On the Second K-group of an Elliptic Curve

by

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The abstract formalism used to define higher algebraic K-groups is more often than not the envy and the despair of the mathematician eager to work with them. In fact, on the one hand, finite generation results tend to be as clearly expected as they are impossible to prove, and, on the other, the general theory makes it usually quite hard to produce explicit non-zero elements in specific K-groups. Recent progress in the spirit of Zagier's polylogarithm conjecture now has provided some (in part conjectural) remedies for the second one of these diametrically opposed problems. Research in this area is moving fast at the moment. We will concentrate here on the group $K_2(E)$, for an elliptic curve E defined over $\overline{\mathbf{Q}}$, where solid progress has been realized by A.987arov, A. Levin, and J. Wildeshaus over the last two years.

The aim of our note is to present a down-to-earth treatment of the case of the elliptic analogue of Zagier's polylogarithm conjecture pertaining to $K_2(E)$—see section 4 below for the final result, which enables one to systematically produce elements of $K_2(E)$ from certain divisors on E. We thus complete the first author's thesis [Rolshausen 1996]. More precisely, outside of K-theory, we use only the classical theory of elliptic functions according to Eisenstein, Kronecker and Weierstrass, as well as algebraic number theory. And from K-theory, except for a few basic exact sequences and well-known properties of K-groups of fields, or of varieties over finite fields, we only need [Schappacher, Scholl 1991, §3]. However, the way we set things up is of course modelled on (earlier versions of) the papers [Gončarov, Levin, 1997] and [Wildeshaus 1997b]. Contrary to Gončarov and Levin, we will systematically neglect the torsion in the groups under consideration, tensoring with $\mathbf{Q}$.

The original motivation of the first author's thesis [Rolshausen 1996] was to check Beilinson's conjecture numerically for certain examples of elliptic curves over $\mathbf{Q}$—see section 5 below as well as the tables in [Rolshausen 1996].

Since our approach is deliberately down-to-earth, we do not discuss the recent preprints [Bloch 1997], [Gončarov 1997]. Nor do we go into the higher conjectures which concern the special values $L(\mathrm{Sym}^n E, n+1)$, for $n \geq 2$. By Beilinson's general formalism, the corresponding absolute cohomology groups are the groups $H^{n+1}_{\mathcal{M}}(\mathrm{Sym}^n E, \mathbf{Q}(n+1))$ attached to the *motive* $\mathrm{Sym}^n E$ which can be defined as the sub-motive of $h(E^{k\prime})$ on which the permutations of $\mathcal{S}_{k+1}$ act via the sign character. Here, $E^{k\prime}$ is the sub-variety of E^{k+1} where the sum of the components is 0. Thus, $H^{n+1}_{\mathcal{M}}(\mathrm{Sym}^n E, \mathbf{Q}(n+1))$ can be described as a sub-vector space of $K_{n+1}(E^{n\prime}) \otimes_{\mathbf{Z}} \mathbf{Q}$. The peculiar problems of these higher cases and some conjectural solutions are discussed in the preprints [Gončarov 1995] and [Wildeshaus 1997c].

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1. Recalling $K_2(E)$.

Let F be a number field over which the elliptic curve E is defined, and let $\mathcal{E}/\mathfrak{o}_F$ be a regular minimal model of E over the ring of integers of F. The second K-groups of $\mathcal{E}$ and of E/F are connected by the localization exact sequence

$$\cdots \longrightarrow \coprod_{\mathfrak{p}} K_2'(\mathcal{E}_{\mathfrak{p}}) \longrightarrow K_2(\mathcal{E}) \longrightarrow K_2(E) \xrightarrow{\partial} \coprod_{\mathfrak{p}} K_1'(\mathcal{E}_{\mathfrak{p}}) \longrightarrow \cdots, \tag{1.1}$$

the direct sums being taken over the finite places of F. As to the fibres $\mathcal{E}_{\mathfrak{p}}$ of $\mathcal{E}$ at the finite primes, it is well-known that all the $K_2'(\mathcal{E}_{\mathfrak{p}})$ are torsion groups, and that $K_1'(\mathcal{E}_{\mathfrak{p}})$ is non-torsion if and only if $K_1'(\mathcal{E}_{\mathfrak{p}}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is a $\mathbf{Q}$-line (and the $\mathfrak{p}$-component of ∂ is surjective), and that this happens if and only if E has split multiplicative reduction at $\mathfrak{p}$—see for instance [Schappacher, Scholl 1991, §3] or [Rolshausen 1996, I.3].

1.2. Beilinson's Conjecture, I. *Denote by r_F the number of infinite places of F. Then one expects that*

$$\dim_{\mathbf{Q}}\Big(K_2(\mathcal{E}/\mathfrak{o}_F) \otimes_Z \mathbf{Q}\Big) \stackrel{?}{=} \mathrm{ord}_{s=0} L(E,s) = r_F.$$

Or, equivalently, that

$$\dim_{\mathbf{Q}}\Big(K_2(E/F) \otimes_Z \mathbf{Q}\Big) \stackrel{?}{=} r + \#\{\mathfrak{p} \mid E \text{ has split multiplicative reduction mod } \mathfrak{p}\}.$$

The conjecture is fleshed out by considering a regulator map. To define it in the case at hand use the following short exact sequence, which is derived from the localization sequence of E with respect to its generic fibre,

$$0 \longrightarrow K_2(E/F) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow K_2(F(E)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\mathcal{T}} \coprod_{P \in E} (k(P)^* \otimes_{\mathbf{Z}} \mathbf{Q}), \tag{1.3}$$

where P varies over the $\overline{F}$-valued points of E, and $k(P)$ denotes the field of definition of P. The zero on the left follows from the fact that the second K-group of a number field is torsion. By Matsumoto's theorem (which the second author learned by presenting it in Martin Kneser's Oberseminar, some twenty years ago),

$$K_2(F(E)) = \frac{\bigwedge^2 F(E)^*}{\langle f \wedge 1 - f \mid f \in F(E), f \neq 0, 1\rangle_{\mathbf{Z}}} \ni \{f, g\}.$$

In terms of these generators, the P-component of the tame symbol $\mathcal{T}$ in (1.3) is explicitly given by the formula

$$\mathcal{T}_P(\{f,g\}) = \Big(-1\Big)^{v_P(f)v_P(g)} \frac{f^{v_P(g)}}{g^{v_P(f)}}(P). \tag{1.4}$$

We can now define the regulator. In the spirit of this note, we simply present it as a mapping

$$\mathrm{reg}_\omega : K_2(E/F) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \mathbf{R}^{r_F} = \bigoplus_{v|\infty} F_v$$

which depends on the choice of a holomorphic differential form ω on E defined over F. It was S. Bloch who first defined it in [Bloch 1977], on the generators $\{f, g\}$ of $K_2(F(E))$. Write the divisors of the functions as $(f) = \sum a_x[x]$, $(g) = \sum b_y[y]$. Fix an embedding $\tau : F \hookrightarrow \mathbf{C}$, and call v the infinite place of F given by τ. Via τ, we have over $\mathbf{C}$, $(E, \omega) \cong (\mathbf{C}/\Gamma, dz)$ with a unique lattice $\Gamma = \omega_1\mathbf{Z} \oplus \omega_2\mathbf{Z} \subset \mathbf{C}$. If $x \in E(\mathbf{C})$, denote by $z(x)$ the corresponding class in $\mathbf{C}/\Gamma$. The v-component of the regulator is defined to be the real part of the integral $\mathrm{reg}_{\omega,v}(\{f, g\}) = \mathrm{Re}\Big(\frac{1}{\pi i}\int_{\mathbf{C}/L} \log|f| \; \overline{\left(\frac{dg}{g}\right)} \wedge dz\Big)$. According to [Bloch 1977], this can be transformed into the following expression—see also [Rohrlich 1987]:

$$\mathrm{reg}_{\omega,v}(\{f, g\}) = -\mathrm{Re}\Big(A(\Gamma)^2 \sum_{x,y} a_x b_y K_1(0, z(x) - z(y), 2, \Gamma)\Big). \tag{1.5}$$

Here we have used the notation from [Weil 1976] for the Kronecker-Eisenstein series: If we take the basis ω_1, ω_2 of the lattice such that ω_1 is real and $y = \mathrm{Im}(\omega_2/\omega_1) > 0$, then $A(\Gamma) = \frac{{\omega_1}^2 y}{\pi}$. The Pontryagin duality between Γ and $\mathbf{C}/\Gamma$ is afforded by $\langle \gamma, x\rangle = \exp(\frac{\gamma\overline{x} - x\overline{\gamma}}{A(\Gamma)})$. Then for $z \in \mathbf{C}/\Gamma$, and an integer $\nu \geq 0$, one has the convergent series in the half-plane $\mathrm{Re}(s) > \frac{\nu}{2} + 1$,

$$K_\nu(0, z, s, \Gamma) = \sum_{\gamma\in\Gamma}{}' \langle \gamma, z\rangle \frac{\overline{\gamma}^\nu}{(\gamma\overline{\gamma})^s}. \tag{1.5.1}$$

(The extra parameter which we set equal to zero here because we will not need it, is relevant for the analytic continuation of these series.)

The analogues of these regulator maps for the finite places are the components of the boundary map ∂ in (1.1). They were determined by Bloch and Grayson, and more generally in [Schappacher, Scholl 1991]. They are zero unless E has split multiplicative reduction at $\mathfrak{p}$. Suppose this is so, i.e., the fibre $\mathcal{E}_\mathfrak{p}$ is a Néron N-gon for some N. Just as we chose a differential ω in order to obtain the regulators at the infinite places as real-valued functions, we now pick a basis of the $\mathbf{Q}$-line $K_1'(\mathcal{E}_\mathfrak{p}) \otimes_{\mathbf{Z}} \mathbf{Q}$. Specifically, we number the components of $\mathcal{E}_\mathfrak{p}$ and take, in the notation of [Schappacher, Scholl 1991], the basis element $\frac{1}{3N} \cdot \Phi_1^1$ in order to identify $K_1'(\mathcal{E}_\mathfrak{p}) \otimes_{\mathbf{Z}} \mathbf{Q} = \mathbf{Q}$. Then the component map $\partial_\mathfrak{p}$ on $K_2(E)$ is obtained by linearity from the following rule on Steinberg symbols $\{f, g\}$ which have the property that the closure of the support of the divisors $(f) = \sum a_x[x]$, $(g) = \sum b_y[y]$ are contained in the smooth part of $\mathcal{E}$:

$$\{f, g\} \mapsto -\sum_{x,y} a_x b_y B_{3,\mathfrak{p}}(x - y), \tag{1.6.1}$$

where

$$(1.6.2) \qquad B_{3,\mathfrak{p}}(x) = \mathbf{B}_3\Big(\Big\langle \frac{\nu}{N} \Big\rangle\Big),$$

if and only if the section of $\mathcal{E}$ defined by x meets the Néron N-gon $\mathcal{E}_{\mathfrak{p}}$ in its ν-th side. Here $\mathbf{B}_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t$ is the third Bernoulli polynomial. Cf. [Schappacher, Scholl 1991, 3.1 (p. 815)].

1.7 Beilinson's Conjecture, II. *(i) Let S be the set of places of F where E has split multiplicative bad reduction. The $\mathbf{Q}$-vector space*

$$\Big(\bigoplus_{v|\infty} \mathrm{reg}_{\omega,v} \oplus \bigoplus_{\mathfrak{p}\in S} \mathbf{B}_{3,\mathfrak{p}}\Big)\Big(K_2(E/F) \otimes_{\mathbf{Z}} \mathbf{Q}\Big)$$

is a $\mathbf{Q}$-structure of $\bigoplus_{v|\infty} \mathbf{R} \oplus \bigoplus_{\mathfrak{p}\in S} \mathbf{R}$.

(ii) $\bigwedge^{r_F}\Big(\mathrm{reg}_\omega(K_2(\mathcal{E}/\mathfrak{o}_F) \otimes_{\mathbf{Z}} \mathbf{Q})\Big) = L^{(r_F)}(E/F, 0) \cdot \mathbf{Q} \subset \mathbf{R} = \bigwedge^{r_F}(\mathbf{R}^{r_F}).$

One may enhance the analogy between the infinite places and those in S by building a factor of $\log(\mathbf{N}\mathfrak{p})$ into the definition of $B_{3,\mathfrak{p}}$, and then relating the total regulator (including the S-components) to the derivative at $s = 0$ of the imprimitive L-function, with Euler-factors at places in S removed. This was proposed, in slightly greater generality, as an "S-Beilinson conjecture" in [Ramakrishnan 1989]. But we find this formal procedure rather pointless in the case at hand.

It is not known at present for a single curve E, that $K_2(\mathcal{E}) \otimes_{\mathbf{Z}} \mathbf{Q}$ or $K_2(E) \otimes_{\mathbf{Z}} \mathbf{Q}$ are finite dimensional. The strongest results towards (ii) known to date still are, on the one hand, Bloch's treatment of the CM case in [Bloch 1977] which was refined in [Rohrlich 1987] and generalized in [Deninger 1989], and on the other, Beilinson's theorem on modular curves, and thus on modular elliptic curves—see [Schappacher, Scholl 1988].

1.8 *Remark on the injectivity of the regulator.* It is sometimes heuristically useful—say, in the case of an elliptic curve $E/\mathbf{Q}$—to assume that the Beilinson regulator reg_∞ is injective on all of $K_2(E) \otimes \mathbf{Q}$—cf. [Bloch, Grayson 1986, p. 84, note (*)] for a case in point. But, as Bloch pointed out to us, this hypothesis inspires less confidence, from a theoretical point of view, than conjecture 1.7.(i), because it postulates the injectivity of a real map on a $\mathbf{Q}$-vector space which is not always going to be one-dimensional. In fact, compare 1.7.(i) to the injectivity of the classical regulator for a number field, i.e., the injectivity of the map on the right in the commutataive triangle of theorem 3.4 below.

2. The basic diagram.

The obvious idea to try and construct elements in $K_2(E)$ is to use the exact sequence (1.3). Let us pass to the direct limit over all number fields. This gives the exact sequence

$$(2.1) \qquad 0 \longrightarrow K_2(E/\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow K_2(\overline{\mathbf{Q}}(E)) \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\mathcal{T}} \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} I_E,$$

where the tame symbol now sends

$$\{f,g\} \mapsto \sum_{x \in E(\overline{\mathbf{Q}})} (-1)^{v_x(f)v_x(g)} \ \left(\frac{f^{v_x(g)}}{g^{v_x(f)}}\Big|_x \otimes x \right).$$

Here we have written (tensoring with $\mathbf{Q}$ a notation used by Gončarov and Levin) I_E for the augmentation ideal of the group algebra $\mathbf{Q}[E(\overline{\mathbf{Q}})]$, i.e., for the divisors of degree 0 on E with $\mathbf{Q}$-coefficients. Weil reciprocity guarantees that the tame symbol does indeed land in this subspace.

Factoring out I_E by principal divisors, i.e., by I_E^2, and inserting Matsumoto's theorem for the middle group, we obtain the exact sequence

$$(2.2)\ \ 0 \longrightarrow K_2(E/\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \frac{\bigwedge^2_{\mathbf{Z}} \overline{\mathbf{Q}}^*}{(\overline{\mathbf{Q}}^* \wedge \overline{\mathbf{Q}}(E)^*)\langle f \wedge (1-f)\rangle_{\mathbf{Z}}} \otimes_{\mathbf{Z}} \mathbf{Q} \xrightarrow{\tilde{\mathcal{T}}} \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} E(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

To see this (cf. the proof of (3.7) in [Gončarov, Levin 1997]), note first that a Steinberg symbol of the form $\{c, g\}$, with a constant $c \in \overline{\mathbf{Q}}^*$, is mapped by $\mathcal{T}$ to $c \otimes \mathrm{div}(g)$, and therefore lies in the kernel of $\tilde{\mathcal{T}}$. But the kernel of the restriction of $\mathcal{T}$ to

$$(\overline{\mathbf{Q}}^* \wedge \overline{\mathbf{Q}}(E)^*) \otimes_{\mathbf{Z}} \mathbf{Q} \longrightarrow \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} I_E^2$$

is $K_2(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} = 0$, again by Matsumoto's theorem, and $\mathcal{T}$ thus restricted to a map $(\overline{\mathbf{Q}}^* \wedge \overline{\mathbf{Q}}(E)^*) \otimes_{\mathbf{Z}} \mathbf{Q} = \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} I_E^2 \longrightarrow \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} I_E^2$ is simply the identity.

Note that, if we work over a field of definition F of E which has finite degree over $\mathbf{Q}$, then (2.2) can be simply cut down by taking invariants under $G_F := G(\overline{\mathbf{Q}}/F)$, because the K-groups tensored with $\mathbf{Q}$ have *Galois descent.* Note also that all the maps $\mathrm{reg}_{\omega,v}$ and $B_{3,\mathfrak{p}}$ vanish on symbols of the form $\{c, h\}$.

The exact sequence (2.2) immediately gives what used to be known as

2.3 Bloch's Lemma. *A Steinberg symbol $\{f, g\}$ such that the divisors of f and g are supported on torsion points of E, can be corrected by elements of the form $\{c, h\}$ to come from $K_2(E) \otimes_{\mathbf{Z}} \mathbf{Q}$.*

For a long time this was the only method to produce explicit elements in $K_2(E)$. If E has complex multiplication (and therefore in particular no primes of bad multiplicative reduction), such elements conjecturally suffice—according to (1.7)—to generate all of $K_2(E)$—see [Rohrlich 1987]. However, for elliptic curves with sufficiently big image of G_F in Aut (E_{tors}), the elements given by Bloch's lemma will all lie in the kernel of $\bigoplus_{v|\infty} \mathrm{reg}_{\omega,v} \oplus \bigoplus_{\mathfrak{p}} \mathbf{B}_{3,\mathfrak{p}}$, and therefore conjecturally be zero—see [Mestre, Schappacher 1991, 1.3.1]. Beilinson's *Eisenstein Symbol*—see [Deninger 1989]—is essentially a generalization and refinement of Bloch's lemma.

A rather short-lived and somewhat frustrating phase in the development of our current knowledge of $K_2(E)$, roughly between 1989 and 1994, saw a number of *ad hoc* constructions of non-trivial elements in $K_2(E)$ from certain points of *infinite* order on special elliptic curves without complex multiplication, working directly from the sequence (2.1) or (2.2) (unpublished constructions by Schappacher-Nekovář, R. de Jeu, J-F. Mestre; also R. Ross obtained an element on the Fermat elliptic curve with CM). We will briefly discuss the oldest such example in section 5 below..

We now write down the commutative diagram which will give us the most general way to obtain elements in $K_2(E)$ from divisors on the elliptic curve. It is suggested by [Gončarov, Levin 1997], although our presentation of it blends in some notions from [Wildeshaus 1997b]. It begins with the line (2.2). The rest will be explained as we go on.

$$\begin{tikzcd}
K_2(E/\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} \arrow[r, hook] & \dfrac{\bigwedge^2_{\mathbf{Z}} \overline{\mathbf{Q}}^*}{(\overline{\mathbf{Q}}^* \wedge \overline{\mathbf{Q}}(E)^*)\langle f \wedge (1-f)\rangle_{\mathbf{Z}}} \otimes_{\mathbf{Z}} \mathbf{Q} \arrow[r, "\tilde{\mathcal{T}}"] \arrow[d, "\beta"] & \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} E(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q} \\
 & \dfrac{I_E^4}{\beta\langle f \wedge (1-f)\rangle_{\mathbf{Q}}} \arrow[r] & \ker d_2 \otimes_{\mathbf{Q}} (E(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}) \arrow[u, "\varphi_2 \otimes \mathrm{id}", "\cong"'] \\
 & I_E^4 \arrow[u, "p"] \arrow[r] \arrow[d, phantom, "\cap"] & \ker d_2 \otimes_{\mathbf{Q}} (E(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}) \arrow[u, equal] \arrow[d, phantom, "\cap"] \\
\ker d_3^{\#} \arrow[r, hook] \arrow[uuu] & \mathcal{L}_3^{\#} \arrow[r, "d_3^{\#}"] & \mathcal{L}_2 \otimes_{\mathbf{Q}} (E(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q})
\end{tikzcd}$$

Commutative diagram (2.4)

The column on the right will be defined and constructed in the next section—see 3.6. Let us discuss the central column. We implicitly encountered the map β when we were discussing regulators. It is defined by

$$\beta(f \wedge g) = \sum_{x,y} a_x b_y \; [x-y] \in \mathbf{Q}[E(\overline{\mathbf{Q}})],$$

where the divisors of the functions are written $(f) = \sum a_x[x]$, $(g) = \sum b_y[y]$. Since the principal divisors make up I_E^2, the image of β coincides with the fourth power I_E^4 of the augmentation ideal.

The vertical map p is the natural projection, and the space at the bottom of the central column is defined, following [Wildeshaus 1997b], by

$$\mathcal{L}_3^{\#} = \mathbf{Q}[E(\overline{\mathbf{Q}}) - \{0\}].$$

The inclusion of I_E^4 into it is defined by forgetting the contribution of 0 in the divisors. Since they are all of degree 0, this does not erase any information.

We now turn to the right column of the diagram. This will take a certain amount of preparation ...

3. The multiplicative group of a number field.

Choose again a number field F over which the elliptic curve E is defined. Put

$$\mathcal{L}_2^{\#} = \mathbf{Q}[E(F) - \{0\}],$$

and write elements of this vector space in the form $\sum_i a_i \{x_i\}_2^{\#}$. Define a $\mathbf{Q}$-linear map by its effect on the basis:

$$\begin{array}{cccc} d_2^{\#} : & \mathcal{L}_2^{\#} & \longrightarrow & E(F) \otimes_{\mathbf{Z}} E(F) \otimes_{\mathbf{Z}} \mathbf{Q} \\ & \{x\}_2^{\#} & \mapsto & x \otimes x \otimes 1. \end{array}$$

The kernel of $d_2^{\#}$ contains a few obvious types of elements: the divisors supported on torsion points of $E(F)$, the relations $\{kx\} - k^2\{x\}$ (for $k \in \mathbf{Z}$ and $x \in E(F)$ not killed by k), and the 'parallelogram equations' 3.1.1 below. It is elementary to show that certain among these elements already suffice to generate all of $\ker d_2$. See for instance [Gončarov, Levin 1997, 4.6].

3.1 Lemma. $\ker d_2^{\#}$ *is generated by all divisors of the following form, where* $x, y \in E(F)$.

(3.1.1)	$\{x+y\}_2^{\#} + \{x-y\}_2^{\#} - 2\{x\}_2^{\#} - 2\{y\}_2^{\#}$	*for* $x \neq \pm y$
(3.1.2)	$\{2x\}_2^{\#} - 4\{x\}_2^{\#}$	*for* $2x \neq O$
(3.1.3)	$\{2x\}_2^{\#} - 2\{x\}_2^{\#} - 2\{-x\}_2^{\#}$	*for* $2x \neq O$
(3.1.4)	$-4\{x\}_2^{\#}$	*for* $2x = O$

3.2 *Remark.* Note that all divisors in $\ker d_2^{\#}$ are such that the global Néron-Tate height $\hat{h}$ on E over F, extended linearly from points to divisors, vanishes on them. — If for a basis $x_1, \ldots, x_\rho$ of the Mordell-Weil group $E(F)$ modulo torsion the heights $\hat{h}(x_i)$ are $\mathbf{Q}$-linearly independent real numbers, then $\ker d_2^{\#} = \{\sum a_i \{x_i\}_2^{\#} \mid \sum a_i \hat{h}(x_i) = 0\}$.

3.3 Notation. For any place v of F, denote by h_v the local Néron height on E/F_v, extended $\mathbf{Q}$-linearly to divisors. For a finite place v, $\log_v : F^* \longrightarrow \mathbf{Z}$ is the additive valuation onto $\mathbf{Z}$, and if v is the infinite place given by the embedding $\tau : F \hookrightarrow \mathbf{C}$, then $\log_v : F^* \longrightarrow \mathbf{R}$ is the function $\log|\ |_\tau$. In either case, we extend by $\mathbf{Q}$-linearity to $F^* \otimes_{\mathbf{Z}} \mathbf{Q}$.

The local Néron height at the infinite places is related to Kronecker double series: If the infinite place v of F is induced by the embedding $\tau : F \hookrightarrow \mathbf{C}$, and if the point $x \in E(F) \overset{\tau}{\hookrightarrow} E(\mathbf{C}) = \mathbf{C}/\Gamma$ corresponds to the complex number z, modulo the lattice Γ, then we have (notations being as in (1.5.1) above)

$$h_v(x) = \frac{A(\Gamma)}{2} K_0(0, z, 1, \Gamma). \tag{3.3.1}$$

This follows for instance from a comparison of [Zagier 1990, Thm. 1.3] with [Silverman 1994, chap. VI].

3.4 Theorem. [Elliptic Analogue of Zagier's Polylogarithm Conjecture for $K_1(F)$.] *There is a unique* $\mathbf{Q}$*-linear map* $\varphi_2^{\#}$ *rendering the following diagram commutative:*

$$\begin{array}{cccc} \varphi_2^{\#}: & \ker d_2^{\#} & \longrightarrow & F^* \otimes_{\mathbf{Z}} \mathbf{Q} \\ & \bigoplus_v h_v \searrow & & \swarrow \bigoplus_v \log_v \\ & & \bigoplus_{v \text{ inf.}} \mathbf{R} \oplus \bigoplus_{v \text{ fin.}} \mathbf{Q} & \end{array}$$

3.4.1 *Remark.* G. Faltings has pointed out to us that this proposition (along with its algebro-geometric prrof via the Poincaré bundle) generalizes *verbatim* to abelian varieties E defined over F. The only additional complication is that the local heights depend in general on the choice of a section of the ample line bundle.

Proof: There are now several proofs of (variants of) this proposition—see [Gončarov, Levin 1997] and [Wildeshaus 1997b, Thm. 2] for a proof using the Poincaré bundle and Deligne's symbol, and [Wildeshaus 1997d] for a motivic approach. We sketch here the proof which seems to us to be the most elementary. It was first envisaged in [Rolshausen 1996].

The uniqueness of $\varphi_2^{\#}$ follows from Dirichlet's Unit Theorem. Also, one can easily define $\varphi_2^{\#}$ on generators of $\ker d_2^{\#}$, say those listed in 3.1, in such a way that the triangle commutes, by invoking standard formulae from the theory of local heights—see for instance [Silverman 1994, chap. VI, in particular ex. 6.3]: Fix a Weierstrass model for E/F of the very classical shape $Y^2 = 4X^3 - g_2X - g_3$ ($g_2, g_3 \in F$), denote by X, Y the coordinate functions with respect to this model, by Δ its discriminant. Then we want to prescribe the following values in $F^* \otimes \mathbf{Q}$.

$$(3.4.2)\qquad \begin{aligned} \varphi_2^{\#}\Big(\{x+y\}_2^{\#} + \{x-y\}_2^{\#} - 2\{x\}_2^{\#} - 2\{y\}_2^{\#}\Big) &= (X(x) - X(y))^{-6}\Delta \otimes \tfrac{1}{6} \\ \varphi_2^{\#}\Big(\{2x\}_2^{\#} - 4\{x\}_2^{\#}\Big) &= (-Y(x))^{-4}\Delta \otimes \tfrac{1}{4} \\ \varphi_2^{\#}\Big(\{2x\}_2^{\#} - 2\{x\}_2^{\#} - 2\{-x\}_2^{\#}\Big) &= (-Y(x))^{-4}\Delta \otimes \tfrac{1}{4} \\ \varphi_2^{\#}\Big(-4\{x\}_2^{\#}\Big) &= (-3X(x)^2 + \tfrac{1}{4}g_2)^{-3}\Delta \otimes \tfrac{1}{3} \end{aligned}$$

So we have to show that there exists a well-defined map on $\ker d_2^{\#}$ which takes all those values. For this we may work over $\mathbf{C}$, fixing any embedding of F. The idea is to exponentiate an analytic formula for the local archimedean height. But since these formulae involve some theta function, one has to be careful about writing down an expression which is independent of the representatives in $\mathbf{C}$ chosen for the points of $\mathbf{C}/\Gamma$, where Γ is a lattice corresponding to E. Specifically, given $\xi = \sum_j a_j\{x_j\}_2^{\#} \in \ker d_2^{\#}$, choose $\Xi = \sum_k A_k[z_k] \in \mathbf{Q}[\mathbf{C} - \Gamma]$ such that $\Xi \mapsto \xi$ under $z_k \mapsto (z_k \bmod \Gamma)$, and such that Ξ lies in the kernel of the $\mathbf{Q}$-linear map $\mathbf{Q}[\mathbf{C} - \Gamma] \longrightarrow \mathbf{C} \otimes_{\mathbf{Q}} \mathbf{C}$ defined by $z \mapsto z \otimes z$. (This last condition was introduced in [Wildeshaus 1997b, p. 374–375], emending the treatment given in [Rolshausen 1996, V.4.4].) For instance, if ξ is a generator of type (3.1.1) and if $x = z \bmod \Gamma$, $y = z' \bmod \Gamma$, then $\Xi = [z+z'] + [z-z'] - 2[z] - 2[z']$ will do. Similarly, one sees that such representatives Ξ exist for all divisors $\xi \in \ker d_2^{\#}$; generators of type (3.1.4) can be recuperated over $\mathbf{C}$ (where all 2-torsion points are rational) from (3.1.1), for various distinct 2-torsion points x and y.

Restricting to such representatives $\Xi = \sum_k A_k[z_k]$ of ξ, the expression

$$(3.4.3) \qquad \varphi_2^{\#}(\xi) = \sum_k \Big[\Big(e^{-6 z_k \eta(z_k)} \sigma(z_k)^{12} \cdot \Delta \Big) \otimes \Big(-\frac{A_k}{12} \Big) \Big] \in \mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{Q},$$

where η, σ are the classical Weierstrass functions relative to the lattice Γ, is well-defined. Indeed, if Ξ, Ξ' are two such representatives of ξ, then $\Xi - \Xi'$ is of the form $\sum_l c_l \Big([z_l] - [z_l + \gamma_l] \Big)$, for suitable $c_l \in \mathbf{Q}$, $z_l \in \mathbf{C}$, and $\gamma_l \in \Gamma$. Write the complex numbers z_l and γ_l in terms of a fixed basis of the lattice: $z_l = A_l \omega_1 + B_l \omega_2$ ($A_l, B_l \in \mathbf{R}$), $\gamma_l = C_l \omega_1 + D_l \omega_2$ ($C_l, D_l \in \mathbf{Z}$). The transformation formula for our theta function—see for instance [Silverman 1994, VI.3.1.(b)]—combined with Legendre's period relation then shows that formula (3.4.3) evaluated on $\Xi - \Xi'$ yields

$$\exp\Big(\sum_l c_l \{ A_l D_l - C_l B_l \} \cdot \pi i \Big) \in \mathbf{C}^* \otimes_{\mathbf{Z}} \mathbf{Q}.$$

So it suffices to show that $\sum_l c_l \{ A_l D_l - C_l B_l \} \in \mathbf{Q}$. On the other hand, $\sum_l c_l \Big([z_l] - [z_l + \gamma_l] \Big)$ goes to 0 in $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{C}$, since this is true by hypothesis for Ξ and Ξ'. This yields in particular the relations

$$\sum_{l, A_l \notin \mathbf{Q}} c_l D_l (A_l \omega_1 \otimes \omega_2) = 0 = \sum_{l, B_l \notin \mathbf{Q}} c_l C_l (\omega_1 \otimes B_l \omega_2),$$

and thus $\sum_{l, A_l \notin \mathbf{Q}} c_l D_l A_l = \sum_{l, B_l \notin \mathbf{Q}} c_l C_l B_l = 0$, which is enough to prove what we want.

To see that (3.4.3) does reproduce in $\mathbf{C}$ all the values prescribed algebraically on individual generators in (3.4.2), use for the type (3.1.1) the divisor $\Xi = [z+z'] + [z-z'] - 2[z] - 2[z']$ and the classical addition theorem for the Weierstrass σ-function:

$$\frac{\sigma(z+z')\sigma(z-z')}{\sigma(z)^2 \sigma(z')^2} = \wp(z) - \wp(z'),$$

and similarly, deriving this addition formula, for the other types of generators.

Note that "Siegel's function", used by Wildeshaus in Proposition 2 in *loc. cit.*, does agree with (3.4.3) for our divisors Ξ, as can be seen by going through the proof of [Silverman 1994, Thm. VI.3.4].

3.5 *Distribution relations.* The map $\varphi_2^{\#}$ satisfies the following compatibility with isogenies of elliptic curves—see [Wildeshaus 1997b] and cf. [Wildeshaus 1997a, I-5], where the term 'norm compatibility' is used: Let $\psi : E \longrightarrow E'$ be an isogeny defined over F and let $\xi = \sum_i a_i \{x_i\}_2^{\#} \in \mathcal{L}_2^{\#}$. Then $\psi(\xi) = \sum_i a_i \{\psi(x_i)\}_2^{\#}$ belongs to $\ker d_2^{\#}$ on E' if and only if one has $d_2^\#(\sum_i a_i \sum_{t\in\ker\psi}\{x_i+t\}_2^\#)=0$ on E. Furthermore, if these equivalent conditions hold, then one has for the maps $\varphi_2^\#$ on E, resp. on E', the following identity in $F^*\otimes_{\mathbf{Z}}\mathbf{Q}$:

$$\varphi_2^\#\Big(\sum_i a_i \sum_{t\in\ker\psi}\{x_i+t\}_2^\#\Big)=\varphi_2^\#\Big(\sum_i a_i\{\psi(x_i)\}_2^\#\Big). \tag{3.5.1}$$

The direct proof of this distribution relation from (3.4.3) is somewhat nontrivial—see [Jarvis 1996], cf. the appendix to [Coates 1991]. Note that, at least if ψ is multiplication by d on E, the distribution relation follows from the corresponding property of the local heights—see [Silverman 1994, ex. 6.4], for the infinite places see also [Mestre, Schappacher 1991, 2.4.2].

3.6 *Back to diagram 2.4.* We can now complete our discussion of diagram 2.4 above. First, divide by the kernel of $\varphi_2^\#$:

$$\begin{array}{llcc} d_2: & \mathcal{L}_2=\mathcal{L}_2^\#/\ker\varphi_2^\# & \longrightarrow & E(F)\otimes_{\mathbf{Z}}E(F)\otimes_{\mathbf{Z}}\mathbf{Q}, \\ \varphi_2: & \ker d_2=\frac{\ker d_2^\#}{\ker\varphi_2^\#} & \longrightarrow & F^*\otimes_{\mathbf{Z}}\mathbf{Q}. \end{array} \tag{3.6.1}$$

Elements of $\mathcal{L}_2$ will be written $\sum_j a_j\{x_j\}_2$. Then, following the usual inductive procedure of Zagier's polylogarithm conjecture, Wildeshaus defines the mapping analogous to $d_2^\#$, one level up, by

$$\begin{array}{lccc} d_3^\#: & \mathcal{L}_3^\#=\mathbf{Q}[E(F)-\{0\}] & \longrightarrow & \mathcal{L}_2\otimes_{\mathbf{Q}}(E(F)\otimes_{\mathbf{Z}}\mathbf{Q}) \\ & \{x\}_3^\# & \mapsto & \{x\}_2\ \otimes\ x\otimes 1. \end{array} \tag{3.6.2}$$

This is the definition over any fixed field of definition F of the curve E. In diagram 2.4, notations refer to the case $F=\overline{\mathbf{Q}}$.

3.6.3. As in the diagram, consider I_E^4 as included in $\mathcal{L}_3^\#$ by forgetting the 0-component. Then

$$d_3^\#(I_E^4)\subset\ker(d_2\otimes\mathrm{id}_{E(\overline{\mathbf{Q}})\otimes_{\mathbf{Z}}\mathbf{Q}})=\ker d_2\otimes_{\mathbf{Q}}(E(\overline{\mathbf{Q}})\otimes_{\mathbf{Z}}\mathbf{Q}).$$

Indeed, to see that $d_2^\#\otimes\mathrm{id}\Big(d_3^\#(\sum_{x\neq y}a_xb_y\{x-y\})\Big)=0$, if $\sum_x a_x\{x\}$ and $\sum_y b_y\{y\}$ are principal divisors, is just an exercise in regrouping terms, and adding the missing contributions for $x=y$ (which average out), once the tensors have been expanded bilinearly. That $d_3^\#$ factors through p will be a byproduct of the commutativity 3.6.5 below.

Thus all solid arrows in diagram 2.4 are defined.

3.6.4 Surjectivity of φ_2 over $\overline{\mathbf{Q}}$. *For every $\alpha\in F^*$, there exists an extension field $F\subseteq F'\subseteq\overline{\mathbf{Q}}$ of degree $[F':F]\le 4$ and an element $\xi\in\Big[(\ker d_2^\#)_{/F'}\Big]^{G(\overline{\mathbf{Q}}/F)}$ such that $\varphi_2^\#(\xi)=\alpha\otimes 1$.*

This follows from the first formula in (3.4.2). As a consequence, φ_2 is an isomorphism for $F=\overline{\mathbf{Q}}$.

3.6.5 The commutativity of diagram 2.4. *For all elliptic functions f, g with divisors $(f) = \sum_x a_x[x]$, $(g) = \sum_y b_y[y]$, putting $\xi = \sum_{x \neq y} a_x b_y \{x-y\}_3^{\#}$, we have*

$$(\varphi_2 \otimes \mathrm{id}) \circ d_3^{\#}\ (\xi) = \tilde{\mathcal{T}}(f \wedge g).$$

Proof. Any expression $f \wedge g$ can be written as a linear combination of $f_i \wedge g_i$ such that, for all i, no inclusion holds between the supports $\mathrm{supp}(f_i), \mathrm{supp}(g_i)$ (multiply f, g by functions introducing new, disjoint zeroes or poles). We are therefore reduced to checking 3.6.5 under this additional assumption on the supports of f and g. Assuming this, we may choose divisors $D(f) = \sum_k A_k[z_k] \in \mathbf{Q}[\mathbf{C}]$ and $D(g) = \sum_l B_l[z'_l] \in \mathbf{Q}[\mathbf{C}]$ such that:

(i) $D(f)$, resp. $D(g)$, projects to the divisor of f, resp. g, on $\mathbf{C}/\Gamma$,
(ii) $\sum_k A_k = \sum_k A_k z_k = \sum_l B_l = \sum_l B_l z'_l = 0$ in $\mathbf{C}$,
(iii) for all k, l, if $z_k - z'_l \in \Gamma$, then $z_k = z'_l$.

By (ii), $\Xi = \sum_{k,l} A_k B_l [z_k - z'_l]$, and therefore also $\tilde{\Xi} = \sum_{k,l; z_k \neq z'_l} A_k B_l [z_k - z'_l]$, goes to zero in $\mathbf{C} \otimes_{\mathbf{Q}} \mathbf{C}$. Then, in view of (i), (iii), $\varphi_2(\sum_{x \neq y} a_x b_y \{x-y\}_2)$ is given by formula (3.4.3) evaluated on $\tilde{\Xi}$. And more importantly: refining in a straightforward manner the arguments of the proof of 3.4, for the well-definedness and the validity of formula (3.4.3), one sees that also the image of $d_3^{\#}(\xi) = \sum_{x \neq y} a_x b_y \Big(\{x-y\}_2 \otimes (x-y)\Big)$ under $\varphi_2 \otimes \mathrm{id}$ is computed by the same theta function, using $\tilde{\Xi}$.

Furthermore, defining

$$F(z) = \prod_k \left(e^{-\frac{1}{2}(z-z_k)\eta(z-z_k)} \sigma(z-z_k) \cdot \Delta^{\frac{1}{12}}\right)^{A_k},$$
$$G(z) = \prod_l \left(e^{-\frac{1}{2}(z-z'_l)\eta(z-z'_l)} \sigma(z-z'_l) \cdot \Delta^{\frac{1}{12}}\right)^{B_l},$$

a classical computation shows (using (i) and (ii)) that F is in fact a meromorphic function on $\mathbf{C}/\Gamma$, with the same divisor as f, and likewise for G and g. It will therefore suffice to check 3.6.5 for $F \wedge G$ instead of $f \wedge g$. Expanding bilinearly

$$d_3^{\#}(\xi) = \Big(\sum_{x \neq y} a_x b_y\ \{x-y\}_2 \otimes (x-y)\Big) \in \mathcal{L}_2 \otimes_{\mathbf{Q}} (E(\overline{\mathbf{Q}}) \otimes \mathbf{Q})$$

and taking first an x in the support of f which is distinct from all the y's, we find in $(\varphi_2 \otimes \mathrm{id})\Big(d_3^{\#}(\xi)\Big)$ the contribution

$$G(x) \otimes x \otimes -\sum_k A_k,$$

with the sum extended over all k's such that $x = z_k \pmod{\Gamma}$. This is precisely the contribution at x of $\tilde{\mathcal{T}}(F \wedge G)$. Similarly, 3.6.5 works out at all points y in the support of g which do not figure among the zeroes or poles of f.

Finally, take $x \in \operatorname{supp}(f) \cap \operatorname{supp}(g)$, say, $x = z_1 \pmod{\Gamma} = z_1' \pmod{\Gamma}$. Then, by (iii), $(\varphi_2 \otimes \mathrm{id})\Big(d_3^\#(\xi)\Big)$ contains the term

$$\Big\{ \Big(\frac{F(z)}{\sigma(z_1 - z)^{A_1}}\Big)^{B_1} \Big(\frac{\sigma(z - z_1')^{B_1}}{G(z)}\Big)^{A_1} \Big\}\Big|_{z = z_1 = z_1'} \otimes x \otimes 1,$$

which again is precisely the contribution at x of $\tilde{\mathcal{T}}(F \wedge G)$—even with the correct sign $(-1)^{A_1 B_1}$ in front (which we could have slobbered working, as we do, in $\mathbf{C}^* \otimes \mathbf{Q}$), because the Weierstrass sigma function is an odd function of z.

This concludes the proof of the commutativity of diagram 2.4, as far as the solid arrows in it are concerned. Next, we turn to the dotted arrow ...

4. The elliptic polylogarithm conjecture for $K_2(E)$.

As before, E denotes an elliptic curve defined over the number field F. Recall the maps $\partial_{\mathfrak{p}}$ from (1.1) (the components of ∂), reg_v from (1.5) (for a fixed choice of differential ω, which is used to determine the lattices Γ_v attached to the elliptic curve with respect to the different embeddings of F into $\mathbf{C}$), and $B_{3,\mathfrak{p}}(x)$ from 1.6. Write $K_{2,1}^{(v)}$ for the $\mathbf{Q}$-linear extension to divisors of the function which, in the notation of 1.5, takes a point $z \bmod \Gamma_v$ on the elliptic curve to the complex number $A(\Gamma_v)^2 K_1(0, z, 2, \Gamma_v)$. Recall also the notation introduced in (3.6.1–2).

4.1 Theorem. [Elliptic Analogue of Zagier's Polylogarithm Conjecture for $K_2(E)$.] *There exists a $\mathbf{Q}$-linear map $\varphi_3^\# : \ker d_3^\# \longrightarrow K_2(E) \otimes_{\mathbf{Z}} \mathbf{Q}$ making diagram 2.4 and the following triangle commute, and satisfying the distribution relation 4.2 below.*

$$\begin{tikzcd}
\ker d_3^\# \arrow[rr, "\varphi_3^\#"] \arrow[dr, "H_{\mathbf{A}} = \bigoplus_{v \text{ inf.}} \mathrm{Re}(K_{2,1}^{(v)}) \oplus \bigoplus_{\mathfrak{p} \text{ fin.}} B_{3,\mathfrak{p}}"'] & & K_2(E) \otimes_{\mathbf{Z}} \mathbf{Q} \arrow[dl, "\bigoplus_{v \text{ inf.}} \mathrm{reg}_v \oplus \bigoplus_{\mathfrak{p} \text{ fin.}} \partial_{\mathfrak{p}} = \mathrm{reg}_{\mathbf{A}}"] \\
& \bigoplus_{v \text{ inf.}} \mathbf{R} \oplus \bigoplus_{v \text{ fin.}} \mathbf{Q} &
\end{tikzcd}$$

4.2 Distribution relation. Let $\psi : E \longrightarrow E'$ be an isogeny defined over F and let $\xi = \sum_i a_i \{x_i\}_3^\# \in \mathcal{L}_3^\#$. Then $\psi(\xi) = \sum_i a_i \{\psi(x_i)\}_3^\# \in \ker d_3^\#$ on E' if and only if one has $d_3^\#(\sum_i a_i \sum_{T \in \ker\psi} \{P_i + T\}_3^\#) = 0$ on E. (This follows from 3.5.) Furthermore, if these equivalent conditions hold, then one has for the maps $\varphi_3^\#$ on E, resp. on E', that

$$\psi_* \circ \varphi_3^\# \Big(\sum_i a_i \sum_{t \in \ker\psi} \{x_i + t\}_3^\#\Big) = \varphi_3^\# \Big(\sum_i a_i \{\psi(x_i)\}_3^\#\Big),$$

where $\psi_* : K_2(E) \otimes \mathbf{Q} \longrightarrow K_2(E') \otimes \mathbf{Q}$ is the Gysin map of ψ. For instance, if ψ is multiplication by d on E, then ψ_* also multiplies the elements of $K_2(E)$ by d.

4.3 Construction of $\varphi_3^\#$. Recall from (3.6.2) that $\ker d_3^\#$ is a subspace of divisors *supported on F-rational points of E*. Denote temporarily by $\mathcal{D}$ the $G(\overline{\mathbf{Q}}/F)$-invariants of the space $(\ker d_3^\#)_{\overline{\mathbf{Q}}}$, taken relative to the ground field $\overline{\mathbf{Q}}$. Then, for any given $\xi \in \ker d_3^\#$, we can find a $\mathbf{Q}$-linear combination $D \in \mathcal{D}$ of divisors of the form $\{x\}_3^\# - d\sum_{d\cdot y=x}\{y\}_3^\#$ (say, with $1 \leq d \leq 7$), such that $\xi - D$ is a $G(\overline{\mathbf{Q}}/F)$-invariant element of the space I_E^4 considered in diagram 2.4. This follows immediately from the fact that I_E^4 is the space of divisors on E which go to zero under the three $\mathbf{Q}$-linear maps $\mathbf{Q}[E(\overline{\mathbf{Q}}) - \{0\}] \longrightarrow E(\overline{\mathbf{Q}})^{\otimes i} \otimes_{\mathbf{Z}} \mathbf{Q}$ given by $x \mapsto x^{\otimes i}$, for $i = 1, 2, 3$. The fact that D goes to zero under $d_3^\#$ follows from 3.5.

Now, map $\xi - D$ up in diagram 2.4 to $p(\xi - D) \in \frac{I_E^4}{\beta\langle f\wedge(1-f)\rangle_{\mathbf{Q}}}$. This element is independent of our choice of D, because any linear combination of divisors of the form $\{x\}_3^\# - d\sum_{d\cdot y=x}\{y\}_3^\#$ that lies in I_E^4 already belongs to $\langle f \wedge (1-f)\rangle_{\mathbf{Q}}$—see lemma 3.20 in [Gončarov, Levin 1997].

Since both ξ and $\mathcal{D}$ belong to the kernel of $d_3^\#$, the commutativity of 2.4 shows that any element of $\beta^{-1}\Big(p(\xi - \mathcal{D})\Big)$ has trivial tame symbol, and therefore lies in $K_2(E/\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}$. Since diagram 2.4 is clearly Galois-equivariant, and our elements are invariant under $G(\overline{\mathbf{Q}}/F)$, they lie in $(K_2(E/\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q})^{G(\overline{\mathbf{Q}}/F)} = K_2(E/F) \otimes_{\mathbf{Z}} \mathbf{Q}$.

Thus, choosing any $\mathbf{Q}$-linear section of β defines some mapping $\varphi_3^\#$ which makes the diagram 2.4 commute, and satisfies 4.2 by what was said in the preceding construction. Any such map also renders the triangle in 4.1 commutative, as we have already remarked in (1.5) and 1.6 above. This completes the proof of the theorem. But to be sure, it does not answer all our questions. Specifically, it does not say anything about the

4.4 Uniqueness of $\varphi_3^\#$. If one believes Beilinson's conjecture 1.7.(i), then the commutativity of the triangle in theorem 4.1 determines $\varphi_3^\#$ uniquely. This conjecture seems inaccessible at the moment. However, refining the analysis of a commutative diagram like 2.4 above, Gončarov and Levin have recently succeeded to prove (even over an arbitrary algebraically closed field) that one has the following sequence, which is exact modulo 2-torsion [Gončarov, Levin 1997, theorem 1.5]:

$$\mathrm{Tor}(\overline{\mathbf{Q}}^*, E(\overline{\mathbf{Q}})) \hookrightarrow \frac{H^0(E,\mathcal{K}_2)}{K_2(\overline{\mathbf{Q}})} \longrightarrow \frac{I^4_{E,\mathbf{Z}}}{\beta\langle f\wedge(1-f)\rangle_{\mathbf{Z}}} \longrightarrow \overline{\mathbf{Q}}^* \otimes_{\mathbf{Z}} E(\overline{\mathbf{Q}}) \longrightarrow \ker\Big(H^1(E,\mathcal{K}_2) \to \overline{\mathbf{Q}}^*\Big) \longrightarrow 0$$

Here $I_{E,\mathbf{Z}}$ is the augmentation of the integral group ring $\mathbf{Z}[E(\overline{\mathbf{Q}})]$, and $\mathcal{K}_2$ is the K_2-sheaf. Tensoring all groups with $\mathbf{Q}$ turns the second term simply into $K_2(\overline{\mathbf{Q}}) \otimes_{\mathbf{Z}} \mathbf{Q}$, and kills the first term of the sequence, thus showing that no choice of splitting of β was needed for our construction …

5. Examples in a family.

Now that we have at our disposal theorem 4.1, which we view in the first place as a systematic way to produce elements of $K_2(E) \otimes \mathbf{Q}$ from certain divisors, we want to illustrate the result in a one-parameter family of elliptic curves which was written down by Jan Nekovář in 1990, following an idea of the second author. The first numerical experiments in this family were performed in the Spring of 1990 by D. Grayson and the second author. This was then later taken up by the first author in his thesis.

Similar numerical computations, for individual elliptic curves, were performed in 1991 by Don Zagier and Henri Cohen, with an intuition guided closely by the formalism of Zagier's polylogarithm conjecture. They saw in particular that local heights impose conditions on the divisors that may create elements in K_2—see section 3 above. Beilinson,987Gončarov, and Levin learned about this. Beilinson and Levin supplied the general motivic theory of elliptic polylogarithms, i.e., Kronecker double series, in [Beilinson, Levin 1994], which was complemented by explicit constructions in [Levin 1994a,b]. And in 1995, Gončarov and Levin, with the initial version of [Gončarov, Levin 1997], gave their first treatment of the elliptic polylogarithm conjecture in the case of K_2.

In contrast to this experimental prehistory of the work of Gončarov and Levin, Wildeshaus came up with the first version of [Wildeshaus 1997b] early in 1995 because he was seeking a motivically inspired elliptic analogue of Zagier's polylogarithm conjecture, generalizing [Beilinson, Deligne 1994] directly. There the rôle of the local heights in the preliminary case of $K_1(F)$ was only recognized by R. de Jeu in subsequent discussions.

5.1 An elementary construction. Let E be an elliptic curves over $\mathbf{Q}$, and assume that $P \in E(\mathbf{Q})$ is a rational point of infinite order. Let $f \in \mathbf{Q}(E)^*$ be a function with divisor $(f) = 2(P) + (-2P) - 3(0)$—take for instance the equation of the tangent to E at P. Assume that we can find a function $g \in \mathbf{Q}(E)^*$ such that, for some $M \geq 2$,

(i) the divisor (g) is supported in E_M;

(ii) $g(P) = -g(-2P) \neq 0$.

For every point Q in the support of (g), let F_Q denote the field generated over $\mathbf{Q}$ by the coordinates of Q, and let F be the composite of these fields F_Q. Then (i) insures that there is a function $\chi_Q \in F_Q(E)^*$ such that $(\chi_Q) = M(Q) - M(O)$.

We find for the tame symbol $\mathcal{T}$ of (2.1):

$$\textbf{(5.1.1)} \qquad \mathcal{T}\Big(M\{f, g\}\Big) = \mathcal{T}\Big(M\{f, g(-2P)\} + \sum_Q \{f(Q)^{v_Q(g)}, \chi_Q\}\Big).$$

Due to Galois descent in $K_2(F(E)) \otimes \mathbf{Q}$, the Galois invariant sum on the right, like every other term in the identity, defines an element of $K_2(\mathbf{Q}(E)) \otimes \mathbf{Q}$. Therefore, correcting $\{f, g\}$ by symbols with a constant component, it comes from an element of $K_2(E) \otimes \mathbf{Q}$.

5.2 Nekovář's family. Take the following special case of 5.1. For $a \in \mathbf{Q}$ with $a \neq -\frac{1}{3}$, let $E_a : (3a+1)\, y^2 = x^3 - (3a^2+1)\, x + (2a^3+a+1)$, and $P_a = (a+1, 1)$. P_a is a point of infinite order except for $a = \frac{1}{3}$, in which case $E_a = X_1(11)$ and P_a is of order 5. Henceforth we assume that $a \neq \pm\frac{1}{3}$. Then $-2P_a = (a-1,-1)$, and we consider the symbol $\{f_a, g_a\}$, where $f_a = y - x + a$, and $g_a = g = y$, so that $M = 2$. By 5.1, $\{f_a, g_a\}$ defines a element of $K_2(E_a) \otimes \mathbf{Q}$. The divisors of our functions are :

$$(f_a) = 2(P_a) + (-2P_a) - 3(0), \qquad (g) = -3(0) + \sum_{i=1}^{3}(Q_i),$$

where $Q_i = (e_i, 0)$ are the 2-torsion points on E_a.

Let $\mathbf{Z} + \tau_a\mathbf{Z}$, $\mathrm{Im}(\tau_a) > 0$, be the lattice corresponding to $(E_a, \frac{\omega_a}{\Omega_a})$, where ω_a is a non zero rational differential on E_a with real period $\Omega_a = |\int_{E_a(\mathbf{R})^0} \omega_a|$. Furthermore, let $z_a \in \mathbf{C}$ correspond to the point P_a under the analytic parametrization $\mathbf{C}/\mathbf{Z} + \tau_a\mathbf{Z} \to E_a$. And let t_i $(i = 1,2,3)$ correspond to the 2-torsion points Q_i. Then the regulator $\mathrm{reg}_\omega(\{f_a, g\})$ of any element of $K_2(E_a) \otimes \mathbf{Q}$ mapping to $\{f_a, g\} \in K_2(\mathbf{Q}(E_a)) \otimes \mathbf{Q}$ equals, in the notation of 4.1:

$$(\mathbf{5.2.1}) \quad \frac{\mathrm{Im}(\tau_a)^2}{\pi^2}\Big(-3\,(2K_{2,1}(z_a) + K_{2,1}(-2z)) + \sum_{i=1}^{3} 2\,K_{2,1}(z_a - t_i) + K_{2,1}(-2z_a - t_i)\Big).$$

By standard properties of the Kronecker series, this transforms into

$$(\mathbf{5.2.2}) \qquad \frac{\mathrm{Im}(\tau_a)^2}{\pi^2}\Big(-8\,K_{2,1}(z_a) + 5K_{2,1}(2z_a)) - \frac{1}{2}K_{2,1}(4z_a)\Big).$$

In order to test Beilinson's conjectures 1.7, the first author computed the expression

$$(\mathbf{5.2.3}) \qquad \frac{\pi^2\ \mathrm{reg}_\omega(\{f_a, g\})}{N_a\ L(E_a, 2)},$$

where N_a is the conductor of the curve E_a, for small values of a using the software package PARI. In fact, in doing so one passes to a minimal model of the curve E_a, and the point on it corresponding to P_a. Beilinson's conjectures (1.7)—together with the functional equation for the L-function $L(E_a, s)$ (which is conjectural in that one cannot prove yet that all elliptic curves in Nekovář's family are modular)—predict that the quantity (5.2.3) is a rational number whenever $\partial_p(\{f_a, g\}) = 0$ at all primes p where E_a acquires split multiplicative reduction—see 1.1. This integrality obstruction is computed explicitly, using [Schappacher, Scholl 1991, §3], in [Rolshausen 1996, IV.3]. It turns out that

$$(\mathbf{5.2.4}) \qquad \forall p \quad \partial_p(\{f_a, g\}) = 0 \qquad \Leftrightarrow \qquad 12a \in \mathbf{Z}.$$

5.3 Relation to $\varphi_3^\#$. In terms of the map $\varphi_3^\# : \mathbf{Q}[E(\mathbf{Q}) - \{0\}] \longrightarrow K_2(E) \otimes \mathbf{Q}$ constructed in section 4 above, the element $-2\{f_a, g\}$ of $K_2(E_a) \otimes \mathbf{Q}$ is given—at least modulo the kernel of the regulator—by the following divisor, suggested by (5.2.2):

$$\xi_a = 16\{P_a\}_3^\# - 10\{2P_a\}_3^\# + \{4P_a\}_3^\#.$$

It is easy to see that $\xi_a \in \ker d_3^{\#}$. The tables in [Rolshausen 1996, chap. VI] show about one hundred curves E_a, for all $a \in \frac{1}{12}\mathbf{Z}$ with $-16 \le a \le +16$, where $\varphi_3^{\#}(\xi_a)$ is integral. In each of these cases, the quantity (5.2.2) is apparently a nonzero rational number.

For values of a where $\varphi_3^{\#}(\xi_a)$ is not integral, (5.2.2) seemed irrational (as far as any number does in the computer). But in these cases, it is more interesting to see whether theorem 4.1 allows us to produce other elements in $K_2(E_a)$ which are integral, and compare their regulator to $L(E_a, 2)$. The search for such elements is greatly facilitated by the fact that we could define $\varphi_3^{\#}$ on all of $\ker d_3^{\#}$, not just on divisors in I_E^4. Let us illustrate this with just one example—for more material, see [Rolshausen 1996, VI.2].

5.4 The example $a = \frac{1}{7}$. The minimal equation of $E = E_{1/7}$ over $\mathbf{Z}$ is $y^2 + y = x^3 - 325x + 6156$. The conductor is $N = N_{1/7} = 4025$. Two points are easy to find on the minimal model: the standard point $P = P_{1/7} = (20, 87)$, and another one, $Q = (45, 287)$. By (5.2.4), $\varphi_3^{\#}\left(16\{P\}_3^{\#} - 10\{2P\}_3^{\#} + \{4P\}_3^{\#}\right)$ does not lie in the integral part of $K_2(E) \otimes \mathbf{Q}$. But an easy search gives the element

$$\xi' = -11\{Q\}_3^{\#} - 2\{2Q\}_3^{\#} + \{3Q\}_3^{\#} \in \ker d_3^{\#}$$

which does pass the integrality test via (1.6.2) at $p = 7$, which is the only prime where split multiplicative reduction occurs for E. Note that ξ' adds up to $-12Q$ in E, so this element of $\ker d_3^{\#}$ does not itself belong to I_E^4. A search for a corresponding element in I_E^4 would have taken longer. Numerically one finds that

$$\frac{\pi^2 \ \mathrm{reg}_\omega\left(\varphi_3^{\#}(\xi')\right)}{N_a \ L(E_a, 2)} = \frac{1}{96}.$$

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