth form on the base space B, we obtain that the differentiation of the L^2-analytic torsion form $d\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is really a smooth form on B. Unfortunately, our methods do not show directly that $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is c^∞. It does have an exterior derivative in a weak sense, which we wish to explain.

To do this we introduce the following. For a smooth manifold W, $\Omega_k(W)$ will denote smooth k-forms on W. We say a continuous k-form α on W has a weak exterior derivative β, β a continuous $k+1$ form on W, if for each smooth map $f : \Delta_{k+1} \to W$,

$$\int_{\Delta_{k+1}} f^*(\beta) = \int_{\partial\Delta_{k+1}} f^*(\alpha).$$

Clearly, weak exterior derivatives are unique if they exist, and we denote in this case $\beta = d\alpha$. Note that $\beta = d\alpha$ has weak exterior derivative 0.

Let $\Omega_k^w(W)$ be the vector space of k-forms with weak exterior derivatives and $\Omega^w(W) = \sum_k \Omega_k^w(W)$. Then $(\Omega^w(W), d)$ is a chain complex containing the chain complex of C^∞ forms $(\Omega(W), d)$. We have an obvious extension of the deRham map $DR : (\Omega(W), d) \to (C^*(W), \delta)$ from forms into the singular cochain complex to $DR^w : (\Omega^w(W), d) \to (C^*(W), \delta)$. In fact, the usual proof goes through to show DR^w induces an isomorphism on cohomology.

Remark 3.1 *The L^2-analytic torsion form $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is an L^2-version of the analytic torsion form defined by Bismut-Lott [BL]. The discussion in this section is related to those in [BL] except that we encountered two new features, the locally Γ-invariant structure and the positive decay rate of $tr^s_\Gamma(e^{\tilde{D}_t^2})$ at infinity . We will study the latter in the next section.*

4 The Family Novikov-Shubin Invariants

In Section 3 we assumed a decay condition in Lemma 3.3 in order to prove an L^2-version of the Riemann-Roch-Grothendieck theorem and the transgression formula. This motivates us to introduce an invariant for fiber bundles with covering spaces as fibers.

Let $P_{\tilde{V}}$ be the orthogonal projection on $Ker(\tilde{V})$ as before. Recall that the decay condition of Lemma 3.3 follows from the assumption that for each compact subset $K \subset B$ there exist constants $\beta^i_K > 0$ such that on K

$$tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}}(\tilde{V}_i)^2P^\perp_{\tilde{V}}}) \to_{t\to\infty} O(t^{-\beta^{(i)}_K/2}), i = 0, 1, \cdots, \dim(\tilde{Z}),$$

where $(\tilde{V}_i)^2$ is the restriction of $\tilde{V}^2$ to $C^\infty(B, \wedge^i(T\tilde{Z}, E|_{\tilde{Z}}))$. Let

$$\alpha^{(i)}_K(M, \tilde{Z}, E) = \sup\{\beta^{(i)}_K : tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}}(\tilde{V}_i)^2P^\perp_{\tilde{V}}}) \to_{t\to\infty} O(t^{-\beta^{(i)}_K/2})\}. \tag{4.1}$$

Then the condition of Lemma 3.3 is guaranteed by $\alpha^{(i)}_K(M, \tilde{Z}, E) > 0$, $i = 0, 1, \cdots, \dim(\tilde{Z})$, $K \subset B$.

Definition 4.1 *We call $\{\alpha^{(i)}_K(M, \tilde{Z}, E)\}, K \subset B$, the family Novikov- Shubin invariants of the fiber bundle M associated with the given local Γ-invariance structure.*

The value of $\alpha^{(i)}_K(M, \tilde{Z}, E)$ may a priori depend on K and the equivalence class of local Γ-invariance structures. If K_1 is a compact subset of K, then $\alpha^{(i)}_{K_1}(M, \tilde{Z}, E) \geq \alpha^{(i)}_K(M, \tilde{Z}, E)$. Taking $K = \{x\}$ defines for each i a function $\alpha^{(i)}_x(M, \tilde{Z}, E)$ on B such that $\alpha^{(i)}_K(M, \tilde{Z}, E) = \inf_{x\in K} \alpha^{(i)}(M_x, E|_{M_x})$, where $\alpha^{(i)}(M_x, E|_{M_x})$ is the Novikov-Shubin invariants of M_x associated with $E|_{M_x}$. When $E|_{\tilde{Z}}$ is a trivial bundle, then by definition $\alpha^{(i)}_x(M, \tilde{Z}, E)$ is the Novikov-Shubin invariants of Z, and thus independent of $x \in B$ and the local Γ-invariance structure. The first result of this section asserts that this is true for any local Γ-invariance structure and thus the family Novikov-Shubin invariants are just the Novikov-Shubin invariants of the fiber.

The key ingredient is the following. Consider two bundles $\tilde{Z}_i \to M_i \to B$ with locally Γ-invariant structures. They are called fiberwise homotopy equivalent if there exist fiberwise maps h_i,

$$M_1 \overset{h_1}{\rightleftharpoons}_{h_2} M_2,$$

such that both compositions are fiberwise homotopic to the identity. In our situation we also require that the maps and the homotopies be locally Γ-invariant, and that $h_1^*(E_2) = E_1$.

Proposition 4.1 *If $\tilde{Z}_i \to M_i \to B$ are fiberwise homotopy equivalent, then $\alpha_x^{(i)}(M_1, \tilde{Z}_1, h_1^*(E_2)) = \alpha_x^{(i)}(M_2, \tilde{Z}_2, E_2)$ and then for each compact subset $K \subset B$, $\alpha_K^{(i)}(M_1, \tilde{Z}_1, h_1^*(E_2)) = \alpha_K^{(i)}(M_2, \tilde{Z}_2, E_2)$.*

Proof. By passing to a finite cover of K, we can assume the bundles $M_i|_K$ are product bundles. Then the proof proceeds exactly as in [GS], the fact that we are incorporating a not necessarily trivial flat bundle over $\tilde{Z}$ requires only a routine modification of the argument. ∎

A priori the $\alpha_x^{(i)}(M, \tilde{Z}, E)$ could be non-constant in x because the metrics vary from fiber to fiber, however, locally the connections are constant and flat fiber bundle is fiber homotopic to a product bundle with constant metric. Hence,

Corollary 4.1 *$\alpha_x^{(i)}(M, \tilde{Z}, E)$ is constant on components of B and thus for each compact subset K in a component of B, $\alpha_K^{(i)}(M, \tilde{Z}, E)$ depends only on a single fiber.*

We have reduced the study of the family Novikov-Shubin invariants to the usual Novikov-Shubin invariants of fibers associated with a flat bundle. We should point out, however, that the difference between a single fiber and the fiber bundle is that the limit in (4.1) is required to be uniform on each compact subset of B. The Novikov-Shubin invariants are also used in [Go 3].

Proposition 4.2 *Let $\tilde{Z}_i \to M_i \to B$ be a fiber bundle, and E_i a flat vector bundle over M_i with Hermitian metric g^{E_i}, $i = 1, 2$. Let M be the fiber product bundle over B with the fiber $\tilde{Z} = \tilde{Z}_1 \times \tilde{Z}_2$ and $P_i : M \to M_i$ be the canonical projection. Assume $E = P_1^* E_1 \otimes P_2^* E_2$ with the Hermitian metric $g^E = P_1^* g^{E_1} \otimes P_2^* g^{E_2}$. Then for a compact subset $K \subset B$,*

$$\alpha_K^{(i)}(M, \tilde{Z}, E) = \min_{p+q=i} \{\alpha_K^{(p)}(M_1, \tilde{Z}_1, E_1), \alpha_K^{(q)}(M_q, \tilde{Z}_2, E_2)\}.$$

Proof. We have

$$C^\infty(B,\mathcal{E}) = C^\infty(B,\mathcal{E}_1\otimes\mathcal{E}_2),$$
$$\tilde{V}^2 = (\tilde{V}_1)^2\otimes 1 + 1\otimes(\tilde{V}_2)^2,$$

where $\tilde{V}_i = d^*_{\tilde{Z}_i} - d_{\tilde{Z}_i}$. More specifically, for the proof below let $\Omega^i = C^\infty(B,\wedge^i(T^*\tilde{Z},E|_{\tilde{Z}}))$ and similarly Ω^p and Ω^q for $\tilde{Z}_1$ and $\tilde{Z}_2$, respectively. Then

$$\tilde{V}^2|_{\Omega^i} = \oplus_{p+q=i}((\tilde{V}_1)^2|_{\Omega^p}\otimes 1 + 1\otimes(\tilde{V}_2)^2|_{\Omega^q}).$$

Hence,

$$\begin{aligned}(Ker(\tilde{V}^2)|_{\Omega^i})^\perp &= \oplus_{p+q=i}[(Ker(\tilde{V}_1)^2|_{\Omega^p})^\perp\otimes(Ker(\tilde{V}_2)^2|_{\Omega^q})^\perp\\ &\oplus\ (Ker(\tilde{V}_1)^2|_{\Omega^p}\otimes(Ker(\tilde{V}_2)^2|_{\Omega^q})^\perp)\oplus((Ker(\tilde{V}_1)^2|_{\Omega^p})^\perp\otimes Ker(\tilde{V}_2)^2|_{\Omega^q})].\end{aligned}$$

We obtain

$$\begin{aligned}tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}}(\tilde{V}_i)^2P^\perp_{\tilde{V}}}) &= \sum_{p+q=i}[tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}_1}(\tilde{V}_{1i})^2P^\perp_{\tilde{V}_1}})tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}_2}(\tilde{V}_{2i})^2P^\perp_{\tilde{V}_2}})\\ &+\ tr_\Gamma(P_{\tilde{V}_1}|_{\Omega^p})tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}_2}(\tilde{V}_{2i})^2P^\perp_{\tilde{V}_2}})\\ &+\ tr_\Gamma(P_{\tilde{V}_2}|_{\Omega^q})tr_\Gamma(e^{\frac{t}{4}P^\perp_{\tilde{V}_1}(\tilde{V}_{1i})^2P^\perp_{\tilde{V}_1}})].\end{aligned}$$

This implies the assertion. ∎

We have so far been considering smooth fiber bundles $\tilde{Z}\to M\to B$ along with covering map $p:\tilde{Z}\to Z$. Given data $\{g^{\tilde{Z}},E,\nabla^E,g^E\}$ and a local Γ-invariance structure, we have defined the family Novikov-Shubin invariants, and under the condition of Lemma 3.3, defined the L^2-analytic torsion forms. Given an open subset U of B, the fiber bundle, the data, the local Γ-invariance structure, the Novikov-Shubin invariants, and the L^2-analytic torsion forms all restrict naturally to U. In this sense, everything is locally determined. At the same time the existence and equivalence class of the local Γ-invariance structure is global in B. It can not be determined by looking in a neighborhood of each point. However given such a structure the Novikov-Shubin invariants are local invariants, indeed as we have just shown are determined at a single point. Thus given a local Γ-invariance structure (which depends on the global structure of the bundle over B) we can study the existence and local properties of the L^2-analytic torsion forms by restricting everything to small open sets in B. In fact, we did this in Section 2 to get our asymptotic formula. This gives us the enormous advantage of having a bundle which covers one with compact fiber.

In the next sections we will study the local properties of our invariants. In this situation we can assume that our data pulls back from a covering map $\rho : M \to \bar{M}$, where $\rho|_{\tilde{Z}} = p$. That is we have a fiber bundle $\pi'' : \bar{M} \to B$ with closed fiber Z such that $\pi = \pi''\rho$. Further our data $\{g^{\tilde{Z}}, E, \nabla^E, g^E\}$ pulls back from data $\{g^Z, \bar{E}, \nabla^{\bar{E}}, g^{\bar{E}}\}$ on $\pi'' : \bar{M} \to B$. Note that the local Γ-invariance structure on $\pi : M \to B$ does not specify the $\rho, \bar{E}, \bar{M}$, etc., uniquely, but does put constraints on our choices. However, $\{\rho, \bar{M}, \bar{E}, \pi''\}$ and the data on π'' do specify the local Γ-invariance structure.

For $x \in B$ we let $\bar{M}_x = (\pi'')^{-1}(x), M_x = \pi^{-1}(x)$ and $\rho_x = \rho : M_x \to \bar{M}_x$. Denote by Γ_x the group of diffeomorphisms a of M_x such that $\rho_x a = \rho_x$. Each Γ_x is isomorphic to Γ but not canonically unless the covering is regular. (If B is simply connected, our covering is regular, and thus from the point of view of studying local invariants we could assume regularity, and for convenience we shortly will). More precisely, given x and y in B, a path in B joining x to y induces a unique isomorphism from $\Gamma_x \to \Gamma_y$ and this isomorphism only depends on the homotopy class of the path. Using this it is simple to construct a flat bundle F over $\bar{M}$ whose restriction to $\bar{M}_x$ is the bundle $M_x \times_{\Gamma_x} l^2(\Gamma_x)$. The importance of this bundle is that there is a natural isomorphism between the fiberwise L^2-E-valued forms $\Omega_2(M, E)$ on M and $\Omega(\bar{M}, \bar{E} \otimes F)$. In this way we can transfer all our operators and construction from fiberwise L^2-forms on M to plain forms on $\bar{M}$ but with values in an infinite dimensional bundle F. Now in general analysis on bundles with infinite dimensional fibers is unpleasant, however, these bundles are very special since the fibers are finite Hilbert modules over finite von Neumann algebras $N(\Gamma_x)$, where $N(\Gamma_x)$ is the completion of the group algebra $\mathbb{C}\Gamma_x$ in the weak operator topology, when considered as $*$-subalgebra of bounded operators on $l^2(\Gamma_x)$. This family of von Neumann algebras itself constitutes a flat bundle of von Neumann algebras over B and thus over $\bar{M}$ and the bundle $\bar{E} \otimes F$ is a Hilbert module over this bundle of algebras. We could continue to work in this generality, however for simplicity we will assume the covering $\rho : M \to \bar{M}$ is regular, which means that we can identify all the Γ_x with Γ and $\bar{E} \otimes F$ as a bundle of finite Hilbert modules over $N(\Gamma)$.

Having reduced to this case we now simplify our notation. Our generic bundle will be a fiber bundle $Z \to M \to B$ with fiber Z a closed manifold. We wish to consider the family Novikov-Shubin invariants of M associated with a flat bundle F of Hilbert $\mathcal{A}$-modules over M, where $\mathcal{A}$ is a finite von Neumann algebra with a faithful normal finite trace $tr_{\mathcal{A}}$ (for example $N(\Gamma)$). This means that F has a flat connection ∇^F which commutes with the action of $\mathcal{A}$ and a Hermitian metric g^F. As in the previous section, we have a super connection d_M on $\Omega(B, \mathcal{F})$, where $\mathcal{F}$ is defined in a similar way to $\mathcal{E}$ in Section 2, $\tilde{V} = d_Z^* - d_Z$, $\tilde{C}_t, \tilde{D}_t$ and

$$f(\tilde{D}_t, g^{\mathcal{F}}) = (2\pi i)^{\frac{1}{2}} \varphi Tr_{\mathcal{A}}(f(\tilde{D}_t)),$$

where $Tr_{\mathcal{A}}$ (resp., $Tr^s_{\mathcal{A}}$) is the tensor product of the usual (resp., super) trace on $\mathcal{L}(\wedge_2(T^*Z))$, the bounded operators on the L^2-forms $\wedge_2(T^*Z)$, and the finite trace $tr_{\mathcal{A}}$ on $\mathcal{A}$. For a compact subset $K \subset B$, let

$$\alpha_K^{(i)}(M, Z, F) = \sup\{\beta_K^{(i)} \geq 0 : Tr_{\mathcal{A}}(e^{\frac{t}{4} P_{\tilde{V}}^{\perp} (\tilde{V}_i)^2 P_{\tilde{V}}^{\perp}}) \to_{t \to \infty} O(t^{-\beta_K^{(i)}/2})\}, i = 0, 1, \cdots, \dim(Z).$$

We also call $\alpha_K^{(i)}(M, Z, F)$ the family Novikov-Shubin invariants of the fiber bundle $Z \to M \xrightarrow{\pi} B$ associated with the bundle F. It enjoys the properties of the usual Novikov-Shubin invariants. In particular, we prove

Theorem 4.1 *Let K be a compact subset of B. If h is a homotopy equivalence of two fiber bundles $Z_i \to M_i \to B, i = 1, 2$, then for a flat Hilbert $\mathcal{A}$-bundle F_2 over M_2,*

$$\alpha_K^{(i)}(M_1, Z_1, h^*(F_2)) = \alpha_K^{(i)}(M_2, Z_2, F_2), i = 0, 1, \cdots, \dim(Z_2).$$

Proof. Let $\{U_i\}$ be a locally finite open covering of B such that M_i and F_2 over U_i are trivial. In view of a general result [GS], we need only to prove that the following two complexes are homotopy equivalent,

$$\begin{aligned} \mathcal{F}_2^* \quad &: \quad 0 \to C^\infty(U_i, \wedge_2^0(T^*Z_2, F_2|_{Z_2})) \xrightarrow{d} C^\infty(U_i, \wedge_2^1(T^*Z_2, F_2|_{Z_2})) \xrightarrow{d} \cdots, \\ h^*(\mathcal{F}_2)^* \quad &: \quad 0 \to C^\infty(U_i, \wedge_2^0(T^*Z_1, h^*(F_2|_{Z_2}))) \xrightarrow{d} C^\infty(U_i, \wedge_2^1(T^*Z_1, h^*(F_2|_{Z_2}))) \xrightarrow{d} \cdots. \end{aligned}$$

Note that $h^* : C^\infty(U_i, \wedge_2^*(T^*Z_2, F_2|_{Z_2})) \to C^\infty(U_i, \wedge_2^*(T^*Z_1, h^*(F_2|_{Z_2})))$ is not in general a bounded operator. To overcome this difficulty, we will employ an embedding method used by Hilsum-Skandalis [HS] and Gromov-Shubin [GS] to produce a homotopy equivalence between

$\mathcal{F}_2^*$ and $h^*(\mathcal{F}_2)^*$. Such a method was also crucial in the proof of equivariant Connes-Gromov-Moscovici theorem [Go 1].

By smooth approximation, we can assume that there are smooth maps $h_1 : Z_1 \to Z_2$ and $h_2 : Z_2 \to Z_1$ such that h_1h_2 and h_2h_1 are homotopic to the identities, respectively. Let $Z_i \xrightarrow{J_i} \mathbb{R}^l$ be an embedding for some $l > 0$ and O_i be a tubular neighborhood of $J_i(Z_i)$ with projection $\pi_i : O_i \to J_i(Z_i)$. We assume that $J_i(Z_i) + B^l \subset O_i$ for the unit ball $B^l \subset \mathbb{R}^l$. Define $\hat{h}_1(x,t) = \pi_2(h_1(x)+t)$ on $Z_1 \times B^l$ and $\hat{h}_2(x,t) = \pi_1(h_2(x)+t)$ on $Z_2 \times B^l$. Then $\hat{h}_i$ is a submersion such that $\hat{h}_1$ is homotopic to $h_1P_1^l$ and $\hat{h}_2$ is homotopic to $h_2P_2^l$, where $P_i^l : Z_i \times B^l \to Z_i$ is the natural projection.

First note that for the projection $P_2 : Z_2 \times I \to Z_2$ and $I = [0,1]$, $\wedge_2^*(T^*Z_2, F_2|_{Z_2})$ and $\wedge_2^*(T^*(Z_2 \times I), P_2^*(F_2|_{Z_2}))$ are homotopy equivalent. Indeed, P_2 induces a bounded operator $P_2^* : \wedge_2^*(T^*Z_2, F_2|_{Z_2}) \to \wedge_2^*(T^*(Z_2 \times I), P_2^*(F_2|_{Z_2}))$. Furthermore, $Q_2 : \wedge_2^*(T^*(Z_2 \times I), P_2^*(F_2|_{Z_2})) \to \wedge_2^*(T^*Z_2, F_2|_{Z_2})$ given by

$$Q_2(\xi) = \int_0^1 i_t^*(\xi)dt$$

is a bounded operator, where $i_t : Z_2 \to Z_2 \times I$ is defined by $i_t(x) = (x,t)$. We have $P_2i_t = Id$. Hence $Q_2P_2^* = Id$. The homotopy H_t between Id and i_tP_2 is given by $H_t(x,u,s) = (x, ts + (1-s)u), 0 \le s \le 1$. $H_t : Z_2 \times I \times I \to Z_2 \times I$ is a submersion and hence induces a bounded map $H_t^* : \wedge_2^*(T^*(Z_2 \times I), P_2^*(F_2|_{Z_2})) \to \wedge_2^*(T^*(Z_2 \times I \times I), \tilde{P}_2^*(F_2|_{Z_2}))$, where $\tilde{P}_2 : Z_2 \times I \times I \to Z_2$ is the natural projection. Let

$$h_t(\xi) = \int_0^1 \iota_{\frac{\partial}{\partial s}} H_t^*(\xi), \quad \xi \in \wedge_2^*(T^*(Z_2 \times I), P_2^*(F_2|_{Z_2})).$$

We get

$$P_2^*i_t^* - Id = d_{Z_2 \times I}h_t + h_t d_{Z_2 \times I}.$$

Taking integral with respect to t on the both sides , we obtain that $P_2^*Q_2$ and Id are homotopic.

Now by repeating the above procedure for $I^l = [0,1]^l$ and using the fact that B^l is Lipschetz homotopic to I^l, we get that $\wedge_2^*(T^*(Z_2 \times B^l), (P_2^l)^*(F_2|_{Z_2})) \overset{Q_2^l}{\underset{(P_2^l)^*}{\rightleftharpoons}} \wedge_2^*(T^*Z_2, F_2|_{Z_2})$ are homotopy equivalent. Similarly, we have the homotopy equivalence

$$\wedge_2^*(T^*(Z_1 \times B^l), (P_1^l)^*(h_1^*(F_2|_{Z_2}))) \overset{Q_1^l}{\underset{(P_1^l)^*}{\rightleftharpoons}} \wedge_2^*(T^*Z_1, h_1^*(F_2|_{Z_2})).$$

Consider now the following maps

$$\wedge_2^*(T^*Z_2, F_2|_{Z_2}) \xrightarrow{\hat{h}_1^*} \wedge_2^*(T^*(Z_1 \times B^l), (P_1^l)^*(h_1^*(F_2|_{Z_2}))) \xrightarrow{Q_1^l} \wedge_2^*(T^*Z_1, h_1^*(F_2|_{Z_2}))$$

and

$$\wedge_2^*(T^*Z_1, h_1^*(F_2|_{Z_2})) \xrightarrow{\hat{h}_2^*} \wedge_2^*(T^*(Z_2 \times B^l), (P_2^l)^*(h_2^*h_1^*(F_2|_{Z_2}))) \xrightarrow{Q_2^l} \wedge_2^*(T^*Z_2, F_2|_{Z_2}).$$

Since $\hat{h}_1(\hat{h}_2 \times Id) : Z_2 \times B^l \times B^l \to Z_2$ is homotopic to the natural projection $P_{1,2}^l : Z_2 \times B^l \times B^l \to Z_2$ and $\hat{h}_1(\hat{h}_2 \times Id)$ and $P_{1,2}^l$ are submersions, by a result of Gromov-Shubin [GS], we can find a submersion homotopy for these two maps. Hence $(\hat{h}_2 \times Id)^*\hat{h}_1^*$ is homotopic to $(P_{1,2}^l)^*$. This implies that $Q_2^l(\hat{h}_2 \times Id)^*(P_1^l)^*(Q_1^l\hat{h}_1^*) \sim Q_2^*(\hat{h}_2 \times Id)^*\hat{h}_1^* \sim Q_2^l(P_{1,2}^l)^* = Id$. In view of $(\hat{h}_2 \times Id)^*(P_1^l)^* = \hat{h}_2^*$, we get $Q_2^l\hat{h}_2^*Q_1^l\hat{h}_1^* \sim Id$. Similarly, we can prove that $Q_1^l\hat{h}_1^*Q_2^l\hat{h}_2^* \sim Id$. Hence they are the required homotopy equivalences.

The above proof shows that the maps $Q_i^l, \hat{h}_i^*$ and the corresponding homotopies depend continuously on U_i. By applying the above fiberwise result to each U_i in the open covering $\{U_i\}$ of B and using the estimate in the proof of Proposition 4.1 [GS], where the constant depends only on the norms of the above mentioned maps, we get that two family Novikov-Shubin invariants over K are equal. Hence the theorem holds. ∎

By the same argument as Corollary 4.1, we obtain

Corollary 4.2 *For B connected, $\alpha_K^i(M, Z, F) = \alpha_x^i(M, Z, F)$ depends only on a single fiber.*

5 L^2-Analytic Torsion Forms for Fibre Bundles

In Section 3 we obtained a transgression formula which gives rise to the L^2-analytic torsion form. The present section is devoted to discussing its properties. Let $\tilde{Z} \to M \xrightarrow{\pi} B$ be a fiber bundle over B with a flat vector bundle E as in Section 2.

Recall that we defined in Definition 3.1 an L^2-analytic torsion form $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ of the fiber bundle M associated with E, which has a weak exterior derivative. Clearly, $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is a real even form on B. By (3.34), we have that if $\dim(\tilde{Z})$ is odd and $H_2(\tilde{Z}, E|_{\tilde{Z}}) = 0$, then $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is a weakly closed form on B. This L^2-analytic torsion form is a generalization of the analytic torsion form [BL] to fiber bundles with covering spaces as fibers. We now consider its properties.

We first gather some facts which will play a role in the discussion of the L^2-analytic torsion form $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$. As in Section 2 we assume $\{g^Z, E, \nabla^E, g^E\}$ is a locally Γ-invariant data on M. The following sums up properties derived in [BL].

Lemma 5.1 *Let the family Novikov-Shubin invariants $\{\alpha_K^j(M, \tilde{Z}, E)\}$ of M be positive.*

(1) *Suppose that $M' = M \times [0,1] \xrightarrow{\pi'} B' = B \times [0,1]$ is the fiber bundle with fiber $\tilde{Z}'$ Γ-equivariantly isomorphic to $\tilde{Z}$, i.e., the fiber over $(x,s) \in B'$ is given by $M'_{(x,s)} = M_x \times \{s\} \simeq \tilde{Z}$, where $\pi'(x,s) = (\pi(x), s)$. Let E' be the lifting of E to M' with the lifted metric $g^{E'}$. Then*

$$-d\mathcal{T}(T^HM', g^{\tilde{Z}'}, g^{E'}) = \int_{\mathcal{F}'} e(T\tilde{Z}', \nabla^{\tilde{Z}'}) f(\nabla^{E'}, g^{E'}) - f(\nabla^{H_2(\tilde{Z}', E'|_{\tilde{Z}'})}, g^{H_2(\tilde{Z}', E'|_{\tilde{Z}'})}), \qquad (5.1)$$

where $\mathcal{F}'$ is a fundamental domain of Γ in $\tilde{Z}'$.

(2) *For each $t > 0$, $tr_\Gamma^s(\tilde{D}_t^2 e^{\tilde{D}_t^2}) = 0$ and $tr_\Gamma^s(f'(\tilde{D}_t)) = \chi_2(\tilde{Z}) rk(E)$.*

(3) *As $t \to 0$,*

$$tr_\Gamma^s(Ne^{t(\tilde{V})^2}) = \begin{cases} \frac{\dim(\tilde{Z}) rk(E) \chi_2(\tilde{Z})}{2} + O(t), & if \quad \dim(\tilde{Z}) \ \ even, \\ \frac{rk(E) a(M, \tilde{Z}, E)}{\sqrt{t}} + O(\sqrt{t}), & if \quad \dim(\tilde{Z}) \ \ odd, \end{cases}$$

where $a(M, \tilde{Z}, E)$ is a constant independent of t.

Proof. (1) Note that the family Novikov-Shubin invariants of $(M', \tilde{Z}', E')$ are positive by Corollary 3.1. (5.1) follows directly from Lemmas 3.2 , 3.3, and (3.33).

(2) Let $f_1(x) = xe^x$. Since $\tilde{D}_t$ is odd in $\Omega(B, End(\mathcal{E}))$,

$$\frac{\partial}{\partial u} tr_\Gamma^s(f_1(u\tilde{D}_t^2)) = tr_\Gamma^s(\tilde{D}_t^2 f_1'(u\tilde{D}_t^2)) = \frac{1}{2} tr_\Gamma^s([\tilde{D}_t, \tilde{D}_t f_1'(u\tilde{D}_t^2)]) = 0.$$

Hence, $tr_\Gamma^s(f_1(\tilde{D}_t^2)) = tr_\Gamma^s(f_1(0)) = 0$.

Let $h(x) = (1+2x)e^x$. Since

$$\frac{\partial}{\partial t} tr_\Gamma^s(f'(\tilde{D}_t)) = tr_\Gamma^s(\frac{\partial}{\partial t}(\tilde{D}_t^2) h(\tilde{D}_t^2)) = tr_\Gamma^s([\tilde{D}_t, \frac{\partial}{\partial t}(\tilde{D}_t) h(\tilde{D}_t^2)]) = 0,$$

$tr_\Gamma^s(f'(\tilde{D}_t))$ is independent of t. Using local index theory techniques as in the proof of Lemma 3.2, we get

$$tr_\Gamma^s(f'(\tilde{D}_t)) = \lim_{t \to 0} tr_\Gamma^s(f'(\tilde{D}_t)) = rk(E)\chi_2(\tilde{Z}).$$

(3) Note that the number operator N on the fiber can be written as

$$N = \frac{1}{2}\sum_{j=1}^{\dim(\tilde{Z})} c(e_j)c_v(e_j) + \frac{\dim(\tilde{Z})}{2}. \tag{5.2}$$

Using the Getzler rescaling $G_{\sqrt{t}}$, i.e., $\nabla_{e_j} \to t^{-1/2}\nabla_{e_j}, c(e_j) \to t^{-1/2}e_j \wedge -t^{1/2}i_{e_j}$, and $c_v(e_j) \to c_v(e_j)$, we get

$$G_{\sqrt{t}}(\frac{\sqrt{t}}{2}\sum_{j=1}^{\dim(\tilde{Z})} c(e_j)c_v(e_j)) \to \frac{1}{2}\sum_{j=1}^{\dim(\tilde{Z})} e_j \wedge c_v(e_j), \quad t \to 0.$$

When $\dim(\tilde{Z})$ is even, the first term of (5.2) will not contribute to the limit that we want to calculate. This follows from the proof of Theorem 7.10 [BZ]. By Lemma 3.2 , the second term of (5.2) yields the required limit. Similarly, one can treat the odd dimensional case. See also the proof of Theorem 7.10 [BZ]. ∎

We now discuss how $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ depends on its arguments $T^HM, g^{\tilde{Z}}$, and g^E. Suppose we have two triples $(T_i^HM, g_i^{\tilde{Z}}, g_i^E), i = 0, 1$. Let $\Omega^{ev}(M,\mathbb{R})$ (resp., $\Omega_d^{ev}(M,\mathbb{R})$) be the real (resp., exact) even forms on M. Choose a secondary class $\hat{e}(T\tilde{Z}, \nabla_i^{\tilde{Z}}) \in \Omega^{ev}(M,\mathbb{R})/\Omega_d^{ev}(M,\mathbb{R})$ such that

$$d\hat{e}(T\tilde{Z}, \nabla_0^{\tilde{Z}}, \nabla_1^{\tilde{Z}}) = e(T\tilde{Z}, \nabla_1^{\tilde{Z}}) - e(T\tilde{Z}, \nabla_0^{\tilde{Z}})$$

for even dimensional $\tilde{Z}$ and $\hat{e}(T\tilde{Z}, \nabla_0^{\tilde{Z}}, \nabla_1^{\tilde{Z}}) = 0$ for $\dim(\tilde{Z})$ odd. Throughout this section we assume that the family Novikov-Shubin invariants of $(M, \tilde{Z}, E)$ are positive.

Lemma 5.2 *In* $\Omega^{ev}(M,\mathbb{R})/\Omega_d^{ev}(M,\mathbb{R})$,

$$-(\mathcal{T}(T^HM, g_1^{\tilde{Z}}, g_1^E) - \mathcal{T}(T^HM, g_0^{\tilde{Z}}, g_0^E)) = \int_{\mathcal{F}} \hat{e}(T\tilde{Z}, \nabla_0^{\tilde{Z}}, \nabla_1^{\tilde{Z}}) f(\nabla^E, g^E) +$$
$$+ \quad \int_{\mathcal{F}} e(T\tilde{Z}, \nabla_1^{\tilde{Z}}) f(\nabla^E, g_0^E, g_1^E) - f(\nabla^{H_2(\tilde{Z},E|_{\tilde{Z}})}, g_0^{H_2(\tilde{Z},E|_{\tilde{Z}})}, g_1^{H_2(\tilde{Z},E|_{\tilde{Z}})}),$$

where for a smooth path of metrics g_s^E *connecting* g_0^E *and* g_1^E,

$$f(\nabla^E, g_0^E, g_1^E) = \int_0^1 \frac{1}{2}\varphi tr_\Gamma^s((g_s^E)^{-1}\frac{\partial g_s^E}{\partial s} f'(\tilde{D}_t(s)))ds \in \frac{\Omega^{ev}(B,\mathbb{R})}{\Omega_d^{ev}(B,\mathbb{R})}.$$

Lemma 5.2 follows from (5.1) as in ([BL], 3.24) by comparing the ds-terms of the two sides of (5.1) and by taking integral with respect to the variable s.

Lemma 5.3 *Let* g^E *be covariantly constant with respect to the connection* ∇^E*. If* Z *is oriented and* $\dim(\tilde{Z})$ *is even, then* $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E) = 0$.

In fact, let $\tau(\xi) = (-1)^{\dim(\tilde{Z})j+\frac{j(j+1)}{2}} * (\xi), \xi \in \wedge^j(T^*\tilde{Z}, E|_{\tilde{Z}})$, where $*$ is the Hodge star operator. Then $d_{\tilde{Z}}^* = (-1)^{\dim(\tilde{Z})}\tau^{-1}d_{\tilde{Z}}\tau$, and $\tilde{D}_t = -\tau\tilde{D}_t\tau^{-1}$. From $N\tau + \tau N = \dim(\tilde{Z})\tau$, and $tr_\Gamma^s(Nf'(\tilde{D}_t)) = tr_\Gamma^s(\tau Nf'(\tilde{D}_t)\tau^{-1})$, one gets $tr_\Gamma^s(Nf'(\tilde{D}_t)) = \frac{\dim(\tilde{Z})}{2}tr_\Gamma^s(f'(\tilde{D}_t))$. By Lemma 5.1(2), $f'(\tilde{D}_t, g^{\mathcal{E}})$ is independent of t. Lemma 3.4 implies then that the integrand in the definition of the L^2-analytic torsion form is zero. We refer to the proof of Theorem 3.26 [BL].

Lemma 5.4 *Let $\dim(\tilde{Z})$ be odd and the L^2-cohomology $H_2(\tilde{Z}, E|_{\tilde{Z}}) = 0$. Then $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ is a weakly closed form on B whose deRham cohomology class is independent of T^HM, $g^{T\tilde{Z}}$, and g^E.*

The proof of this lemma follows from (3.34) and the arguments of [BL]. Note that the $\mathcal{T}(T^HM, g^{\tilde{Z}}, g^E)$ depends on the equivalent class of local Γ-invariant structure. Let Z_i be two closed Riemannian manifolds and $\tilde{Z}_i$ be regular covering spaces of Z_i with covering groups Γ_i, $i = 1, 2$. Suppose $\tilde{Z}_i \to M_i \to B$ is a fiber bundle, $i = 1, 2$, and $\{g_i^{\tilde{Z}}, E_i, \nabla^{E_i}, g^{E_i}\}$ is a locally Γ_i-invariant data on M_i. Let $\tilde{Z}_1 \times \tilde{Z}_2 \to M \to B$ be the fiber product bundle of M_1 and M_2 , $E = P_1^*(E_1) \otimes P_2^*(E_2)$ and $g^E = P_1^*(g^{E_1}) \otimes P_2^*(g^{E_2})$, where $P_i : M \to M_i$ is the natural projection. We have the following product formula.

Proposition 5.1 *Assume that the family Novikov-Shubin invariants of $(M_i, \tilde{Z}_i, E_i)$ are positive, $i = 1, 2$. Then*

$$\mathcal{T}(T^HM, g^{\tilde{Z}_1\times\tilde{Z}_2}, g^E) = rk(E_1)\chi_2(\tilde{Z}_1)\mathcal{T}(T^HM_2, g^{\tilde{Z}_2}, g^{E_2}) + rk(E_2)\chi_2(\tilde{Z}_2)\mathcal{T}(T^HM_1, g^{\tilde{Z}_1}, g^{E_1}).$$

Proof. First note that by Proposition 4.1, $\alpha_K(M, \tilde{Z}_1 \times \tilde{Z}_2, E) > 0$. We have

$$\begin{aligned}
\chi_2(\tilde{Z}_1 \times \tilde{Z}_2, E) &= \sum_i (-1)^i i tr_{\Gamma_1\times\Gamma_2}(I|_{\oplus_{p+q=i}(H_2^p(\tilde{Z}_1,E_1|_{\tilde{Z}_1})\otimes H_2^q(\tilde{Z}_2,E_2|_{\tilde{Z}_2}))}) \\
&= \chi_2(\tilde{Z}_1, E_1)\sum_q(-1)^q tr_{\Gamma_2}(I|_{H_2^q(\tilde{Z}_2,E_2|_{\tilde{Z}_2})}) \\
&+ \chi_2(\tilde{Z}_2, E_2)\sum_p(-1)^p tr_{\Gamma_1}(I|_{H_2^p(\tilde{Z}_1,E_1|_{\tilde{Z}_1})}) \\
&= \chi_2(\tilde{Z}_1, E_1)\chi_2(\tilde{Z}_2)rk(E_2) + \chi_2(\tilde{Z}_2, E_2)\chi_2(\tilde{Z}_1)rk(E_1).
\end{aligned}$$

The last step follows from the fact that $\sum_p(-1)^p tr_{\Gamma_1}(I|_{H_2^p(\tilde{Z}_1,E_1|_{\tilde{Z}_1})}) = \chi_2(\tilde{Z}_1)rk(E_1)$, which is derived from the proof of Theorem 3.21 [BL]. By Lemma 5.1(2), the rest of the proof is similar to that of Proposition 3.28 [BL]. ∎

Recall that the L^2-analytic torsion invariant [Lo][Ma] on each fiber M_x associated with vector bundle $\tilde{E}_x$ is defined as

$$\mathcal{T}(M_x, \tilde{E}_x) = e^{\frac{1}{2}\frac{\partial}{\partial u} tr^s_\Gamma (N(-P^\perp_{\tilde{V}_x}(\tilde{V}_x)^2 P^\perp_{\tilde{V}_x}))^{-u}|_{u=0}}.$$

The following lemma says that the zero-form component $\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)_{[0]}$ of $\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)$ at $x \in B$ is equal to $\log \mathcal{T}(M_x, \tilde{E}_x)$.

Lemma 5.5 *Assume that the family Novikov-Shubin invariants of $(M, \tilde{Z}, E)$ are positive. Then*

$$\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)_{[0]}(x) = \log \mathcal{T}(M_x, \tilde{E}_x),$$

where $\tilde{E}_x = E|_{M_x}$.

The proof of this lemma follows from Lemma 5.1(3) and the definitions of the L^2-analytic torsions. We refer to the proof of Theorem 3.29 [BL] for the detail. We should point out that for the odd dimensional case one has to utilize a meromorphic extension of relevant functions because of the factor $t^{-1/2}$ in Lemma 5.1(3). We close this section by stating a mapping property of the L^2-analytic torsion form.

Proposition 5.2 *Let $u : B_1 \to B_2$ be a smooth map of two Riemannian manifolds and $\tilde{Z}_2 \to M_2 \to B_2$ a fiber bundle with regular covering space $\tilde{Z}_2$ of a closed Riemannian manifold Z_2 as typical fiber. Assume $\{g^{\tilde{Z}_2}, E_2, \nabla^{E_2}, g^{E_2}\}$ is a locally Γ-invariant data on M_2, where Γ is the covering group of $\tilde{Z}_2$. If the family Novikov-Shubin invariants of $(M_2, \tilde{Z}_2, E_2)$ are positive, then*

$$u^*(\mathcal{T}(T^H M_2, g^{\tilde{Z}_2}, g^{E_2})) = \mathcal{T}(T^H u^*(M_2), g^{u^*(\tilde{Z}_2)}, g^{u^*(E_2)}).$$

Proof. The torsion form of the right hand side of the above identity is determined by the pullback of operators on M_2, as we see below. Let $u^*(M_2)$ be the pull back of the fiber bundle M_2 to B_1. Then $u^*(M_2)$ is a fiber bundle over B_1 with fiber $u^*(\tilde{Z}_2) \simeq \tilde{Z}_2$ and $u^* : u^*(M_2) \to M_2$ is a map of fiber bundles whose restriction to each fiber is an identity map. Let $u^*(E_2)$ be the pull back of E_2 to $u^*(M_2)$ with the pull back metric $u^*(g^{E_2})$ and connection $u^*(\nabla^{E_2})$. We can also pull back the superconnection d_{M_2} of M_2 to $u^*(M_2)$, denoted it by $u^*(d_{M_2})$. Since we take a fiberwise adjoint and u is an identity map on each fiber, we get $(u^*(d_{M_2}))^* = u^*(d^*_{M_2})$ and then $\tilde{D}_{t,1} = u^*(\tilde{D}_{t,2})$, where $\tilde{D}_{t,2}$ is the operator $\tilde{D}_t$ in Section 2 for M_2 and

$$\tilde{D}_{t,1} = \frac{1}{2}(t^{-N/2}(u^*(d_{M_2}))^* t^{N/2} - t^{N/2}(u^*(d_{M_2})) t^{-N/2}).$$

Therefore,

$$f'(\tilde{D}_{t,1}, g^{u^*(E_2)}) \equiv \varphi tr^s_\Gamma(\frac{N}{2} f'(\tilde{D}_{t,1})) = u^*(\varphi tr^s_\Gamma(\frac{N}{2} f'(\tilde{D}_{t,2}))).$$

This implies that for $K_1 = u^{-1}(K), K \subset B$ compact,

$$\alpha_{K_1}(u^*(M_2), u^*(\tilde{Z}_2), u^*(E_2)) = u^*(\alpha_K(M_2, \tilde{Z}_2, E_2)).$$

We have

$$\begin{aligned} \mathcal{T}(T^H u^*(M_2), g^{u^*(\tilde{Z}_2)}, g^{u^*(E_2)}) &= \int_0^\infty [f'(\tilde{D}_{t,1}, g^{u^*\mathcal{E}}) - \frac{1}{2}\chi_2(u^*(\tilde{Z}_2), u^*(E_2)) \\ - & (\frac{1}{4}\dim(u^*(\tilde{Z}_2)) rk(u^*(E_2)) \chi_2(u^*(\tilde{Z}_2)) - \frac{1}{2}\chi_2(u^*(\tilde{Z}_2), u^*(E_2))) f'(\frac{i\sqrt{t}}{2})] \frac{dt}{t} \\ = & u^*(\mathcal{T}(T^H M_2, g^{\tilde{Z}_2}, g^{E_2})) \end{aligned}$$

∎

We apply Proposition 5.2 to the case where B_1 is a submanifold of B_2 and u is the inclusion. Note that $u^*(M_2) = M_2|_{B_1}$ and $u^*(E_2) = E_2|_{M_2|_{B_1}}$. We have the following.

Corollary 5.1 *In Proposition* 5.2*, if B_1 is a submanifold of B_2 and u is the inclusion, then*

$$\mathcal{T}(T^H M_2|_{B_1}, g^{\tilde{Z}_2}, g^{E_2|_{M_2|_{B_1}}}) = u^* \mathcal{T}(T^H M_2, g^{\tilde{Z}_2}, g^{E_2}).$$

We point out that the L^2-analytic torsion form and its properties could be extended to the general situation in Section 4 where a bundle of finite Hilbert-von Neumann algebra modules over a compact fiber bundle was considered.

6 Analytic Torsion Forms for Fibre Bundles with Fibers Homogeneous Spaces

In this section we will consider fiber bundles with homogeneous spaces as fibers. The purpose of this section is to compute some examples which show that the L^2-torsion forms are really non-trivial forms on the base space. Let G be a reductive Lie group with countably many components and K a maximal compact subgroup of G. The homogeneous space G/K is defined to be the space of cosets $gK, g \in G$. Suppose Γ is a finitely generated discrete subgroup of G such that $Z = \Gamma\backslash G/K$ is a compact manifold. We are interested in fiber bundles with fiber $\tilde{Z} = G/K$. Thus let H be a compact subgroup of G which normalizes Γ and P be a principle H-bundle over a Riemannian manifold B. Since H acts naturally on $\tilde{Z}$, we can form a fiber bundle $M = P \times_H \tilde{Z}$ over B with the fiber $\tilde{Z}$. Let $\tilde{E} = G/K \times \mathbb{C}^n$ and $E = M \times \mathbb{C}^n$ be trivial vector bundles over G/K and M, respectively. Note that since H normalizes Γ, M factors over a compact bundle with fiber Z and thus has a well defined local Γ-invariance structure. Then using a Haar measure on G and a normalized Haar measure on K, we can define $\wedge_2(T^*\tilde{Z}, \tilde{E})$, the space of $\tilde{E}$-valued L^2-smooth forms on $\tilde{Z}$.

Let $(\Omega(P)\hat{\otimes} \wedge^* (T^*\tilde{Z}, \tilde{E}))_b$ be the set of all forms $\xi \in \Omega(P)\hat{\otimes} \wedge^* (T^*\tilde{Z}, \tilde{E})$ such that ξ is H-equivariant and $i_X\xi = 0$ for all $X \in \mathcal{H}$, where $\mathcal{H}$ is the Lie algebra of H and i_X acts on the factor $\Omega(P)$. Using the notation of Section 2, we have

$$\Omega(M, E) = (\Omega(P)\hat{\otimes} \wedge^* (T^*\tilde{Z}, \tilde{E}))_b.$$

Let $\{x_\alpha\}$ be a basis of the Lie algebra $\mathcal{H}$ and $\theta = \theta^\alpha x_\alpha$ and $\Omega = \Omega^\alpha x_\alpha$ be connection and corresponding curvature forms on P, respectively. Then the operators $d_{\tilde{Z}}$, $\nabla^{\mathcal{E}}$ and $i_{\tilde{T}}$ on $\Omega(M, E)$ are the restrictions to $(\Omega(P)\hat{\otimes} \wedge^* (T^*\tilde{Z}, \tilde{E}))_b$ of the following operators on $\Omega(P)\hat{\otimes} \wedge^* (T^*\tilde{Z}, \tilde{E})$, respectively,

$$\begin{aligned}
\hat{d}_{\tilde{Z}}(\xi_1 \otimes \xi_2) &= (-1)^{deg\xi_1}\xi_1 \otimes d_{\tilde{Z}}\xi_2, \\
\hat{\nabla}^{\mathcal{E}}(\xi_1 \otimes \xi_2) &= d_P\xi_1 \otimes \xi_2 + (\theta^\alpha \wedge \xi_1) \otimes L_{X_\alpha}\xi_2, \\
\hat{i}_{\tilde{T}}(\xi_1 \otimes \xi_2) &= -(\Omega^\alpha \wedge \xi_1) \otimes i_{X_\alpha}\xi_2,
\end{aligned}$$

where L_{X_α} and i_{X_α} are the Lie derivative and contraction operator of the vector X_α on $\wedge^*(T^*\tilde{Z}, \tilde{E})$ which comes from the basis $\{x_\alpha\}$ of $\mathcal{H}$. The fiberwise adjoints of these operators as components of the super connection $\hat{d}_{P\times\tilde{Z}}$ on $P \times \tilde{Z}$ are

$$\begin{aligned}
\hat{d}^*_{\tilde{Z}}(\xi_1 \otimes \xi_2) &= (-1)^{deg\xi_1}\xi_1 \otimes d^*_{\tilde{Z}}\xi_2, \\
(\hat{\nabla}^{\mathcal{E}})^*(\xi_1 \otimes \xi_2) &= d_P\xi_1 \otimes \xi_2 + (\theta^\alpha \wedge \xi_1) \otimes L_{X_\alpha}\xi_2, \\
\hat{i}^*_{\tilde{T}}(\xi_1 \otimes \xi_2) &= (\Omega^\alpha \wedge \xi_1) \otimes (X_\alpha \wedge \xi_2).
\end{aligned}$$

Hence, the operators $\tilde{C}_t$ and $\tilde{D}_t$ are the restrictions of the following operators on $\Omega(P)\hat{\otimes} \wedge^*(T^*\tilde{Z}, \tilde{E})$

to $(\Omega(P)\hat{\otimes}\wedge^*(T^*\tilde{Z},\tilde{E}))_b$,

$$\begin{aligned}\tilde{C}'_t &= \frac{\sqrt{t}}{2}(\hat{d}^*_{\tilde{Z}}+\hat{d}_{\tilde{Z}})+\hat{\nabla}^{\mathcal{E}}+\frac{1}{2\sqrt{t}}c(\Omega),\\ \tilde{D}'_t &= \frac{\sqrt{t}}{2}(\hat{d}^*_{\tilde{Z}}-\hat{d}_{\tilde{Z}})+\frac{1}{2\sqrt{t}}c_v(\Omega).\end{aligned}$$

We obtain that the restriction of $\tilde{D}'_t$ to fibers is Γ-equivariant simply because each element in the Lie algebra $\mathcal{G}$ of G is left invariant and $\mathcal{H}$ is a subalgebra of $\mathcal{G}$. Thus we are able to define an analytic torsion form. The special feature of this analytic torsion form is that it is always closed.

Lemma 6.1 *If the family Novikov-Shubin invariants of M associated with E are positive, then the analytic torsion form $\mathcal{T}(T^HM,g^{\tilde{Z}},g^E)$ is a closed even form on B.*

The proof of this lemma is straightforward (see [BL]). We now use equivariant methods to relate the torsion class to characteristic forms of bundle P. Let us first recall the Chern -Weil homomorphism. Let W be a finite dimensional vector space. Consider the space of polynomial functions on W, $C(W)$. It is generated by W^* and the constant functions. If $\{x_\alpha\}$ is a basis of W, then $\{x^*_\alpha\}$ is a set of polynomial generators for W^*. When $W=\mathcal{H}$, the Lie algebra of H, we have the linear map: $\mathcal{H}^*\to\Omega(P)$ given by $x^*_\alpha\to\Omega^\alpha$ extends to a unique homomorphism $\Phi: C(\mathcal{H})\to\Omega(P)$ of algebras which is independent of the basis $\{x_\alpha\}$. The homomorphism Φ is called the Chern-Weil homomorphism. It is equivariant with respect to the actions of H on the domain and range.

If $\mathcal{A}$ is an arbitrary algebra, we consider $C(W)\otimes\mathcal{A}$. Each element of $W^*\otimes\mathcal{A}$ defines a function from W to $\mathcal{A}$ defined on generator $f\otimes a$ by $x\to f(x)a$. This extends to a unique monomorphism of algebras $C(W)\otimes\mathcal{A}\to Maps(W,\mathcal{A})$, where the algebra structure on the range comes from that on $\mathcal{A}$. Using this identification one defines the algebra of polynomial functions from W to $\mathcal{A}$ as $C(W)\otimes\mathcal{A}$. In the case where $W=\mathcal{H}$, the Chern-Weil map Φ extends to a homomorphism of algebras $C(W)\otimes\mathcal{A}\to\Omega(P)\otimes\mathcal{A}$, which we also denote by Φ and refer to as the Chern-Weil map. If the group H acts on $\mathcal{A}$, this is an H-equivariant map.

Let

$$\begin{aligned}\wedge^*(T^*\tilde{Z})_H &= (\mathbb{C}[\mathcal{H}]\otimes\wedge^*(T^*\tilde{Z}))^H,\\ \wedge^*(T^*\tilde{Z},\tilde{E})_H &= (\mathbb{C}[\mathcal{H}]\otimes\wedge^*(T^*\tilde{Z},\tilde{E}))^H,\end{aligned}$$

and

$$\wedge^*(pt, End^{\Gamma}(\wedge_2(T^*\tilde{Z},\tilde{E})))_H = (\mathbb{C}[\mathcal{H}] \otimes End^{\Gamma}(\wedge_2(T^*\tilde{Z},\tilde{E})))^H,$$

where $End^{\Gamma}(\wedge_2(T^*\tilde{Z},\tilde{E}))$ is the space of Γ-trace class operators on $\wedge_2(T^*\tilde{Z},\tilde{E})$. Then using the above Chern-Weil homomorphism $\Phi : \wedge(T^*\tilde{Z})_H \to (\Omega(P)\hat{\otimes} \wedge (T^*\tilde{Z},\tilde{E}))_b$, we have

$$\wedge^*(pt, End^{\Gamma}(\wedge_2(T^*\tilde{Z},\tilde{E})))_H \xrightarrow{tr_\Gamma} \mathbb{C}[\mathcal{H}]^H \xrightarrow{\Phi} \Omega(P)_b \simeq \Omega(B).$$

Let

$$\tilde{D}_t'(x) = \frac{\sqrt{t}}{2}(\hat{d}^*_{\tilde{Z}} - \hat{d}_{\tilde{Z}}) + \frac{1}{2\sqrt{t}} c_v(\frac{X}{2\pi i}),$$

where X is a tangent vector on $\tilde{Z}$ corresponding to $x \in \mathcal{H}$. This is a polynomial function from $\mathcal{H}$ to $End(\wedge_2(T^*\tilde{Z},\tilde{E}))$. We have on $\wedge^*(T^*\tilde{Z},\tilde{E})$

$$(\tilde{D}_t'(x))^2 = -\frac{t}{4}\hat{\Delta} + \frac{1}{8\pi i}(-L_X^* - L_X - [\hat{d}_{\tilde{Z}}, X] - [\hat{d}_{\tilde{Z}}, X]^*) + \frac{c_v(X)^2}{4t(2\pi i)^2}, \tag{6.1}$$

where $\hat{\Delta} = \hat{d}^*_{\tilde{Z}}\hat{d}_{\tilde{Z}} + \hat{d}_{\tilde{Z}}\hat{d}^*_{\tilde{Z}}$ and $L_X = \hat{d}_{\tilde{Z}} i_X + i_X \hat{d}_{\tilde{Z}}$. As in [BL], we also obtain

$$f'(\tilde{D}_t, g^{\mathcal{E}}) = \varphi tr^s_\Gamma(\frac{N}{2} f'(\tilde{D}_t)) = \Phi(\varphi tr^s_\Gamma(\frac{N}{2} f'(\tilde{D}_t'(x)))).$$

This implies that the L^2-analytic torsion form $\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)$ is the pushdown of a linear combination of characteristic forms of the principle H-bundle P. However, it is not clear at the moment how to explicitly compute the L^2-analytic torsion form even for fiber bundles with hyperbolic spaces as fibers.

We now compute the L^2-analytic torsion form for the following special case. Let $Z_1 = S^1$ and $G = S^1$. G acts on S^1 via multiplication by r-th powers, $r > 1$. Let P be a S^1-principal bundle over B and G' be the extension of G on $\tilde{Z}_1 \simeq \mathbb{R}^1$. G' can be written as

$$\begin{aligned} G' &= \{(g,\beta) \in G \times Diff(\tilde{Z}_1) : \pi_{Z_1}\beta = g\pi_{Z_1}\} \\ &= \{(g,\beta) \in G \times Diff(\tilde{Z}_1) : g^r = e^{i\beta}\} \\ &\simeq \mathbb{Z}/r\mathbb{Z} \times \mathbb{R}, \end{aligned}$$

where $\pi_{Z_1} : \tilde{Z}_1 \to Z_1$ is the projection, and $\beta \in \mathbb{R}^1$ acts on $\mathbb{R}^1$ by translation. The last isomorphism is given by $(g,\beta) \to (ge^{-i\beta/r}, \beta)$. Let $\tilde{Z}_2$ be the universal covering space of a closed even dimensional Riemann manifold Z_2. We take the trivial action of G on Z_2. Take $M = (P \times \tilde{Z})/G'$, where $\tilde{Z} = \tilde{Z}_1 \times \tilde{Z}_2$. Let ρ_r be the unitary representation of $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{C}$ given by multiplication of a r-th root $\zeta \neq 1$ of unity. We get a unitary representation ρ_r' of G' on $\mathbb{C}$ by combining ρ_r with the projection $\mathbb{Z}/r\mathbb{Z} \times \mathbb{R} \to \mathbb{Z}/r\mathbb{Z}$. G' acts on $P \times \tilde{Z} \times \mathbb{C}$ by $(g, \beta)(p, \tilde{z}, \xi) = (pg^{-1}, \beta\tilde{z}, \rho_r'(\beta)\xi)$. Let $E = (P \times \tilde{Z} \times \mathbb{C})/G'$. Then E is a flat vector bundle over M with fiber $\mathbb{C}$. Let $\Gamma = \pi_1(S^1) = \mathbb{Z}$. The restriction $E_{Z_1 \times \tilde{Z}_2}$ of E to a fiber is the flat bundle $\tilde{Z} \times_\Gamma \mathbb{C}$ over $Z_1 \times \tilde{Z}_2$ with the Hermitian metric inherited from the standard metric on $\mathbb{C}$.

Proposition 6.1 *Let $G = S^1, Z_1 = S^1, \tilde{Z} = \tilde{Z}_1 \times \tilde{Z}_2, M = (P \times \tilde{Z})/G'$ and $E = (P \times \tilde{Z} \times \mathbb{C})/G'$ as above. Assume that the Novikov-Shubin invariants of $\tilde{Z}_2$ are positive. Then for $r > 1$, the L^2-analytic torsion form $\mathcal{T}(T^H M, g^{\tilde{Z}}, g^E)$ is the pushdown to B of the following form on P:*

$$\tilde{\mathcal{T}} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{(2j+1)!}{2^{2j}(j!)^2} \sum_{m \neq 0} \frac{\zeta^m}{m^j |m|} (\frac{r\Omega}{4\pi^2})^j \chi_2(\tilde{Z}_2).$$

Proof. Let $M_1 = (P \times \tilde{Z}_1)/G'$ and $M_2 = (P \times \tilde{Z}_2)/G' \simeq P/G \times \tilde{Z}_2$. Then M is the fiber product of M_1 and M_2. By Proposition 5.1 and Proposition 4.13 [BL], we get the result. ∎

7 Appendix

In Section 2 we introduced the notion of a local Γ-invariance structure, which we have used throughout this paper. The invariants we have constructed are invariants of such structures. In this appendix we would like to discuss existence and uniqueness of such structures. We will sketch, without proof, a method to reduce these questions to those in homotopy theory, where hopefully well developed techniques in algebraic topology should apply.

Recall that we start with the following data: a smooth fiber bundle $\tilde{Z} \to M \xrightarrow{\pi} B$, where $p : \tilde{Z} \to Z = \tilde{Z}/\Gamma$ is a regular covering with covering group Γ, and a complex flat vector bundle $E \to M$. We will denote the structure group of the fiber bundle M by G which we assume is given as a subgroup of $Diff(\tilde{Z})$. We are interested in finding a metric, $g^{\tilde{Z}}$, on the vertical tangent bundle $T\tilde{Z}$, and a metric on E, g^E, and bundle charts $\{U_i\} \times \tilde{Z}$, which satisfy the conditions of Definition 2.1. As is usual in bundle theory we will "solve" this problem, by reformulating it as a problem in lifting the group of the fiber bundle to a more refined structure. To do this we need to introduce the notion of a structure groupoid of a fiber bundle.

Definition A.1 *A groupoid $\hat{G}$ is a category in which all maps have left and right inverses.*

We will identify a category with its set of maps, and consider only topological categories where the maps form a topological space such that composition and inversion are continuous. Morphisms of groupoids are just functors which will always be assumed continuous.

A group is just a groupoid with one element. An action of a groupoid $\hat{G}$ on a smooth manifold W is just a morphism of groupoids $\hat{G} \xrightarrow{\lambda} Diff(W)$. Given a fiber bundle as $\tilde{Z} \to M \to B$ with structure group G and a morphism $\lambda : \hat{G} \to G$, we can define what it means to lift the structure groupoid of the fiber bundle to $\hat{G}$.

Definition A.2 *A lifting of the structure of the fiber bundle $\tilde{Z} \to M \to B$ to $\hat{G}$ consists of coordinate data: an open cover $\{U_i\}$ of B, bundle coordinates $\phi_i : M|_{U_i} \simeq U_i \times \tilde{Z}$, and continuous maps $\phi_{ik} : U_i \cap U_k \to \hat{G}$ satisfying gluing equation: $\lambda(\phi_{ik}) = \phi_i \phi_k^{-1}$, and coherence equation: $\phi_{ik}\phi_{kl} = \phi_{il}$.*

Here we compose two elements of a groupoid provided the domain of the first element is the range of the second one . If we refine the covering of B we get an induced lifting using the refined covering. We will identify two liftings if they have a common refinement.

Conversely, if we are given a morphism $\lambda : \hat{G} \to Diff(Z)$, and gluing data, i.e. $\{U_i, \phi_{ik}\}, \phi_{ik} : U_i \cap U_k \to \hat{G}$, which satisfy the coherence equation, we can construct, just as in the usual bundle situation, the bundle $\tilde{Z} \to M \to B$ for which the initial data is coordinate data for a lifting to the groupoid $\hat{G}$.

Working out the classification theory for fiber bundles with structure groupoid $\hat{G}$ is similar to the case for groups, with a couple of important complications. The first is that the notion of a principle bundle must be revised since a groupoid does not act on itself. Further covering homotopy in the usual sense need not hold.

We adopt the following definition of equivalence of liftings. Note it depends only on the gluing data.

Definition A.3 *Two sets of gluing data on B define equivalent $\hat{G}$-fiber bundles if there is gluing data on $B \times I$ which restricts on $B \times \{i\}$ to our given data, $i = 0, 1$.*

To define the classifying space of $\hat{G}$, we can simply take the classifying space of the category $\hat{G}$. We can also modify the more elementary Milnor construction. Since a groupoid doesn't act on itself, the usual Milnor construction of a classifying space for $\hat{G}$ must be modified. Instead of taking the limit of n fold joins of $\hat{G}$, we consider the limit of restricted n fold joins, where a point in the restricted n fold join is a point in the usual n fold join such that all the $\hat{G}$-coordinates have a common range. One can transform a point in this restricted join by an element of $\hat{G}$ whose domain is equal to this common range. This defines an equivalence relation on the restricted joins. The orbit space of the restricted joins under this relation has a limit which we denote by $B\hat{G}$. When the base space B of the fiber bundle M is a cell complex and $\hat{G}$ satisfies some mild topological conditions we have:

Theorem A.1 *There is a natural equivalence between $[B, B\hat{G}]$ and equivalence classes of $\hat{G}$-fiber bundles on B. Here as usual, $[,]$ denotes homotopy classes of maps.*

Note that $\pi_0(B\hat{G})$ is in $1-1$ correspondence with set of isomorphic classes of objects of $\hat{G}$, and thus $B\hat{G}$ is not in general connected.

Theorem $A.1$ reduces the question of existence and classification of liftings to the following homotopy question. Suppose as above $\tilde{Z} \to M \to B$ is a fiber bundle with structure group G, and $\lambda : \hat{G} \to G$ is a morphism of groupoids. Then λ induces $B(\lambda) : B\hat{G} \to BG$, which up to homotopy we can assume is a fibration. The fiber bundle M defines a map $\psi : B \to BG$, unique up to homotopy. The equivalence classes of liftings of the fiber bundle to $\hat{G}$ correspond $1-1$ to homotopy classes of liftings of ψ, $\hat{\psi} : B \to B\hat{G}$, where two liftings are homotopic if they are homotopic through liftings of ψ.

Obstruction theory now kicks in to tell us that we need to study the homotopy groups of the fiber of the maps $B(\lambda)$. For a group the n-th homotopy group of BG is just the $n-1$ homotopy group of G. For a groupoid however this is no longer true, and the homotopy groups of $B\hat{G}$ are quite mysterious. There is a situation where this complication does not arise.

Theorem A.2 *Let $\hat{G}$ be a groupoid such that the domain map $d : \hat{G} \to Objects(\hat{G})$ is a fiber bundle. Then $\pi_{n+1}(B\hat{G}) = \pi_n(\hat{G})$, and the isomorphism is natural in the full subcategory of groupoids which satisfy the hypothesis.*

Example A.1 Given $\tilde{Z} \to M \to B$ and G as above. let $\hat{G}$ be the following groupoid. The objects are the Riemannian metrics on $\tilde{Z}$, $Rie(\tilde{Z})$. Maps$(<,>_1, <,>_2)$ are elements g of G such that $<,>_1$ is the pullback metric of $<,>_2$ under g. Given $<,>_1$ and g, $<,>_2$ is uniquely determined. Therefore as a space $\hat{G} = G \times Rie(\tilde{Z})$. Thus this G satisfies the condition of

Theorem $A.2$.

The projection λ of $\hat{G}$ into G is a fibration with contractible fiber $Rie(\tilde{Z})$, and thus induces an isomorphism on homotopy groups. By Theorem $A.2$, $B(\lambda)$ is also a weak homotopy equivalence, and thus we conclude that the fiber bundle has a lifting to $\hat{G}$ unique up to equivalence. What is a lifting to this $\hat{G}$? It is exactly a metric on the vertical tangent bundle $T\tilde{Z}$. Hence we have proven in a rather round about way that $T\tilde{Z}$ has a metric unique up to deformation. This example isn't very interesting, but it shows how the machinery works. The next example gets us closer to what we are after.

Example A.2 With the assumption of Example $A.1$, we now bring into play that we have a regular covering $p : \tilde{Z} \to Z = \tilde{Z}/\Gamma$. Let $\hat{G}_\Gamma$ be the full subcategory of $\hat{G}$ whose objects are Γ-invariant metrics on $\tilde{Z}$, i.e. metrics which pull back under p from metrics on Z. Notice that in general the projection $\lambda_\Gamma : \hat{G}_\Gamma \to G$ is not very nice. It may not be onto and $\hat{G}_\Gamma$ will not satisfy the hypothesis of Theorem $A.2$. It does have however the property that the inverse image of each point is either empty or an affine space and thus contractible. If for a given G, λ_Γ is onto, then with appropriate geometric hypothesis we should be able to conclude that λ_Γ is a homotopy or at least a homology equivalence. If we can conclude this for $B(\lambda_\Gamma)$, we could prove existence and uniqueness of liftings.

Example A.3 We now further complicate our example by taking into account our flat bundles. We assume given a flat complex bundle of dimension n over Z, $E_0 \to Z$, and will denote the pullback flat bundle over $\tilde{Z}$ by E_1. E_1 comes with a Γ-action and a flat connection invariant under Γ. We consider the group G^1 of all bundle isomorphisms on E^1 which preserve the connection and cover an element $g \in G, g : \tilde{Z} \to \tilde{Z}$. Note that the natural map $G^1 \to G$ has as image the union of certain components of G and a discrete kernel. We will assume it is onto and thus G^1 is a covering group of G. Through the covering map the group G^1 acts on $\tilde{Z}$ and we can consider the groupoid $\hat{G}^1$ whose objects are pairs of invariant metrics, the first on the tangent bundle of $\tilde{Z}$ and the second on E^1. G^1 acts as affine transformations on the spaces of metrics of the two bundles and thus we can define $Maps((a, b), (a', b'))$ of $\hat{G}^1$ to be the set of all g in G^1 with $g(a) = a'$ and $g(b) = b'$, where a, a' are invariant metrics on $T\tilde{Z}$ and b, b' are invariant metrics on E^1. As in the previous example the inverse image of a point in G^1 under projection map $\hat{G}^1 \to G^1$ is either empty or an affine space. When this projection is onto it is natural to look for conditions for which it is a homotopy equivalence.

Note that a lifting of $\tilde{Z} \to M \to B$ to the groupoid $\hat{G}^1$ produces a flat bundle E over M and data which satisfies Condition 1 in Definition 2.1. To satisfy Condition 2 of Definition 2.1, we consider the subgroupoid of $\hat{G}^1$, $\hat{G}^2$, generated by special elements of $\hat{G}^1$. $g \in Maps((a,b),(a',b'))$ is called special if there exists a bundle isomorphism $\psi : E^1 \to E^1$, not necessarily in $\hat{G}^1$, covering an isometry $\psi' : (\tilde{Z}, a') \to (\tilde{Z}, a')$ such that ψ preserves b' and such that $\psi' g : \tilde{Z} \to \tilde{Z}$ takes some fundamental domain of Γ to a fundamental domain of Γ.

Liftings to the groupoid $\hat{G}^2$ induce local Γ-invariance structures on $\tilde{Z} \to M \to B$, and equivalent liftings induce equivalent local Γ-invariance structures. Thus the groupoid $\hat{G}^2$ allows us to reduce questions on the existence and uniqueness of such structures to questions of homotopy theory. Of course in general the groupoid $\hat{G}^2$ is not very tractible, however in some reasonable geometric situations the morphisms $\hat{G}^2 \to \hat{G}^1 \to G^1 \to G$ are rather simple, for example fibrations. In such situations $B\hat{G}^2$ becomes acessible, and we can say something about the existence and uniqueness of local Γ-invariance structures.

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