ON REGULAR GROUPS OF AUTOMORPHISMS OF TRIVALENT POLYGONAL COMPLEXES WITH NON-POSITIVE CURVATURE

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Polygonal complex is a 2–dimensional polyhedral cell complex which is homogeneous, i.e. each its cell is contained in a 2–cell. We shall refer to 0–, 1– and 2–cells as vertices, edges and cells respectively. Polygonal complex X is *trivalent* if each its edge is contained in exactly three cells.

A group Γ of automorphisms of a polygonal complex X is *regular* if it acts transitively on *flags* in X, i.e. on incident triples $(vertex, edge, cell)$. Polygonal complex X which admits a regular group Γ of automorhisms is called itself *regular*. Of course, if X is regular, then the group $Aut\, X$ of all its automorphisms is regular, since it contains Γ.

Simply connected regular polygonal complexes with nonpositive curvature (see (1.2) below) are the 2–dimensional geometric analogs of regular trees; we will call them shortly *regular 2-trees*. The class of regular 2–trees is very rich, and the classification of the trivalent case has been obtained in [Poly]. The classification of regular 2–trees with higher valency is, according to our knowledge, an open problem.

The present paper is motivated by the result of D. Djokovič and G. Miller (see [DM]) – classification of a class of regular (i.e acting transitively on the set of all oriented edges) groups of automorphisms of the trivalent tree. We study regular groups of automorphisms of trivalent 2–trees, and give a method to construct examples of such groups which are proper subgroups in full automorphism groups. In many cases the full automorphism group of a regular polygonal complex is uncountable, but a subgroup obtained by our method is discrete.

Here is a short description of our method.

Given a vertex v in a polygonal complex X, the *link* of X at v, denoted $L(v, X)$, is a graph whose vertices and edges represent respectively the edges and cells adjacent to v in X, and a relation of being a face is induced from that in X.

If X is a regular polygonal complex, then there exist a natural number $k \geq 3$, and a trivalent regular graph L, such that

(i) each cell of X is a k–gon, i.e. its boundary consists of k vertices and k edges;

(ii) links of X at all vertices are isomorphic to L.

We will call any simply connected polygonal complex satisfying (i) and (ii) a (k, L)–*complex.*

Given a regular subgrup $G \subset Aut\, L$ we define an additional structure on a (k, L)–complex, which we call G–structure. We define also a notion of G–isomorphism, i.e. combinatorial isomorphism respecting G–structures.

Each regular group of automorphisms of a (k, L)–complex X induces canonically a

G–structure on X, with some regular $G \subset Aut\, L$. In our approach we take the opposite direction: we construct and classify regular groups of automorphisms of non–positively curved (k, L)–complexes, which arise as groups of G–automorphisms of G–structures.

The idea behind our formalism is to view a regular graph L equipped with a regular subgroup $G \subset Aut\, L$ as a "less symmetric" regular graph. It turns out that the methods developed in [Poly] apply to the study of regular polygonal complexes with such "reduced" graphs as links, and these are nothing else than (k, L)–complexes with regular G–structures. On the other hand, the results of [Poly] are contained in those of this paper as a case when $G = Aut\, L$.

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0. Trivalent graphs with regular groups of automorphisms.

0.1. Definitions and general results.

A graph L is 3–*valent*, if its each vertex is contained in exactly three edges. A group G of automorphisms of L is *regular*, if it acts transitively on the set of all incident pairs $(vertex, edge)$ in L. If L admits a regular group of automorphisms, we say that L itself is regular. Of course, if L is regular, then the group $Aut\, L$ of all its automorphisms is regular.

Given a natural number s, an s–*arc* in a graph L is a sequence $(v_0, v_1, \ldots, v_s)$ of its vertices, such that

(i) (v_i, v_{i+1}) is an edge of L, for $i = 0, \ldots, s-1$;

(ii) $v_i \neq v_{i+2}$ for $i = 0, \ldots, s-2$.

Any group G of automorphisms of L clearly acts on the set of all s–arcs in L, for any s.

Definition. A group G of automorphisms of L is called s–*arc–transitive*, if it acts transitively on the set of all s–arcs in L. It is s–*regular*, if it acts simply transitively on s–arcs. Graph L is s–*regular*, if so is the group $Aut\, L$ of all its automorphisms.

In the paper [T] W. Tutte initiated the study of regular groups of automorphisms of trivalent graphs by proving the following.

Theorem [Tutte, 1947]. Let L be a finite, connected, trivalent and regular graph. Then L is s–regular for some $s \in \{1, 2, 3, 4, 5\}$.

The study culminated in the work of D. Djokovič and G. Miller [D–M], where they indicated seven classes of s–regularity, according as $s \in \{1, 2', 2'', 3, 4', 4'', 5\}$. In both cases of $s = 2$ or $s = 4$ the s'– and s''–regularity are the subcases of s–regularity, distinguished by means of edge stabilizers. The main result of [D–M] is the following.

Theorem [Djokovič–Miller, 1980].
Let L be a finite, connected and trivalent graph, and G a regular group of its automorphisms. Then G is s–regular for some $s \in \{1, 2', 2'', 3, 4', 4'', 5\}$.

Moreover, Theorem 3 on page 211 of [D–M] shows the possible regularity types of pairs (G, G') of a regular group G and its regular subgroup G'.

0.2. Some examples.

The main source of examples for us is "The Foster Cenzus" [F] of trivalent regular graphs. For the reader's convenience we mention here several examples.

Many regular trivalent graphs correspond to 1–skeletons of regular tesseations of surfaces by polygons. A tesselation is called *regular* if the group of its combinatorial symmetries acts transitively on the set of all *flags*, i.e. incident triples $(vertex, edge, polygon)$, of this tesselation. The most obvious examples are the tesselations of the sphere corresponding to the boundaries of platonic solids: tetrahedron, cube and dodecahedron. The corresponding 1–skeletons are 2–regular graphs.

We mention several non–spherical examples.

1. The Petersen graph of girth 5 and 10 vertices, which is 3–regular, corresponds to a tesselation of the projective plane by 6 pentagons. This tesselation can be obtained from dodecahedron by the antipodal identification.

2. The incidence graph of the Desargues configuration, which is 3–regular, consists of 20 vertices and has girth 6, corresponds to a tesselation with six 10–gons of the non–orientable surface with $\chi = -4$. This tesselation can be obtained from the previous tesselation of the projective plane by the 2–fold cover branched over barycenters of all six pentagonal cells.

3. The full bipartite graph $K(3,3)$, which is 3–regular, corresponds to a tesselation of the torus by three hexagons.

4. The Möbius–Kantor graph of girth 6 and 16 vertices, which is 2–regular, corresponds to a tesselation of the genus 2 oriented surface by six octagons. This tesselation is obtained from the cube by the 2–fold cover branched over the barycenters of all six square faces.

5. There is also a tesselation of the torus by 7 hexagons, the 1–skeleton of which is isomorphic to the 4–regular Heawood graph having 14 vertices and girth 6. (This graph can be also described as the incidence graph of the projective plane over the field F_2 of order 2; it is also a spherical building of type A_2.) However, the symmetry group of this tesselation is transitive only on flags with given orientation, i.e. it does not contain orientation reversing mappings of the torus.

For any graph represented as a 1–skeleton of a tesselation, each symmetry of this tesselation induces an automorphism of the graph. If a tesselated surface is orientable, and its tesselation regular, then the group G of all symmetries of this tesselation, and the group G^+ of all orientation–preserving symmetries, induce the groups of automorphisms of the graph, which are $2'$– and 1–regular respectively. The 1–regular subgroup appears also in case of the Heawood graph (example 5 above), as induced from the full symmetry group of the tesselation corresponding to this graph.

If a regularly tesselated surface is non–orientable, we get by the same procedure a $2'$–regular group of automorphisms of the corresponding graphs.

In many cases the groups described above are the proper subgroups in full automorphism groups of corresponding graphs.

1. G–structures, G–isomorphisms, and the Main Theorem.

Consider a trivalent graph L, an s–regular group $G \subset Aut\, L$ of its automorphisms and a (k, L)–complex X (see definitions in the introduction and in section 0). Given a vertex v of X, a collection A_v of isomorphisms $\gamma : L(v, X) \to L$ will be called a *G–atlas at v*, if for any two maps $\gamma, \gamma' \in A_v$, we have $\gamma' \circ \gamma^{-1} \in G$, and A_v is a maximal collection with this property. Note that a G–atlas A_v is in fact determined by any its element γ by $A_v = \{g \circ \gamma : g \in G\}$. Note also, that the number of distinct G–atlasses at v corresponds to the index of G in $Aut\, L$. The maps of any G–atlas will be called *charts*.

A *G–structure* on X, is a set $\mathcal{G} = \{A_v : v \in X^{(0)}\}$ of G–atlasses, one for each vertex v of X. A (k, L)–complex X equipped with a G–structure $\mathcal{G}$ will be called a *(k, L, G)–complex*.

Let $T : X \to Y$ be a combinatorial isomorphism of (k, L)–complexes. Denote by $T_v : L(v, X) \to L(T(v), Y)$ the induced isomorphism of links. If the complexes are equipped with G–structures $\mathcal{G} = \{A_v : v \in X^{(0)}\}$ and $\mathcal{G}' = \{A'_w : w \in Y^{(0)}\}$, then T is said to *respect* G–structures $\mathcal{G}$ and $\mathcal{G}'$ (or, it is a *G–isomorphism*), if for any vertex v of X, and any charts $\gamma \in A_v$ and $\gamma' \in A'_{T(v)}$, we have

$$\gamma' \circ T_v \circ \gamma^{-1} \in G. \tag{1.1}$$

Note that if (1.1) holds for some $\gamma \in A_v$ and $\gamma' \in A'_{T(v)}$, then it holds for any such γ and γ'.

We denote the group of all G–automorphisms of a complex X with a G–structure $\mathcal{G}$ by $Aut(X, \mathcal{G})$. A (k, L, G)–complex $(X, \mathcal{G})$ is said to be *flag–symmetric*, if the group $Aut(X, \mathcal{G})$ is flag–transitive on X. Note, that if X is a flag–symmetric (k, L)–complex, and G is a proper subgroup of $Aut\, L$, then $Aut(X, \mathcal{G})$ is a proper subgroup of $Aut\, X$, becouse its elements induce less isomorphisms on links. On the other hand, in the trivial case of $G = Aut\, L$, there is only one G–structure $\mathcal{G}$ on X, which in fact gives no additional structure, and we have $Aut(X, \mathcal{G}) = Aut\, X$.

We say that a group Γ of automorphisms of a (k, L)–complex X is *rigid*, if its elements are determined by restriction to the star of any vertex (called also a 1–*ball* in X); it is *weakly rigid*, if it is not rigid, but its elements are determined by restriction the star of star of any vertex (we will call such a piece a 2–*ball* in X). If a group Γ is vertex–transitive, then it follows from its rigidity (or weak rigidity) that the stabilisers of vertices (which are all isomorphic) are finite, the group is finitely generated, and it acts properly discontinuously and cocompactly on X.

A group Γ is *flexible*, if for any finite subcomplex $P \subset X$ there exists an automorphism F in Γ such that $F|_P = id_P$ and $F \neq id_X$. In such a case stabilisers of vertices are uncountable, and the compact–open topology on Γ is nondiscrete.

1.1. Main Theorem.

Let L be a finite connected regular and trivalent graph, $k \geq 3$ a natural number, and assume that the following non–positive curvature condition is satisfied (compare Remark

0.5 in [Poly]):

$$g(L) \geq \frac{2k}{k-1}, \tag{1.2}$$

where $g(L)$ is the length of shortest nontrivial circuit in L. Moreover, let $G \subset Aut\, L$ be an s–arc–regular-subgroup of automorphisms of graph L.

In the following statements (k, L, G)–complexes are considered to be equal, if they are G–isomorphic.

(1) If G is s–regular with $s \in \{3, 4', 4'', 5\}$, and if $k \geq 4$, then there exists a unique (k, L, G)–complex $(X, \mathcal{G})$, it is flag–symmetric, and the group $Aut(X, \mathcal{G})$ is flexible.

(2) If G is 1–regular, then there are two distinct flag–symmetric (k, L, G)–complexes, and in each case the group $Aut(X, \mathcal{G})$ is rigid.

(3) If one of the following conditions is satisfied:

(i) G is $2'$–regular;

(ii) G is $2''$–regular, and k is even;

then there are two distinct flag–symmetric (k, L, G)–complexes, and in each case the group $Aut(X, \mathcal{G})$ is rigid.

(4) If G is $2''$–regular, and k is odd, then no (k, L, G)–complex is flag–symmetric.

(5) If G is 3–regular, and $k = 3$, then there are as many distinct (k, L, G)–complexes as G–invariant elements in the cohomology group $H^1(L, Z_2)$. The group $Aut(X, \mathcal{G})$ is in each such case weakly rigid.

(6) If G is s–regular with $s \in \{4', 4'', 5\}$, and $k = 3$, then there are two distinct flag–symmetric (k, L, G)–complexes, and in each case the group $Aut(X, \mathcal{G})$ is flexible.

1.2. Remark on chart changes in L.

Let $g : L \to L$ be an element of G. For any maps $f : Y \to L$, $h : L \to Y$ and $\varphi : L \to L$, consider the maps $f' = g \circ f$, $h' = h \circ g^{-1}$ and $\varphi' = g \circ \varphi \circ g^{-1}$. Then we say that f', h' and φ' are the results of a *chart change* in L, or that they are *equal* to f, h and φ respectively, up to a chart change in L. It is important that an isomorphism g used to define a chart change in L belongs to the subgroup G; the similar operations with use of isomorphisms which are not in G are not chart changes.

Note the following facts, the proofs of which come straightforward from the definition of a chart change.

1. The subgroup G itself does not depend on a chart change in L.

2. Since any G–atlas consists of isomorphisms obtained from a fixed one by all chart changes in L, it clearly does not depend on a chart change in L. Similarly, the notions of G–structure and G–isomorphism are independent on chart changes in L.

2. Inductive construction of a (k, L, G)–complex.

In this section we recall briefly the general construction of a (k, L)–complex of non-positive curvature, as presented in section 3 of [Poly]. We indicate the additional feature of this construction, so that it produces not only a complex, but also a G–structure on it.

2.1. Definition of a (k, L)–complex with convex boundary.

Let K be a 2-dimensional cell complex, with all 2-cells k–gonal. A vertex w of K is called 1–*free*, if the link $L(w, K)$ is a single edge; it is 2–*free*, if $L(w, K)$ is isomorphic to a star of vertex in a 3–valent tree, i.e. it consists of three edges adjacent with one endpoint to a common vertex; it is 3–*free*, if $L(w, K)$ is isomorphic to a star of edge in a 3–valent tree. An edge of K is called *free*, if it is contained in exactly one cell of K, and it is *interior*, if it is contained in three cells of K. A vertex w is *interior*, if the link $L(w, K)$ is isomorphic to L.

K is a *(k, L)-complex with convex boundary*, if each its vertex is either interior or m–free, for some $m \in \{1, 2\}$ if $k \geq 4$, and for some $m \in \{1, 2, 3\}$ if $k = 3$. Note that then each edge of K is either interior or free, and denote by ∂K the subcomplex of K consisting of all free edges, and call it the *boundary* of K.

A G–structure on a (k, L)–complex K with convex buondary, is a collection $\mathcal{G}_K$ of G–atlasses at interior vertices of K. A combinatorial isomorphism between two such complexes is a G–isomorphism, if it satisfies (1.1) for all interior vertices of the first complex.

2.2. Initial step of the construction.

Arrange a disjoint collection of k–gonal cell, labelled by edges of the graph L, around an initial vertex v, by glueing them along edges according to the pattern provided by L, so that the link of resulting complex at v is isomorphic to L, and only the edges which are adjacent to v have been glued. We will denote the complex obtained in this way by B_1, since it corresponds to a 1–ball in a (k, L)–complex. Note that the labelling of cells of B_1 provides the isomorphism $\gamma : L(v, B_1) \to L$. We take the G–atlas $A = \{g \circ \gamma : g \in G\}$ at v, getting a G–structure on B_1.

2.3. General inductive step.

Note that the complex B_1 constructed above is a (k, L)–complex with convex boundary, and therefore we can apply everything that follows below in this subsection to it, thus initiating a process of an inductive construction.

Let K be a finite (k, L)–complex with convex boundary, and assume $\partial K \neq \emptyset$. Denote by $\widetilde{K}$ the complex obtained from K by glueing two new k–gonal cells to each its free edge, without performing any other glueings. For each free (i.e. not interior) vertex w of K consider a *label map* $\lambda_w : L(w, \widetilde{K}) \to L$, which is by definition locally injective, i.e. injective on star of each vertex in $L(w, \widetilde{K})$. Denoting by $V_\partial K$ the set of the boundary vertices of K,call a collection $\Lambda = \{\lambda_w : w \in V_\partial K\}$ of such label maps a *label system* for K.

Remark. Note that under non–positive curvature condition (1.2), the only cases when a label map λ_w is not injective (globally) are:
(i) w is 2–free and $g(L) \leq 4$;
(ii) w is 1–free and $g(L) = 3$;
since 3–free vertices w, by convexity, appear only when $k = 3$, but then $g(L) \geq 6$ by (1.2), which implies injectivity of λ_w.

In any case, we have

$$\lambda_w|_{L(w,K)} \text{ is injective.} \tag{2.1}$$

For each free vertex w of K consider a copy B_1^w of a complex isomorphic to B_1, together with a chart $\gamma_w : L(v_w, B_1^w) \to L$ defining a G–structure on it (where v_w is the interior, i.e. central vertex of B_1^w). Consider the composition map $(\gamma_w)^{-1} \circ \lambda_w : L(w, \widetilde{K}) \to L(v_w, B_1^w)$, and denote by $\psi_w : st(w, \widetilde{K}) \to B_1^w$ the naturally induced map on stars of the corresponding vertices. Define a complex $\overline{K} = \overline{K}(\Lambda)$ by

$$\overline{K} = \widetilde{K} \cup_{\psi_{w_1}} B_1^{w_1} \cup \ldots \cup_{\psi_{w_r}} B_1^{w_r}, \tag{2.2}$$

where $w_1, \ldots, w_r$ are all the vertices of ∂K.

Note that, due to (2.1), K is naturally embedded in $\overline{K}$. Moreover, as it is proved in 3.6 of [Poly], $\overline{K}$ is again a finite (k, L)–complex with nonempty convex boundary.

Assume we have a G–structure $\mathcal{G}_K$ on K, i.e. a collection of G–atlasses at interior vertices of K. We shall extend $\mathcal{G}_K$ to a G–structure in $\overline{K}$, in a canonical way. Note that the complexes $B_1^{w_i}$ are naturally embedded in $\overline{K}$, since a target space of a glueing map always embeds into a composition space obtained by a glueing. Thus, we can think of charts γ_{w_i} of those balls as of charts $\gamma_{w_i} : L(w_i, \overline{K}) \to L$. These determine G–atlasses $A_{w_i} = \{g \circ \gamma_{w_i} : g \in G\}$ at vertices w_i, and we define

$$\mathcal{G}_{\overline{K}} = \mathcal{G}_K \cup \{A_{w_i} : w_i \in V_\partial K\} \tag{2.3}$$

Since the only new interior vertices of $\overline{K}$ are those contained in the boundary of K, $\mathcal{G}_{\overline{K}}$ is a G–atlas for $\overline{K}$.

2.4. Remark. Note that, denoting by $\kappa_w : L(w, \widetilde{K}) \to L(w, \overline{K})$ the map induced on links by the canonical map $\widetilde{K} \to \overline{K}$, we have $\gamma_w \circ \kappa_w = \lambda_w$. This means that our distinguished charts γ_w coincide with the label maps λ_w on the parts of links corresponding to $\widetilde{K}$. We shall use this fact later.

To summarize, we construct a (k, L, G)–complex by an infinite sequence of succesive steps from K to $\overline{K}$, starting with $K = B_1$. The initial chart for B_1, together with the sequence of label systems used at all steps, determine uniquely the G–structure $\mathcal{G} = \bigcup_K \mathcal{G}_K$ on the resulting (k, L)–complex X.

2.5. Lemma. A (k, L, G)–complex constructed as above does not depend, up to G-isomorphism, on chart changes in L, for the label maps used at all steps of the construction.

More precisely, if Λ' is a label system obtained by chart changes in L from a system Λ, then the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ are G–isomorphic by an isomorphism extending the identity on K.

The proof of above Lemma consists of straightforward manipulations with chart changes, and we omit it.

2.6. Remark. Note that any (k, L, G)-complex $(X, \mathcal{G})$ can be obtain by means of the general construction above, if the label maps at the corresponding steps of the construction are chosen appropriately.

To see this, consider a subcomplex K in X, with convex boundary, and a G–structure $\mathcal{G}_K$ on it, restricted from the one on X. We shall interpret the star $st(K, X)$ of K in X, as a complex $\overline{K}(\Lambda)$ obtained from K by procedure described in 3.3, with use of some label system Λ.

Recall that the star $st(K, X)$ is the subcomplex of X consisting of all cells of K, and cells adjacent to the boundary vertices of K. We then have the obvious map $\mu : \widetilde{K} \to st(K, X)$, extending id_K, well defined up to transpositions at pairs of cells not contained in K, and adjacent to its boundary edges. For $w \in V_\partial K$, let $\mu_w : L(w, \widetilde{K}) \to L(w, st(K, X))$ be the induced map on links. Consider any chart γ_w from the G–atlas at w for the G–structure $\mathcal{G}$ on X, and put

$$\lambda_w = \gamma_w \circ \mu_w, \tag{2.4}$$

thus getting the label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$. A standart checking shows that the complexes $st(K, X)$ and $\overline{K}(\Lambda)$ are then canonically G–isomorphic by the G–isomorphism extending id_K.

Remark 2.6 follows by applying above interpretation to n–balls in X, i.e. to the subcomplexes of X obtained by the succesive iteration of the operation of taking star, first applying it to a vertex in X.

3. Order systems and characteristic functions.

As it is explained in Remark 2.6 of the previous section, each (k, L, G)–complex can be obtained by a variant of the inductive construction there described. The choices of label systems provide freedom in the construction, allowing non–G–isomorphic complexes to appear. In this section we introduce characteristic functions of label systems – the notion usefull in the study of the G–isomorphism question for resulting (k, L, G)–complexes.

3.1. Definitions of order systems in L.

Let p be a vertex, and e an edge of L.

3.1.1. Let x, y be the vertices of edge e. Denote by O_x and O_y the pairs of edges in L, distinct from e, adjacent to x and y respectively, with distinguished order for each pair. We will call them *ordered pairs of peripheral edges at edge* e, and the whole system (e, O_x, O_y) an *order–system at* e.

3.1.2. Let x, y, z be all the vertices of $st(p, L)$ distinct from p. Denote by O_x, O_y and O_z the pairs of edges in L, not contained in $st(p, L)$, adjacent to x, y and z respectively, with distinguished order for each pair. We will call them *ordered pairs of peripheral edges at star of vertex* p, and the whole system (p, O_x, O_y, O_z) an *order–system at star of* p.

3.1.3. Let x, y, z and u be all the vertices of $st(e, L)$ not contained in e. The system (e, O_x, O_y, O_z, O_u) consisting of ordered pairs of edges in L, adjacent to x, y, z and u respectively, and not contained in $st(e, L)$, will be called an *order–system at star of* e, and its elements *ordered pairs of peripheral edges at star of* e.

3.1.4. Denote by O_p the cyclically ordered triple of edges in L adjacent to p. We will call (p, O_p) a *cyclic order at* p.

All orders at pairs of peripheral edges, or at a triple in the last case, will be called *peripheral orders*.

3.2. Definition of good label systems.

Assume we have fixed order–systems in L of all four types 3.1.1–3.1.4, and let K be a finite (k, L)–complex with convex boundary, as defined in 2.1. Denote by $V_\partial K$ the set of vertices of K contained in the boundary ∂K.

We will say that a label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$ is *good*, if its label maps satisfy the following conditions:

(i) if w is 1–free in K, then $\lambda_w(L(w, K)) = e$, where e is the edge appearing in the corresponding order–system in L;

(ii) if w is 2–free in K, then $\lambda_w(L(w, K)) = st(p, L)$, where p is the vertex appearing in the corresponding order–system in L;

(iii) if w is 3–free in K, then $\lambda_w(L(w, K)) = st(e, L)$, where e is the edge appearing in the corresponding order–system in L.

3.3. Remark. Note that, due to transitivity of G on vertices and edges of L, each label system can be made good by the appropriate chart changes in L, for its label maps.

3.4. Characteristic functions of good label systems.

Let $E_\partial K$ denotes the set of boundary edges of K. Given a good label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$, for each edge $d \in E_\partial K$ and its endpoint w, consider the order on the pair of cells in $\widetilde{K} \setminus K$ adjacent to d, induced by the label map λ_w from the order system of appropriate type in L. Since d has two endpoints, we have two such induced orders. We define a *characteristic function* $\chi_\Lambda : E_\partial K \to \{0, 1\}$ by

$$\chi_\Lambda(d) = \begin{cases} 0 & \text{if the two corresponding induced orders agree;} \\ 1 & \text{otherwise.} \end{cases} \tag{3.1}$$

3.5. *K–equivalences of label maps and label systems.*

Given $w \in V_\partial K$, we say that label maps $\lambda, \lambda' : L(w, \widetilde{K} \to L)$ are *K–equivalent*, if $\lambda|_{L(w,K)} = \lambda'|_{L(w,K)}$, i.e. their restrictions to the link of K coincide. Label systems $\Lambda = \{\lambda_w : w \in V_\partial K\}$ and $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$ are *K–equivalent*, if for each $w \in V_\partial K$ the label maps λ_w and λ'_w are K–equivalent.

The significance of the notions of K–equivalence and and characteristic function of a good label system becomes apparent due to the following lemma, which will be used later in the classifications up to G–isomorphism. We shall also use characteristic functions to define local invariants of G–isomorphism in the next section.

3.6. Lemma. If two good label systems $\Lambda = \{\lambda_w : w \in V_\partial K\}$ and $\Lambda' = \{\lambda'_w : w \in V_\partial K\}$ are K–equivalent, and have equal characteristic functions, then there exists a unique combinatorial automorphism $T : \overline{K}(\Lambda) \to \overline{K}(\Lambda')$ satisfying the following conditions:

(i) $T|_K = id_K$;

(ii) $T_w = (\gamma'_w)^{-1} \circ \gamma_w$ for each $w \in V_\partial K$.

Moreover, T is a G–isomorphism with respect to G–structures $\mathcal{G}_{\overline{K}(\Lambda)}$ and $\mathcal{G}_{\overline{K}(\Lambda')}$.

(Recall that T_w is the isomorphism induced on the links at w by isomorphism T, and that γ_w and γ'_w are the charts at vertex w naturally appearing during constructions of $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ respectively.)

Proof: By K–equivalence of label systems Λ and Λ', in view of Remark 2.4, we have

$$((\gamma'_w)^{-1} \circ \gamma_w)|_{L(w,K)} = id_{L(w,K)}, \tag{3.2}$$

for each $w \in V_\partial K$. This means that condition (i) is compatible with conditions of form (ii), for all vertices $w \in V_\partial K$. It remains to check, wheather conditions of form (ii), for any two adjacent vertices, are compatible.

By the equality of characteristic functions χ_Λ and $\chi_{\Lambda'}$ at a boundary edge $d = (v, w)$, we get that either both $(\gamma'_v)^{-1} \circ \gamma_v$ and $(\gamma'_w)^{-1} \circ \gamma_w$ induce a tranposition at cells of $\widetilde{K} \setminus K$ adjacent to d, or they both induce the identity on them. In any of the cases, this gives the compatibility of conditions (ii) for v and w, since the stars $st(v, \overline{K})$ and $st(w, \overline{K})$ intersect only at three cells adjacent to d.

Since the conditions (i) and (ii) together determine T at stars of all interior vertices of $\overline{K}$, by their compatibility we get both the existence and uniqueness of T.

To prove that T is a G–isomorphism, we check (1.1) at each vertex $w \in V_\partial K$ by putting $\gamma = \gamma_w$ and $\gamma' = \gamma'_w$, thus getting

$$\gamma' \circ T_w \circ \gamma^{-1} = \gamma'_w \circ ((\gamma'_w)^{-1} \circ \gamma_w) \circ \gamma_w^{-1} = id_L \in G. \tag{3.3}$$

This finishes the proof of Lemma.

4. Local invariants of G–isomorphism.

4.1. Action of G on order systems.

It is clear that G acts on the sets of: vertices, edges, oriented edges, stars of vertices and stars of edges in L, and consequently on the corresponding order systems in L. We extract the following results concerning these actions from Propositions 1–5 of [DM] (compare Properties 2.2.1, 2.3.1, 2.4.1 and 2.5.1 in [Poly]).

4.2. Proposition.

(1) Any element of the stabiliser $Stab(p, G) \subset G$
  (a) preserves a cyclic order at p, if G is 1–regular;
  (b) either preserves all three peripheral orders at star of p, or reverses them all, if G is 3–regular, and orders are properly chosen (it is not true for each choice).
  (c) either preserves all three peripheral orders at star of p, or reverses exactly two of them, if G is s–regular for some $s \in \{4', 4'', 5\}$.

(2) Any element of the subgroup $Stab(e, G) \subset G$ of automorphisms fixing e, but not necessarily its endpoints either preserves both peripheral orders at e, or reverses them both, if G is $2'$–regular;

(3) Any element of the subgroup $Stab^+(e, G) \subset G$ of automorphisms fixing edge e as oriented edge, i.e. together with its endpoints, either preserves both peripheral orders at e, or reverses them both, if G is $2''$–regular.

(4) any element of G fixing edge e, but reversing its orientation, reverses exactly one of the peripheral orders, if G is $2''$–regular.

(5) The converse of any of above statements is true, i.e. for any change of peripheral orders mentioned in any of the cases (1)–(4) above, there exists a corresponding automorphism from G, which results with exactly this change.

4.3. Local invariants of G–isomorphism.

Invariants described in this subsection are closely related to those described in section 2 of [Poly]. To describe them by means of characteristic functions, we shall view local pieces in (k, L)–complexes, such as stars of edges, of cells, or of 1–balls, as been constructed out of those edges, cells or 1–balls respectively, by means of procedure of section 2.3, as it is explained in Remark 2.6. Note that, by Remark 3.3, the label system of form (2.4) can be always chosen to be good, and Proposition 4.2 clearly applies in this situation.

G, as always, denotes an s–arc–regular subgroup of $Aut\, L$.

4.3.1. Case $s = 1$.

One easily extends all the notions, constructions and results introduced so far in this paper, to a little bit degenerate case of K equal to a single edge $d = (w_1, w_2)$. Define $\widetilde{K}$, to consist of three k–gonal cells glued to d along some of their edges, and fix a cyclic order (p, O_p) in L, as defined in 3.1.4. Define a label system Λ for K, to consist of two label maps $\lambda_i : L(w_i, \widetilde{K}) \to L$. Λ is said to be good, if $\lambda_i(L(w_i, K)) = p$, for $i = 1, 2$. Given such a good label system, induce twice the cyclic order from L to the set of three cells of $\widetilde{K}$, by pulling back with respect to the label maps λ_i, and define characteristic function $\chi_\Lambda : \{d\} \to \{0, 1\}$ by

$$\chi_\Lambda(d) = \begin{cases} 0 & \text{if the two induced cyclic orders agree;} \\ 1 & \text{otherwise.} \end{cases} \tag{4.1}$$

The value of χ_Λ does not depend on the chart changes in L for which the label maps λ_i remain good, since the stabilizer of vertex p in G preserves the cyclic order O_p (see Proposition 4.2(1)(a)).

Then, viewing the star $st(e, X)$ of any edge e as been constructed in the way as above, put

$$\varepsilon_{X,\mathcal{G}}(e) = \chi_\Lambda(e), \tag{4.2}$$

for some good label system Λ of form (2.4).

4.3.2. Case $s = 2'$.

Given a cell c in a (k, L)-complex X, put $K = c$, and consider a good label system Λ of form (2.4) for K. Since by Proposition 4.2, an element $[\chi_\Lambda] \in H^1(\partial K, Z_2)$ does not depend on the choice of Λ, put

$$\xi_{X,\mathcal{G}}(c) = [\chi_\Lambda]. \tag{4.3}$$

4.3.3. Case $s = 2''$.

Consider a variant of the order system of form 3.1.3 at an edge e in L, to consist of an oriented edge e, and two orders at peripheral pairs at e. Then given an oriented cell c^+ in a (k, L)-complex X, put $K = c$, and note that c induces the orientations on edges d_c corresponding to c, in links $L(w, X)$ at vertices w of c. We wiil say that a label map λ_w for K is *good*, if it maps d_c onto e preserving orientations.

Consider a good label system Λ of form (2.4) for K (which clearly exists due to the transitivity of G on the set of oriented edges in L), and note that due to Proposition 4.2(3), the cohomology element

$$\xi''_{X,\mathcal{G}}(c^+) = [\chi_\Lambda] \tag{4.4}$$

does not depend on the chart changes in L, for which Λ remains good in the above explained sense.

Note also that due to Proposition 4.2(4), if $c^\pm$ denote the opositely oriented cells related to a nonoriented cell c, then

$$\xi''_{X,\mathcal{G}}(c^+) \neq \xi''_{X,\mathcal{G}}(c^-) \text{ if } k \text{ is odd, and } \xi''_{X,\mathcal{G}}(c^+) = \xi''_{X,\mathcal{G}}(c^-) \text{ if } k \text{ is even.} \tag{4.5}$$

4.3.4. Case $s = 3$.

For a vertex v in X, let K be a 1–ball centered at v, and Λ a good label system of form (2.4) for K. Since by Proposition 4.2, an element $[\chi_\Lambda] \in H^1(\partial K, Z_2)$ does not depend on the choice of Λ, put

$$\eta_{X,\mathcal{G}}(v) = [\chi_\Lambda]. \tag{4.6}$$

4.3.5. Case $s \in \{4', 4'', 5\}$ and $k = 3$.

Let v, K and Λ be as in 4.3.4. Then by Propostion 4.2, the number

$$\sigma_{X,\mathcal{G}}(v) = \sum_{d \in E_\partial K} \chi_\Lambda(d) \pmod 2 \tag{4.7}$$

does not depend on the choice of Λ.

4.3.6. Lemma. In all four cases above, the quantities ε, ξ, η and σ are the invariants of G–isomorhism of the corresponding complexes $st(K, X)$.

In case of $s = 3$ this requires the following more carefull statement. If $T : B_2(v) \to B_2(w)$ is a G–isomorphism of 2–balls in (k, L)–complexes, and $T_1 = T|_{\partial B_1(v)}$ is the restriction of T to the boundary of the 1–ball $B_1(v)$, then T_1 pulls back $\eta(w)$ to $\eta(v)$, i.e. $T_1^*(\eta(w)) = \eta(v)$.

The proof of Lemma 4.3.6 consists of the straightforward manipulations with G–structures, and we omit it.

5. Conditions for label systems, related to local invariants.

The general construction of (k, L, G)–complexes, as presented in section 2, does not provide any control on local invariants of resulting complexes. In this section we describe how to provide such a control, in terms of restrictive conditions for label systems used at corresponding steps of construction. We shall use letter C as a variable, speaking about condition C, whenever refering to conditions of this genaral form. We shall prove that conditions of this type are consistent, i.e. that there exist label systems which satisfy them. We shall also prove that these conditions determine resulting complexes uniquely, up to G–isomorphism.

5.1. Description of conditions.

Given k and L satisfying nonpositive curvature condition (1.2), we shall call an *n–ball* a (k, L)–complex with convex boundary, equipped with a G–structure, obtained from a 1–ball as described in 2.2, by succesive application of $n - 1$ steps of construction described in 2.3.

In the following descriptions of conditions C, K will denote an n–ball with its G–structure $\mathcal{G}_K$. We shall denote by $\mathcal{L}_K^C$ the set of good label systems for K, satisfying given condition C.

5.1.1. The case when G is 1–regular.

Denote by $E_{out}K$ the set of all *outer* edges in K, i.e. those which have at least one vertex on the boundary ∂K. Consider any function $\varepsilon_C : E_{out}K \to \{0,1\}$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \varepsilon_{\overline{K(\Lambda)}}(e) = \varepsilon_C(e) \text{ for each } e \in E_{out}K. \tag{5.1}$$

Remark. Note that a G–structure $\mathcal{G}_K$, together with a condition C, determine values of invariant ε at all edges of K, so that no stronger condition can be expressed in terms of this local invariant.

5.1.2. The case when G is $2'$–regular, or when G is $2''$–regular and k is even.

Denote by $C_{out}K$ the set of all *outer* cells of K, i.e. those which contain at least one boundary vertex. Consider any function $\xi : C_{out}K \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \xi_{\overline{K(\Lambda)}}(c) = \xi_C(c) \text{ for each } c \in C_{out}K. \tag{5.2}$$

In the above equation we use the canonical identification of $H^1(\partial c, Z_2)$ with Z_2.

Note that due to (4.5), when G is $2''$–regular and k is even, we can view ξ'' as an invariant of nonoriented cells. Then define condition C in this case in exactly the same way, replacing ξ by xi'' in (5.2).

Remark. Note that $\mathcal{G}_K$ together with C, determine values of invariant ξ (or ξ'' respectively) at each cell of K, so that no stronger condition can be expressed in terms of this invariant.

5.1.3. The case when G is $2''$–regular and k is odd.

For each cell $c \in C_{out}K$ choose an orientation, and denote the set of so oriented cells by $C_{out}^{or}K$ (note that only one of the two oriented cells corresponding to a given cell of $C_{out}K$ is contained in $C_{out}^{or}K$). Consider a function $\xi_C'' : C_{out}^{or}K \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \xi''_{\overline{K(\Lambda)}}(c) = \xi''_C(c) \text{ for each } c \in C_{out}^{or}K. \tag{5.3}$$

Remark. Note that, due to (4.5), if the value of ξ'' is determined at an oriented cell, then it is also determined at this cell with opposite orientation. Thus, a G–structure $\mathcal{G}_K$, together with a condition C, determine values of ξ'' at all oriented cells of K, so that no stronger condition can be expressed in terms of this invariant.

5.1.4. The case when G is 3–regular.

Denote by K^- the $(n-1)$–ball contained in K, corresponding to the one step shorter construction, i.e. centered at the same point, and such that $\overline{K^-} = K$. For each $v \in$

$V_\partial K^-$, consider the boundary $\partial B_1(v)$ of the 1–ball centered at v, and an element $\eta_C(v) \in H^1(\partial B_1(v), Z_2)$, thus getting a *sheaf* η_C of cohomology elements. Define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \eta_{\overline{K}(\Lambda)}(v) = \eta_C(v) \text{ for each } v \in V_\partial K^-. \tag{5.4}$$

5.1.5. The case when G is s–regular for some $s \in \{4', 4'', 5\}$.
Under notation of 5.1.4, consider a function $\sigma_C : V_\partial K^- \to Z_2$, and define condition C by

$$\Lambda \in \mathcal{L}_K^C \text{ iff } \sigma_{\overline{K}(\Lambda)}(v) = \sigma_C(v) \text{ for each } v \in V_\partial K^-. \tag{5.5}$$

Remark. Note that a G–structure $\mathcal{G}_K$ and a condition C, determine valeues of invariants η or σ (in the corresponding cases of s–regularity of G) at all vertices of K^-, so that no stronger conditions can be expressed in terms of these invariants.

The rest of this section is devoted to proving the following two results concerning conditions C described above. For the second result, recall the notion of K–equivalence of label systems, introduced in 3.5.

5.2. Proposition. Let C be any of the conditions described in 5.1. Then the set $\mathcal{L}_K^C$ of good label systems satisfying this condition is nonempty.

5.3. Proposition. Any two label systems $\Lambda_1, \Lambda_2 \in \mathcal{L}_K^C$ are K–equivalent, after appropriate chart changes in L for label maps of one of them.

Proposition 5.2 means that any condition C of 5.1 is consistent, while Proposition 5.3, in view of Lemmas 2.5 and 3.6, is the first step towards the proof that such condition determines the resulting complex $\overline{K}$ uniquely up to G–isomorphism. The proof of this last fact will be completed in section 6.

Before strarting to prove Propositions 5.2 and 5.3, we need to establish some facts concerning the notion of an order structure in a trivalent graph.

5.4. Order structures in trivalent graphs.
By a *trivalent graph* we shall mean a connected graph Q, each vertex of each is adjacent either to one or to three edges of Q. Q is a *tree*, if it is simply connected, i.e. contains no nontrivial cycle of edges. We shall denote by ∂Q the set of all boundary vertices of Q, i.e. those which are adjacent to only one edge of Q.

5.4.1. Three types of order structures.

Type 1. If v is an interior (i.e. not boundary) vertex of Q, an *order atlas at* v consists of a choice of a cyclic order at v (i.e. a cyclic order for the edges of Q adjacent to v).

An *order structure* of this type in Q consists of a collection of fixed order atlasses at all interior edges of Q.

Type 2. Let e be an *interior* edge of Q, i.e. one with both endpoints interior. An *order atlas at* e consists of two order systems at e (as described in 3.1.1), that differ by the simultaneous change of both peripheral orders. An *order structure* in Q of this type consists of a collection of fixed order atlesses at all interior edges of Q.

Type 3. We shall need also a variant of order structure consisting of order atlasses at oriented interior edges of Q. Order atlasses of this structure consist of two order systems that differ by the simultaneous change of both peripheral orders. Moreover, order atlasses of this structure, at any two edges with opposite orientation, differ from each other by a change of any one of their peripheral orders (peripheral pairs of edges are the same at such two edges).

5.4.2. Examples: G–invariant order structures in L.

We define such an order structure by simply taking an orbit of the action of G in the space of order systems at vertices, edges or oriented edges in L (compare 4.1), arranging elements of this orbit into a collection of order atlasses in the obvious way. More precisely

(i) if G is 1–regular, we define an order structure of type 1;
(ii) if G is $2'$–regular, we define an order structure of type 2;
(iii) and finally, if G is $2''$–regular, we define an order structure of type 3.

Remark. All above order structures are well defined due to Proposition 4.2.

5.4.3. Proposition. Consider a G–invariant order structure in L, as in Example 5.4.2, and a trivalent tree Q equipped with an order structure of the same type. Then there exists a combinatorial immersion $i : Q \to L$ preserving order structures. Moreover, this immersion is unique up to chart change in L.

Proof: Consider an s–arc in Q, where $s = 1$ for type 1, and $s = 2$ for types 2 and 3. (If there is no such arc in Q, then there is no order structure in Q, and you can immerse it whatever you like, getting uniqueness by the regularity of G.) Immerse this s–arc arbitrarily in L, and notice that the requirement of order structures to be preserved, determines an extension of immersion to the whole of Q uniquely, in all three cases of types.

The uniqueness of immersion up to chart change in L follows from above uniqueness of extension.

We omit further details.

5.5. Proof of Proposition 5.2.

We will provide the proof separately for the corresponding cases of s–regularity of G and the kind of invariant used to define condition C.

5.5.1. Remark. Note that using order structure in L, it is possible to induce order systems, or rather classes of order systems corresponding to order atlasses, by means of any label maps, not necessarily good ones. If an order structure in L is G–invariant, as in examples 5.4.2, then the class of induced order systems does not depend on chart changes for a label map.

The case when G is 1–regular.

In this case condition C is defined in terms of invariant ε, as in 5.1.1.

Consider a vertex $w \in V_\partial K$, and an edge $e = (w, u)$ adjacent to w and not contained in ∂K. The vertex $p_e \in L(w, K)$, which represents edge e, is then interior in the link $L(w, K)$ viewed as a trivalent tree.

Since vertex u of e is interior in K, we have the G–atlas of G–structure $\mathcal{G}_K$ at it, and an induced (by some chart of this G–atlas), well defined (by Remark 5.5.1 and Proposition 4.2(1)(a)) cyclic order on the triple of cells in K adjacent to e. By condition C, we want to have $\varepsilon_{\overline{K}}(e) = \varepsilon_C(e)$, and thus the value of $\varepsilon_C(e)$ determines the cyclic order at vertex p_e in $L(w, \widetilde{K})$, namely the one which has to be induced by a label map at w used in construction of $\overline{K}$.

By repeating above procedure for all outer but not buondary edges of K, we get the order structures determined by condition C in all links $L(w, K)$ at boundary vertices w, and we know that restrictions $\lambda_w|_{L(w,K)}$ of label maps used in construction of $\overline{K}$, have to preserve these order structures, with respect to the G–invariant order structure in L, determined by the order system chosen in L (the one used in definition of invariant ε).

Immerse links $L(w, K)$ into L, in a way preserving order structures, which is possible by Proposition 5.4.3, and extend arbitrarily these immersions (after chart change if necessary) to good label maps λ_w at all $w \in V_\partial K$, getting a good label system Λ. Construct the complex $\overline{K}(\Lambda)$, and note that $\varepsilon_{\overline{K}(\Lambda)}(e) = \varepsilon_C(e)$ for all outer not boundary edges of K. Now, for each boundary edge d of K, compare the values of $\varepsilon_{\overline{K}(\Lambda)}(d)$ and $\varepsilon_C(d)$, and if they are different, change the label map at one endpoint w of d, only at the peripheral pair of edges in $L(w, \widetilde{K})$ corresponding to the peripheral pair of cells in $\widetilde{K}$ adjacent to d.

Note that after the changes as above, the new label system Λ' belongs to $\mathcal{L}_K^C$, and thus the proof for this case follows.

The case when G is $2'$– or $2''$–regular.

In these two cases conditions C are defined in terms of invariants ξ and ξ'' respectively, as in 5.1.2 and 5.1.3.

Consider a cell c in K, which is outer, but has no edge contained in the boundary ∂K, and denote by w its unique vertex contained in ∂K. Note that the edge e_c representing c is inner in the link $L(w, K)$ viewed as a trivalent tree.

Since all the vertices of c other than w are inner in K, we have G–atlasses of the G–structure $\mathcal{G}_K$ at them, and we have order systems (or rather classes of them) in the corresponding links at those vertices, induced by charts of these G–atlasses. Of particuliar interest to us are order systems induced at edges representing cell c in these links, since they take part in determining invariant ξ (or ξ'') at cell c.

It is easy to realise that the requirement $\xi_{\overline{K}}(c) = \xi_C(c)$ (or $\xi''_{\overline{K}}(c^\pm) = \xi''_C(c^\pm)$, where $c^\pm$ are the two oriented cells corresponding to a nonoriented cell c) implies that label maps $\lambda_w : L(w, \widetilde{K}) \to L$ which fulfill it, induce order systems at the edge e_c (at oriented edges

$e_c^{\pm}$*respectively*) which form a certain order atlas at this edge.

Having equipped $L(w,K)$ with the order structure determined by condition C in above way, we choose any its immersion into L preserving order structures, and extend it arbitrarily (after a chart change if necessary) to a good label map $\lambda_w : L(w,\widetilde{K} \to L)$, thus getting a label system Λ by repeating above procedure at all vertices $w \in V_\partial K$. By construction of Λ, we have $\xi_{\overline{K}(\Lambda)}(c) = \xi_C(c)$ for all outer cells c of K with no edge contained in ∂K (and the same holds for such oriented cells, and for invariant ξ'', in the corresponding second case considered parallelly).

We now consider an outer cell b, with at least one edge contained in ∂K. If $\xi_{\overline{K}(\Lambda)}(b) \neq \xi_C(b)$, choose one endpoint w of one boundary edge e of c, and modify λ_w only at the peripheral pair of edges in $L(w,\widetilde{K})$ representing the peripheral pair of cells in $\widetilde{K}$ adjacent to e. Repeating this for all cells b as above, we get a new label system Λ', which now belongs to $\mathcal{L}_K^C$.

The case when G is $4'$–, $4''$– or 5–regular, and $k = 3$.

Note that condition C in this case is expressed in terms of invariant σ, as in 5.1.5.

Choose an arbitrary good label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$, and for each $v \in V_\partial K^-$, compare the numbers $\sigma_{\overline{K}(\Lambda)}(v)$ and $\sigma_C(v)$. If they are different, then consider a boundary edge e of K, which belongs to $\partial B_1(v)$ (such edge always exists by convexity of the boundary of K, becouse the nonpositive curvature condition (1.2) implies $g(L) \geq 6$ in this case). Choose one endpoint of e, say w, and modify the label map λ_w at the peripheral pair of edges in $L(w,\widetilde{K})$ corresponding to the peripheral pair of cells in $\widetilde{K}$ adjacent to e. The so obtained new label system Λ' clearly belongs to $\mathcal{L}_K^C$.

The case when G is 3–regular.

Note that condition C in this case is expressed in terms of invariant η, as in 5.1.4.

Choose an arbitrary good label system $\Lambda = \{\lambda_w : w \in V_\partial K\}$, and for each vertex $v \in V_\partial K^-$, compare the cohomology elements $\eta_{\overline{K}(\Lambda)}(v)$ and $\eta_C(v)$. If they are different, denote by $A(v)$ the subgraph in $\partial B_1(v)$ consisting of all edges not contained in ∂K.

5.5.2. Claim. $A(v)$ is a connected contractible subgraph in $\partial B_1(v)$.

Proof of Claim: Denote by L' the graph obtained from $L(v,K)$ by subdivision of each its edge into $k-2$ smaller edges (where k is the number characterising our (k,L)–complex), and by $\widetilde{L}'$, its part corresponding to $L(v,K^-)$. Then $A(v)$ is isomorphic to $st(\widetilde{L}',L')$. Since $A(v)$ has then a form of subdivided $L(v,K^-)$ with pairs of peripheral edges attached disjointly to all former boundary vertices, the Claim follows in any of the following cases, which exhast all the possibilities:

(i) $k \geq 5$, since then all the boundary vertices of K^- are m–free with $m \leq 2$;
(ii) $k \geq 4$, since then $g(L) \geq 4$ by (1.2), and all the boundary vertices of K^- are m–free with $m \leq 2$;
(iii) $k = 3$, since then $g(L) \geq 6$ by (1.2), and $diam\, A(v) \leq 5$ by convexity.

This finishes the proof of Claim.

Note that, by contractibility of $A(v)$, it follows from the exact cohomology sequence for the pair $(\partial B_1(v), A(v))$, that any 1–cochain in $A(v)$ can be extended to a 1–cochain in $\partial B_1(v)$, representing any cohomology element from $H^1(\partial B_1(v), Z_2)$ (see the proof of Proposition 7.4.6, and the Property 7.4.7 in [Poly], for more details). Thus, we can modify label maps of Λ at peripheral pairs of edges in $L(w, \widetilde{K})$, so that a new label system Λ' satisfies equalities $\eta_{\overline{K}(\Lambda')}(v) = \eta_C(v)$ for each $v \in V_\partial K^-$, and thus belongs to $\mathcal{L}_K^C$.

This finishes the proof of Proposition 5.2.

5.6. Proof of Proposition 5.3.

In the cases when G is 1–, 2′– or 2″–regular, Proposition follows from Proposition 5.4.3 applied to links $L(w, K) : w \in V_\partial K$, equipped with order structures determined by condition C in the way described in the proof of Proposition 5.2 (and more precisely, from the uniqueness part of Proposition 5.4.3).

In the cases when G is 3–, 4′–, 4″– or 5–regular, Proposition 5.3 follows even more easily from the following observation.

5.6.1. Claim. If a buondary vertex w is m–free, and G is m–arc–transitive, then any two good label maps at w are K–equivalent up to a chart change.

Proof of Claim: This follows from transitivity of G on m–arcs, and thus on restrictions of good label maps to $L(w, K)$ (since $m \leq 3$ by convexity).

This finishes the proof of Proposition 5.3.

6. Local conditions determine (k, L, G)–complexes uniquely.

This section is devoted to the proof of the following.

6.1. Proposition. Let K be an n–ball, as defined at the beginning of 5.1, with a triple (k, L, G) as local data, where $G \subset Aut\, L$ is an s–regular subgroup, for some $s \in \{1, 2', 2'', 3, 4', 4'', 5\}$. Moreover, let C be a condition of one of the forms described in 5.1 (or just the empty condition, if $s \in \{3, 4', 4'', 5\}$ and $k \geq 4$). Then for any two label systems $\Lambda, \Lambda' \in \mathcal{L}_K^C$, the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ constructed by means of these label systems, are G–isomorphic, by a G–isomorphism extending the identity automorphism of K. Moreover, if $s \in \{1, 2', 2''\}$, or if $s = 3$, $k = 3$ and $n \geq 2$, then this G–isomorphism is unique, while there is more than one such G–isomorphism in the other cases.

6.1.1. Remark. In the case when $s \in \{3, 4', 4'', 5\}$ and $k \geq 4$, we consider the empty condition C (no restrictions for label systems), in which case the space $\mathcal{L}_K^C$ consists of all good label systems for K. The Proposition says then, that any two $(n+1)$–balls of form $\overline{K}(\Lambda)$ are G–isomorphic by an isomorphism extending id_K.

To prove Proposition 6.1, we introduce the notion of modifications of characteristic functions, and give two preparatory lemmas.

6.2. Modifications of characteristic functions.

Consider the following *elementary modifications* of a function $\chi : E_\partial K \to \{0, 1\}$:

(i) change the values of of χ at both edges of ∂K adjacent to a 1–free vertex of K;
(ii) change the value of χ at any edge of ∂K adjacent to a 1–free vertex of K;
(iii) change the values of χ at all three edges of ∂K adjacent to a 2–free vertex of K;
(iv) change the values of χ at any two of the three edges of ∂K adjacent to a 2–free vertex of K;
(v) change the values of χ at all four edges of ∂K adjacent to a 3–free vertex of K;
(vi) change the valeus of χ at any two of the four edges of ∂K adjacent to a 3–free vertex w of K, such that the vertices in $L(w, K)$ corresponding to those two edges are at distance two in $L(w, K)$.

A *modification* is a finite sequence of elementary modifications of the forms as below:
(1) when $s = 1$, no modifications;
(2) when $s = 2'$ or $s = 2''$, form (i) only;
(3) when $s = 3$, forms (ii) and (iii);
(4) when $s = 4'$ or $s = 4''$, forms (ii), (iv) and (v);
(5) when $s = 5$, forms (ii), (iv) and (vi).

6.3. Lemma. Let $\chi' : E_\partial K \to \{0, 1\}$ be a function obtained from a characteristic function $\chi = \chi_\Lambda$ of a label system Λ by a modification (suitable for the corresponding regularity type of G). Then there exists a K–equivalent to Λ label system Λ', obtained from Λ by a chart change in L, such that $\chi' = \chi_{\Lambda'}$.

Proof: The Lemma follows from Propositions 1–5 of [DM], where the subgroups of G fixing poitwise an edge, a star of vertex, and a star of edge in L (the pieces which correspond to the links of K at the boundary vertices) are described, together with their actions on edges of peripheral pairs.

6.4. Lemma. Under the notation of Proposition 6.1, characteristic functions of any two K–equivalent label systems from $\mathcal{L}_K^G$ differ at most by a modification (suitable for the corresponding case of regularity of G).

The proof of Lemma 6.4 splits in fact into five separate cases. Each of the cases brings another type of difficulty into play. We do not present here these five proofs, since they are implicit in the proofs of the corresponding results in [Poly]. For the reader's convenience, we give the following detailed references:
(1) for the case of $s = 1$, see Lemma 7.1.3;
(2) for the case of $s = 2'$ or $s = 2''$, see Lemma 7.2.4, together with preceding Lemma 7.2.3;
(3) for the case of $s = 3$ and $k = 3$, see Lemma 7.4.10 together with Lemma 7.4.6(ii);
(4) for the case of $s \in \{4', 4'', 5\}$ and $k = 3$, see Lemma 7.3.6 together with 7.3.3–7.3.5;
(5) for the case of $s \in \{3, 4', 4'', 5\}$ and $k \geq 4$, see 6.1–6.5.

6.5. Proof of Proposition 6.1.

Given label systems $\Lambda, \Lambda' \in \mathcal{L}_K^G$, we know by Lemma 3.5 that the complexes $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ do not depend, up to G–isomorphisms extending id_K, on chart changes in L. Thus, due to Proposition 5.3, we can assume that Λ and Λ' are K–equivalent. Then, Lemma 6.4 tells us that characteristic functions χ_Λ and $\chi_{\Lambda'}$ differ by a modification, and using Lemma 6.3 we again modify, say Λ, by a chart change in L, so that now Λ and Λ' are K–equivalent and have equal characteristic functions. But this allows to apply Lemma 3.6, which finishes the proof of the existence part of Proposition 6.1.

To see the uniqueness in the cases when $s \in \{1, 2', 2''\}$, consider G–isomorphisms $T_1, T_2 : \overline{K}(\Lambda) \to \overline{K}(\Lambda')$, both extending the identity automorphism of K. Note that in these cases there is no nontrivial isomorphism of G, fixing pointwise a star of vertex or a star of edge in L. This implies that, if w is a 2–free or a 3–free vertex of K, then $T_1|_{st(w,\overline{K}(\Lambda))} = T_2|_{st(w,\overline{K}(\Lambda))}$, and hence the isomorphisms T_1 and T_2 coincide on the subcomplex

$$M = K \cup \bigcup \{st(w, \overline{K}(\Lambda)) : w \text{ is 2–free or 3–free in } K\}.$$

But then, the links $L(v, M)$ at those 1–free vertices v of K, which have a 2–free or a 3–free neighbour, are big enough to imply the coincidence of T_1 and T_2 on the stars $st(v, \overline{K}(\Lambda))$. Repeating this argument, we get $T_1 = T_2$ by the fact that the boundary ∂K is connected.

The similar argument as above applies to the case when $s = 3$ and $k = 3$, with the following changes. There is no nontrivial isomorphism of G, fixing pointwise a star of edge in L. Thus, G–isomorphisms T_1 and T_2 coincide on stars of 3–free vertices of K. The following Claim allows then to extend this coincidence to the whole of $\overline{K}(\Lambda)$, by the arguments of the previous cases.

Claim. If $k = 3$ and K is an n–ball with $n \geq 2$, then at least one vertex in K is 3–free, and no one is 1–free.

Proof of Claim: Constructing an n–ball from an $(n-1)$–ball, we glue 1–balls to it, according to the pattern provided by a label system. The boundary of this n–ball consists then of parts of boundaries of the glued 1–balls. If $k = 3$, no boundary vertex of the 1–ball is 1–free, and so this is also true for K. On the other hand, if $k = 3$ then all the vertices of $\widetilde{B}_{n-1} \setminus B_{n-1}$ become 3–free in B_n, and the Claim follows.

Now, we proceed to prove that there is more than one G–isomorphism between $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$, extending id_k, in the following three remaining cases:

(1) $s = 3$, $k = 3$ and K is a 1–ball;
(2) $s = 3$ and $k \geq 4$;
(3) $s \in \{4', 4'', 5\}$.

We start with the observation that in the case (1) only 2–free vertices appear at ∂K, and in the case (2) only 2–free and 1–free ones. In any of the cases, there exists an isomorphism of G fixing pointwise an edge (a star of vertex respectively) in L, and transposing all pairs of its peripheral edges; if the case (3) happens, the same is true also for a star of edge in L (see [DM] Propositions 1–5, and note that Proposition 6.3 is also related to those properties).

Using these facts we can construct in any of the cases (1)–(3) a G–isomorphism between $\overline{K}(\Lambda)$ and $\overline{K}(\Lambda')$ that differs from a given one at all pairs of peripheral cells in $\widetilde{K}$. We omit further details.

7. Proof of the Main Theorem.

The necessary condition for a (k, L, G)–complex $(X, \mathcal{G})$ to be flag–symmetric, is that the value of the appropriate local invariant (viewed as a function on the set of vertices, edges or cells of X, or as a sheaf of cohomology elements, respectively) is constant. It is clear what this constantness means in all the cases when $s \neq 3$ or $k \geq 4$, since then the invariants (can be canonically viewed to) have values in the set $Z_2 = \{0, 1\}$. In the case when $s = 3$ and $k = 3$, it is necessary that the value $\eta_{X,\mathcal{G}}(w) \in H^1(\partial B_1(w), Z_2)$, at any vertex $w \in X$, is a G–invariant cohomology element (with respect to the action of G on $\partial B_1(w)$, induced by any map of the G–atlas A_w of $\mathcal{G}$, from L). It is also clear how to compare such elements at distinct vertices of X, since the 1–spheres $\partial B_1(w)$ are canonically G–equivariant. The necessary condition for the flag–symmetry of the complex $(X, \mathcal{G})$, is that the sheaf $\eta_{X,\mathcal{G}}$ is constant with respect to the above G–equivariance.

The existence and uniqueness properties of flag–symmetric (k, L, G)–complexes, as stated in the Main Theorem, are immediately implied by the following.

7.1. Lemma. Given any local data satisfying (1.2) (with the exception for $2''$–regular G and odd k), and a value for the local invariant (of the type suitable for the corresponding regularity type of G), there exists a (k, L, G)–complex $(X, \mathcal{G})$, with the local invariant constant and equal to the given value. Moreover, this (k, L, G)–complex is unique up to G–isomorphism, and flag–symmetric.

7.1.1. Remark. The above Lemma is obviously not valid in the case when G is $2''$–regular and k is odd, due to (4.5). This means that no flag–symmetric (k, L, G)–complex with these local data exists, which coincides with the statement of part (4) of the Main Theorem.

Proof: The existence of an appropriate (k, L, G)–complex follows from the possibility to construct it. The requirement for the local invariant, to be constant and equal to the given value, can be expressed in terms of the appropriate conditions C for label systems used during the construction. The existence part of the Lemma follows then from Proposition 5.2, by which the appropriate label systems at each stage of the construction exist.

The uniqueness up to G–isomorphism follows from Proposition 6.1, by which we can construct inductively a G–isomorphism between any two (k, L, G)–complexes with constant and equal local invariants. The same construction proves the flag–symmetry of any such (k, L, G)–complex.

This finishes the proof of Lemma.

We now proceed to prove the properties of the automorphism groups $Aut(X, \mathcal{G})$. They all follow easily from the uniqueness–nonuniqueness part of Proposition 6.1.

By the existence part of this Proposition, we can construct inductively a G–automorphism of a flag–symmetric (k, L, G)–complex, starting from any G–isomorphism of some of its 1–balls. In the cases when $s \in \{1, 2', 2''\}$, this initial G–isomorphism extends uniquely at each inductive step, giving the G–automorphism uniquely determined by its restriction to the initial 1–ball. In the next case when $s = 3$ and $k = 3$, this initial G–isomorphism extends in several ways to a G–isomorphism of 2–balls, but since the further extensions are unique, any resulting G–automorphism is determined by its restriction to the initial 2–ball. In the remaining cases, there are many possibilities of extensions at any stage of the inductive construction of a G–automorphism.

From above remarks, the properties of automorphism groups $Aut(X, \mathcal{G})$ stated in the Main Theorem follow, which finishes the proof.

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