VANISHING LINE FOR THE DESCENT SPECTRAL SEQUENCE

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Abstract. *We prove that if $f : X \to Y$ is a surjective and proper map between locally compact spaces and if additionally it is cohomologically $(k-1)$-connected, for some $k > 0$, then for the cohomological descent spectral sequence [1] one has $E_2^{pq} = 0$ provided $q < pk$.*

Mathematics Subject Classifications (2000). 18G40, 55N30

Key words. descent, spectral sequence, sheaf, vanishing line.

1. Introduction. Let $f : X \to Y$ be a continuous map of topological spaces and let $\mathcal{F}$ be an abelian sheaf on Y. One defines the simplicial space X_* by

$$X_n = \{(x_0, ..., x_n) \in X^{n+1} \mid f(x_0) = ... = f(x_n)\}, n \geq 0,$$

$$\partial_i(x_0, ..., x_n) = (x_0, ..., \widehat{x}_i, ..., x_n), 0 \leq i \leq n$$

$$s_i(x_0, ..., x_n) = (x_0, ..., x_i, x_i, ..., x_n), 0 \leq i \leq n.$$

We let $f_n : X_n \to Y$ be the map given by $f_n(x_0, ..., x_n) = f(x_0) = ... = f(x_n)$. Moreover we put $\mathcal{F}_n = f_n^*\mathcal{F}$. Thus $X_0 = X$, $f_0 = f$ and $\mathcal{F}_0 = f^*\mathcal{F}$. Then $(f_n)_{n\geq 0}$ defines the morphism from the simplicial space X_* to the constant simplicial space Y and $\mathcal{F}_*$ is a sheaf on the simplicial space X_*. The following spectral sequence is known as the cohomological descent spectral sequence (see for example 5.3 of [1] and references given there).

Theorem 1. *Let $f : X \to Y$ be a surjective and proper map, then there exists a spectral sequence*

$$E_1^{pq} = \mathsf{H}^q(X_p, \mathcal{F}_p) \Longrightarrow \mathsf{H}^{p+q}(Y, \mathcal{F}).$$

Moreover E_2^{pq} is isomorphic to the p-dimensional cohomology of the cochain complex associated to the cosimplicial abelian group E_1^{*q}.

The aim of the paper is to prove the following vanishing result for the cohomological descant spectral sequence. In this paper all maps between topological spaces are continuous and all sheafs are abelian. Moreover a space Z is called *cohomologically k-connected* if for any constant sheaf A one has

$$\mathsf{H}^0(Z, A) \cong A, \quad \text{and} \quad \mathsf{H}^i(Z, A) = 0 \quad \text{for} \quad 0 < i \leq k.$$

Similarly, a map $f : X \to Y$ is called *cohomologically k-connected* if for each $y \in Y$ the space $f^{-1}(y)$ is cohomologically k-connected.

Theorem 2. *Let $f : X \to Y$ be a surjective and proper map between locally compact spaces. If f is cohomologically $(k-1)$-connected, for some $k > 0$, then for the cohomological descent spectral sequence one has $E_2^{pq} = 0$ provided $q < pk$.*

Actually we will prove much strong fact: the normalization of the cosimplicial abelian group E_1^{*q} vanishes in dimension p as soon as $q < kp$.

We refer the reader to [4] for different sort of vanishing results in group cohomology framework.

The author was partially supported by the grant INTAS-99-00817 and by the TMR network K-theory and algebraic groups, ERB FMRX CT-97-0107. Special thanks to MPI at Bonn for hospitality.

3. Preliminaries on cosimplicial abelian groups. Let $(A^*, \delta^*, s^i)_{0\leq i\leq n}$ be a cosimplicial abelian group. It can be considered as a cochain complex with the coboundary map given by $d = \sum_i (-1)^i \partial^i$. The normalization of A^* is a subcomplex N^*A^*, given by

$$N^n(A^*) = \{x \in A^n \mid s^i x = 0, i = 0, \cdots, n-1\}.$$

It is well-known that N_nA^* is isomorphic to the quotient of

$$A_n/(Im(\delta^0) + \cdots + Im(\delta^{n-1}))$$

and the inclusion $N^*A^* \subset A^*$ is a homotopy equivalence. Thus $H^*(A^*, d) \cong H^*(N^*A^*)$. Moreover a theorem of Dold and Kan asserts that the functor N^* establishes an equivalence between the category of cosimplicial abelian groups and the category of nonnegative cochain complexes. In particular N^* is an exact functor and therefore for any short exact sequence of cosimplicial abelian groups

$$0 \longrightarrow A_1^* \longrightarrow A^* \longrightarrow A_2^* \longrightarrow 0$$

one has

$$N^i(A^*) = 0, \text{iff } N^i(A_1^*) = 0 = N^i(A_2^*).$$

Now we consider the following example. Let R be an algebra over a commutative ring K. We let $C^*(R)$ be the following cosimplicial K-module

$$C^n(R) = R^{\otimes(n+1)},$$

$$\delta^i(x_0, \cdots, x_n) = (x_0, \cdots, x_{i-1}, 1, x_i, \cdots, x_n),$$

$$s^i(x_0,\cdots,x_n)=(x_0,\cdots,x_ix_{i+1},\cdots,x_n).$$

It is clear that for any R one has

$$N^n(C^*(R))=\bar{R}^{\otimes n}\otimes R,$$

where $\bar{R}=R/K.1$. If R is graded $R=\bigoplus_{i\geq 0}R^i$, then one has an isomorphism of cosimplicial modules $C^*(R)\cong\bigoplus_{m\geq 0}C^{*m}$, where $C^{nm}=\bigoplus R^{i_0}\otimes\cdots\otimes R^{i_n}$ and the sum is taken over all $i_0+\cdots+i_n=m, i_s\geq 0$. It is now clear that, if R is graded and connected (that is $R^0=K$), then $N^nC^{*m}=0$ if $m<n$. More generally, we have $N^nC^{*m}=0$ if $m<kn$, as soon as R is $(k-1)$-connected for some $k\geq 1$ (that is connected and $R^i=0$ for $0<i<k$).

We will use this observation in the following situation. Let S be a compact space, which is assumed to be cohomologically $(k-1)$-connected. We let $c_*(S)$ be the simplicial space given by

$$c_n(S)=S^{n+1},$$

$$\partial_i(x_0,\cdots,x_n)=(x_0,\cdots,\hat{x}_i,\cdots,x_n),$$

$$s_i(x_0,\cdots,x_{n-1})=(x_0,\cdots,x_i,x_i,\cdots,x_{n-1}).$$

Now applying the functor $\mathsf{H}^m(-,A)$ we get a cosimplicial abelian group $\mathsf{H}^m(c_*(S),A)$. Here A is an abelian group, considered as a constant sheaf. We claim that the normalization of this cosimplicial abelian group vanishes in dimension n provided $m<nk$. It is well known that the cohomology of compact space with constant coefficients is isomorphic to the Čech cohomology and therefore one has the following natural short exact sequence

$$0\to\mathsf{H}^i(S,\mathbf{Z})\otimes A\to\mathsf{H}^i(S,A)\to\mathsf{H}^{i-1}(S,\mathbf{Z})*A\to 0. \tag{4}$$

Indeed, since the tensor and the torsion products preserves colimits, it suffice to consider the case, when S is a finite polyhedron. In this case Čech cohomology is isomorphic to the simplicial cohomology, and hence $\mathsf{H}^*(S,A)$ is the homology of $\mathsf{Hom}(C_*(S),A)$ with degreewise free and finitely generated $C_*(S)$. Now it is enough to observe that the natural map

$$\mathsf{Hom}(C_*(S),\mathbf{Z})\otimes A\to\mathsf{Hom}(C_*(S),A)$$

is an isomorphism. It follows from the exact sequence (4) that to prove the claim it suffice to consider the case, when $A=K$ is a field. In this case on can use the Künnent theorem for Ĉech cohomology (see for example Theorem 7.2 of [3]) to obtain the isomorphism that $C^*(\mathsf{H}^*(S,K))\cong\mathsf{H}^*(c_*(S),K)$. Thus $N^m(\mathsf{H}^n(c_*(S),K)=0$ provided $n<mk$ and the claim is proved.

5. Preliminaries on the Lerey spectral sequence. Let $g : W \to Z$ be a map of topological spaces and let $\mathcal{G}$ be a sheaf on W. Then there exists a spectral sequence

$$E_2^{pq} = \mathsf{H}^p(Z, \mathcal{R}^q g_*(\mathcal{G})) \Longrightarrow \mathsf{H}^{p+q}(W, \mathcal{G})$$

known as the Lerey spectral sequence. Here $\mathcal{R}^q g_*(\mathcal{G})$ is the sheaf on Z associated to the presheaf $U \mapsto \mathsf{H}^q(g^{-1}(U), \mathcal{G})$. Here U is an open subset of Z. It is well-known (see page 201 of [2]) that

$$\mathcal{R}^q g_*(\mathcal{G})_z \cong \mathsf{H}^q(g^{-1}(z), \mathcal{G})$$

as soon as g is proper and W, Z are locally compact. Now we take $\mathcal{G} = g^*\mathcal{B}$, where $\mathcal{B}$ is a sheaf on Z. Then for each $w \in W$ one has $\mathcal{G}_w = \mathcal{B}_{g(w)}$. Thus for each $z \in Z$ the sheaf $\mathcal{G}$ is a constant sheaf on $g^{-1}(z)$ associated to the abelian group $\mathcal{B}_z$. In this case we allow ourself to rewrite the Leray spectral sequence as follows

$$E_2^{pq} = \mathsf{H}^p(Z, z \mapsto \mathsf{H}^q(g^{-1}(z), \mathcal{B}_z)) \Longrightarrow \mathsf{H}^{p+q}(W, g^*\mathcal{B}).$$

6. Proof of Theorem 2. For each $n \geq 0$ we consider the Leray spectral sequence corresponding to f_n:

$${}_nE_2^{pq} = \mathsf{H}^p(Y, y \mapsto \mathsf{H}^q(f_n^{-1}(y), \mathcal{F}_y) \Longrightarrow \mathsf{H}^{p+q}(X_n, \mathcal{F}_n).$$

Varying n one obtains the spectral sequence of cosimplicial abelian groups. Since the normalization is an exact functor it suffices to show that for fixed p and q the normalization of the cosimplicial abelian group $[n] \mapsto {}_nE_2^{pq}$ vanishes in the dimension m, provided $q < mk$. Let us consider the cosimplicial object in the category of sheafs over Y given by

$$[n] \mapsto \mathcal{R}^q_*(f_n)_*(f_n^*\mathcal{F}),$$

which we denote for simplicity by $[n] \mapsto (\mathcal{R}^q)^n$. Thus

$${}_*E_2^{pq} = \mathsf{H}^p(Y, (\mathcal{R}^q)^*).$$

It follows from the exactness of the normalization that

$$N^m({}_*E_2^{pq}) = \mathsf{H}^p(Y, N^m(\mathcal{R}^q)^*).$$

Therefore it suffices to show that $N^m(\mathcal{R}^q)^* = 0$ provided $m < qk$. Since the stalk at y is an exact functor, we have

$$(N^m(\mathcal{R}^q)^*)_y \cong N^m((\mathcal{R}^q_y)^*).$$

The last group is the same as $N^m\mathsf{H}^q(f_*^{-1}(y), \mathcal{F}_y)$. It suffice to show that the last group vanishes as soon as $q < km$. But this follows from the fact that

$$f_n^{-1}(y) = f^{-1}(y) \times \cdots \times f^{-1}(y)$$

(and therefore $f_*^{-1}(y) = c_*(f^{-1}(y)))$ and from the example of Section 3.

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