Locally symmetric connections on possibly degenerate affine hypersurfaces

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One of the important results in affine diffferential geometry in the last decade is the characterization of quadrics of dimension ≥ 3 as nondegnerate hypersurfaces M^n in R^{n+1} whose induced connections are locally symmetric [V-V]. In this paper we generalize this theorem in two directions. On the one hand, we allow an entirely arbitrary transversal vector field instead of restricting it to the affine normal field or an equiaffine transversal field; on the other hand, we allow affine hypersurfaces to be degenerate (that is, the rank of the affine fundamental form h is not maximal). In this second direction, we shall use the idea and results developed in our earlier paper [N-O].

The present paper is constructed as follows.

In Section 1 we treat nondegenerate hypersurfaces and extend the theorem in [V-V] to the case of an arbitrary transversal vector field. Our main result here is the following.

Theorem 1. *Let $f : M^n \to R^{n+1}, n \geq 3$, be a nondegenerate hypersurface, endowed with a transversal vector field ξ. If the connection ∇ induced by ξ is locally symmetric, then ∇ is locally flat or it is the Blaschke connection and $f(M^n)$ is an open part of a quadric with center.*

In Section 2, we first prove two key lemmas which will reduce the case of degenerate hypersurfaces to that of nondegnerate hypersurfaces. In Section 3, we carry out this reduction and prove the second main result in the following form. Note that there is an open dense subset Ω of M^n such that the rank of h is constant in a neighborhood of each point of Ω (cf. [N-P2], p.358).

Theorem 2. *Let $f : M^n \to R^{n+1}$, rank $h \geq 2$, be a connected hypersurface endowed with a transversal vector field ξ. If the connection induced by ξ is locally symmetric, then the shape operator S is identically 0 (and ∇ is flat) or each point $x_0 \in \Omega$ has a neighborhood U of the form $M^r \times W$, where W is an open subset of an affine subspace R^s and M^r is immersed by f into an affine subspace R^{r+1} transversal to R^s as a nonsingular hypersurface with a transversal vector field $\overline{\xi}$ which induces a locally symmetric connection $\overline{\nabla}$ on M^r. If rank $h \geq 3$ at x_0, then either $\overline{\nabla}$ is flat or it is the Blaschke connection on M^r and $f(M^r)$ is an open part of a quadric with center.*

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1. Nondegenerate hypersurfaces.

Let $f : M^n \to R^{n+1}$ be a connected orientable n-dimensional manifold immersed in the affine space R^{n+1} provided with a fixed determinant function (volume element parallel relative to the standard flat connection D). Let ξ be an arbitrarily chosen transversal vector field. As usual, we write

$$(1) \qquad D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi$$

and

$$(2) \qquad D_X \xi = -f_*(SX) + \tau(X)\xi,$$

where X, Y are vector fields on M^n, ∇ is the induced connection on M^n, h the affine fundamental form, S the shape operator, and τ the transversal connection form, all depending on the chosen ξ. As is well-known (see, for example, [N-P]) we have the fundamental equations:

$$(3) \qquad R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY \quad \text{Gauss},$$

where R is the curvature tensor of ∇;

$$(4) \qquad h(X,SY) - h(SX,Y) = d\tau(X,Y) \quad \text{Ricci}$$

$$(5) \qquad \nabla h(X,Y,Z) - \nabla h(Y,X,Z) = h(X,Z)\tau(Y) - h(Y,Z)\tau(X) \quad \text{Codazzi for } h,$$

where $\nabla h(X,Y,Z)$ means $(\nabla_X h)(Y,Z)$;

$$(6,) \qquad \nabla S(X,Y) - \nabla S(Y,X) = \tau(X)SY - \tau(Y)SX \quad \text{Codazzi for } S$$

where $\nabla S(X,Y)$ means $(\nabla_X S)(Y)$.

Recall that ξ is said to be equiaffine if τ vanishes everywhere. For a nondegenerate hypersurface (that is, h is nondegenerate), there is a unique choice of ξ, up to sign, called the affine normal (of Blaschke), such that the induced volume element

$$\theta(X_1, \cdots, X_n) = \det(f_*(X_1), \cdots, f_*(X_n), \xi)$$

coincides with the volume element ω_h of the affine metric h. The connection induced by the affine normal is called the Blaschke connection. If a transversal vector field has the same direction as the affine normal at each point, then the induced connection ∇ coincides with the Blaschke connection.

In order to prove Theorem 1 we first observe that the assumption $\nabla R = 0$ implies that for any $X, Y \in T_x(M^n)$ we have $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on R as derivation. For all $X, Y, Z, W \in T_x(M^n)$ we obtain by using the Gauss equation

$$
\begin{aligned}
(7)\qquad & (R(X,Y)\cdot R)(Z,V)W \\
&= [h(V,W)h(Y,SZ) - h(Z,W)h(Y,SV]SX \\
&\quad +[h(Z,W)h(X,SV) - h(V,W)h(X,SZ)]SY \\
&\quad +[h(Y,V)h(Z,W) - h(Y,Z)h(V,W)]S^2X \\
&\quad +[h(X,Z)h(V,W) - h(X,V)h(Z,W)]S^2Y \\
&\quad +[h(X,V)h(SY,W) - h(Y,V)h(SX,W) \\
&\quad +h(X,W)h(SY,V) - h(Y,W)h(SX,V)]SZ \\
&\quad +[h(Y,Z)h(SX,W) - h(X,Z)h(SY,W) \\
&\quad +h(Y,W)h(SX,Z) - h(X,W)h(SY,Z)]SV.
\end{aligned}
$$

We first prove

Lemma 1. *At each point x of M^n the endomorphism S_x is nonsingular or vanishes on $T_x(M^n)$.*

Proof. For any $X \in \ker S_x$ the equation (7) gives

$$
\begin{aligned}
(8)\qquad 0 &= [h(Z,W)h(X,SV) - h(V,W)h(X,SZ)]SY \\
&\quad +[h(X,Z)h(V,W) - h(X,V)h(Z,W)]S^2Y \\
&\quad +[h(X,V)h(SY,W) + h(X,W)h(SY,V)]SZ \\
&\quad -[h(X,Z)h(SY,W) + h(X,W)h(SY,Z)]SV.
\end{aligned}
$$

Assume that $\ker S_x \neq \{0\}$. In the sequel we omit the letter x. Consider the two cases I and II.

Case I: $h|_{\ker S} = 0$. Then rank $S \geq 2$, because $n \geq 3$ and h is nondegenerate. Let $0 \neq X \in \ker S$. Take $W = X, Y = Z$. (8) yields

$$(9)\qquad h(Y,X)h(X,SV)SY - h(X,Y)h(SY,X)SV = 0$$

for every Y and V. Therefore

$$(10)\qquad h(X,Y)h(SY,X) = 0$$

for every Y. Now there exists a basis $e_1, \cdots, e_n$ of T_xM^n such that $h(X,e_i) \neq 0$ for every $i = 1, \cdots, n$, as we see from the following argument. Let $< X >$ be the space spanned by X and $< X >^*$ the space of all vectors that are h-orthogonal to X. Then $\dim < X >^* = n-1$ and $X \in < X >^*$. Let $e_1 \notin < X >^*$ and let $< X >'$ be an algebraic complement to $< X >$ in $< X >^*$. Then $< X > \bigoplus < X >' \bigoplus < e_1 > = T_xM^n$. Let $\overline{e}_2, \cdots, \overline{e}_{n-1}$ be a basis of $< X >'$. Set $\overline{e}_n = X$. Then $\{e_1, e_2 = e_1 + \overline{e}_2, \cdots, e_n = e_1 + \overline{e}_n\}$ is a basis satisfying the desired condition.

By substituting vectors of this basis as Y into (10) we get

$$(11)\qquad h(SY,X) = 0 \text{ for every } Y \in T_xM^n \text{ and for every } X \in \ker S.$$

Let us go back to (8) and set $X = Z$. By using also (11) we obtain

$$h(X,V)h(X,W)S^2Y = 0$$

for every Y, V and W. Take $V = W$ so that $h(X,V) \neq 0$. Then $S^2Y = 0$. Hence $S^2 = 0$ on $T_x(M^n)$. Using now once again (8), (11), we get

$$\begin{aligned} (12) \qquad & [h(X,V)h(SY,W) + h(X,W)h(SY,V)]SZ \\ & -[h(X,Z)h(SY,W) + h(X,W)h(SY,Z)]SV = 0 \end{aligned}$$

for every Y, Z, W, V. Let $e_1, \cdots, e_n$ be a basis of T_xM^n such that $h(e_i, X) \neq 0$ for every $i = 1, \cdots, n$ and let V be an arbitrary vector of this basis.

If $SV \neq 0$, then there is a vector Z of the basis such that SV and SZ are linearly independent. By substituting these V and Z into (12) and setting $W = V$ we get $h(X,V)h(SY,V) = 0$. Since $h(X,V) \neq 0$, we have $h(SY,V) = 0$. If $SV = 0$, then we still have $h(SY,V) = 0$ because of (11). Hence for every $V \in T_xM^n$ we have

$$(13) \qquad h(SY,V) = 0.$$

It follows that S vanishes on T_xM^n, which fact is impossible in the case under consideration. Thus Case I does not occur.

Case II: $h|_{\ker S}$ is not identically 0. There is an $X \in \ker S$ such that $h(X,X) \neq 0$. By putting such X into (8) and setting $Z = X$ we obtain

$$\begin{aligned} (14) \qquad 0 = & h(X,W)h(X,SV)SY \\ & +[h(X,X)h(V,W) - h(X,V)h(X,W)]S^2Y \\ & -h(X,X)h(SY,W) + h(X,W)h(SY,X)]SV, \end{aligned}$$

for every Y, V, W. Take any $Y \notin \ker S$. Since $n \geq 3$ and h is nondegenerate, there exist $W \neq 0$ and V such that $h(X,W) = 0, h(SY,W) = 0$ and $h(V,W) \neq 0$. Then by (14) we get $S^2Y = 0$. Thus

$$(15) \qquad S^2 = 0 \text{ on } T_xM^n.$$

Take now an arbitrary Y and $V = Y$. By (14) and (15) we get $h(SY,W)SY = 0$ for every W. Then $SY = 0$. Consequently, S vanishes on $T_x(M^n)$. This completes the proof of Lemma 1.

We now prove Theorem 1. From the Gauss equation and from rank $h \geq 2$ we see

$$(16) \qquad \text{im}S_x = \text{im}R_x = \text{span}\{R(X,Y)Z; X, Y, Z \in T_x(M^n)\}.$$

Since R is parallel relative to ∇ by assumption, so is $\text{im}S_x$. Thus dim $\text{im}S_x$ is constant on M^n. By Lemma 1, it follows that either S vanishes on M^n (and thus ∇ is flat) or S is nonsingular at every point $x \in M^n$. Assume the second alternative. Let X, Y, Z be mutually h-orthogonal and set $V = X, W = Y$. Using (7) we obtain

$$(17) \qquad 0 = [h(X,X)h(SY,Y) - h(Y,Y)h(SX,X)]SZ + h(Y,Y)h(SX,Z)SX.$$

Let $e_1, \cdots, e_n$ be an h-orthonormal basis of $T_x(M^n)$ and let X, Y, Z be distinct vectors of the basis. Since S is nonsingular, (17) implies the equality $h(SX, Z) = 0$. Hence for every vector X of the basis, the vector SX is parallel to X. Since the basis is arbitrary, it follows that S is a multiple of the identity: $S = \rho\,\mathrm{id}$ for some nowhere-vanishing function ρ. Using the Ricci equation we get $d\tau = 0$. Therefore for every $x \in M^n$ there is a neighborhood U of x and a function ψ on U such that $d\psi = \tau|_U$. Let $\overline{\xi} = e^{-\psi}\xi|_U$. It is easily seen that $\overline{\xi}$ is equiaffine and induces the same connection ∇ on U. Let $\overline{h}, \overline{S}$ correspond to ξ as in (1) and (2). By applying all the arguments above to the objects induced by $\overline{\xi}$ and using Codazzi's equation for S we obtain $\overline{S} = \overline{\rho}\,\mathrm{id}$, where $\overline{\rho}$ is a constant. The assumption $\nabla R = 0$ and the Gauss equation yield

$$0 = (\nabla_V R)(X, Y)Z = \overline{\rho}[\nabla\overline{h}(V, Y, Z) - \nabla\overline{h}(V, X, Z)Y].$$

Hence $\nabla\overline{h} = 0$. Since ξ and $\overline{\xi}$ have the same direction, it follows that ∇ is the Blaschke connection on M^n. Therefore form the beginning fo the proof we could take ξ to be the affine normal. Now using the classical theorem of Pick-Berwald, we conclude the proof of Theorem 1.

Remark. For $n = 2$, we can show that the condition $R \cdot R = 0$ is equivalent to the symmetry of the Ricci tensor of ∇.

2. Key lemmas.

In this section we prove two key lemmas which will reduce the question to the nondegenerate case. We denote the null space (i.e. kernel) of h by T^0. We start with

Lemma 2. *Under the assumption $R(X, Y) \cdot R = 0$ we have $S(T^0) \subset T^0$.*

Proof. Let $Z \in T^0$. By (7) we get

$$\begin{aligned} (18) \quad 0 &= (R(Z, Y) \cdot R)(Z, V)Y \\ &= h(V, Y)h(Y, SZ)SZ - [h(Y, V)h(SZ, Y) + h(Y, Y)h(SZ, V)]SZ \\ &= -h(Y, Y)h(SZ, V)SZ. \end{aligned}$$

If $SZ = 0$, then $SZ \in T^0$. Hence we can assume that $SZ \neq 0$. Then by (18) we have $h(Y, Y)h(SZ, V) = 0$ for every Y and V. When $h \neq 0$, there exists Y such that $h(Y, Y) \neq 0$. Hence $h(SZ, V) = 0$ for all V, showing that $SZ \in T^0$ and completing the proof of Lemma 2.

Next we prove

Lemma 3. *If $\nabla R = 0$ and if rank h is constant and ≥ 2, then $S = 0$ on M^n or T^0 is $\nabla-$parallel.*

Proof. From the Gauss equation we get

$$\begin{aligned} (19) \quad 0 &= (\nabla_W R)(X, Y)Z \\ &= \nabla h(W, Y, Z)SX - \nabla h(W, X, Z)SY \\ &\quad + h(Y, Z)\nabla S(W, X) - h(X, Z)\nabla S(W, Y), \end{aligned}$$

for all X, Y, Z, W. Since $\nabla R = 0$ and rank $h \geq 2$, it follows that rank S is constant on M^n, because im $S =$ im R is $\nabla-$parallel. We observe that T^0 is parallel if and only if

(20) $\nabla h(X, Y, Z) = 0$ for every $Z \in T^0$ and for arbitrary X, Y.

It is also trivial that $(\nabla h)(X, Y, Z) = 0$ provided $Y, Z \in T^0$.

Now let $Z \in T^0$. Then by (21) we get

(21) $0 = \nabla h(W, Y, Z)SX - \nabla h(W, X, Z)SY$ for every X, Y, W.

If rank $S \geq 2$, then (21) gives condtion (20).

From now on assume that rank $S = 1$. Let $X, Z \in T^0$. By (21) we have $\nabla h(W, Y, Z)SX = 0$ for every W, Y. If there is $X \in T^0$ such that $SX \neq 0$, then $\nabla h(W, Y, Z) = 0$ for all W, Y, that is, we have (20). Thus remains the case where $T^0 \subset \ker S$.

In this case, let $\{e_1, \cdots, e_n\}$ be a basis of $T_x M^n$ such that $\{e_1, \cdots, e_k\}$ is a basis of T^0 and $\{e_1, \cdots, e_k, \cdots, e_{n-1}\}$ is a basis of $\ker S$. Since rank $h \geq 2$ we can assume that $h(e_{n-1}, e_{n-1}) \neq 0$. The vector $e_n \notin \ker S$ can be chosen so that the form $\alpha : v \in \ker S \rightarrow h(v, e_n)$ is not 0. If it is, then we can replace e_n by $e_n + e_{n-1}$ and get

$$h(e_{n-1}, e_n + e_{n-1}) = h(e_{n-1}, e_{n-1}) \neq 0.$$

Since α is surjective, there is an $X \in \ker S$ such that

(22) $$h(X, e_n) = h(e_n, e_n).$$

Take $j \leq k$. By (19) we have

$$\begin{aligned} 0 &= (\nabla_{e_n} R)(e_j, X - e_n)e_n \\ &= \nabla h(e_n, X - e_n, e_n)Se_j - \nabla h(e_n, e_j, e_n)S(X - e_n) \\ &\quad + h(X - e_n, e_n)\nabla S(e_n, e_j) - h(e_j, e_n)\nabla S(e_n, X - e_n) \\ &= \nabla h(e_n, e_j, e_n)Se_n - (h(X, e_n) - h(e_n, e_n))\nabla S(e_n, e_j) \\ &= \nabla h(e_n, e_j, e_n)Se_n, \end{aligned}$$

since $Se_j = 0, h(e_j, e_n) = 0$, and $h(X, e_n) - h(e_n, e_n) = 0$ by (22). Thus we obtain

(23) $\nabla h(e_n, Z, e_n) = 0$ for all $Z \in T^0$.

If $Z \in T^0$ and $X \in \ker S$, then by (21) we have $\nabla h(W, X, Z)SY = 0$ for every Y, W. Since rank $S > 0$, we get $\nabla h(W, X, Z) = 0$ for every W. Hence we have

(24) $\nabla h(W, X, Z) = 0$ for $Z \in T^0, X \in \ker S$, and for arbitrary W.

Note that $\nabla h(X, W, Z) = \nabla h(W, X, Z)$ for $Z \in T^0$ because, by Codazzi's equation for h, we have

$$\nabla h(W,X,Z) - \nabla h(X,W,Z) = h(W,Z)\tau(X) - h(X,Z)\tau(W) = 0.$$

Now combining (24) and (23) we get (20), completing the proof of Lemma 3.

3. Proof of Theorem 2.

Since Theorem 2 is about points of Ω, we may assume that the rank of h is constant. Suppose S is not identically 0. Then by Lemma 3 we see that T^0 is parallel. Let s be $\dim T^0$. By the theorem in [N-O] we may obtain a local cylinder representation $M^n = M^r \times W$, where W is an open subset of an affine subspace R^s of R^{n+1} and M^r is a nondegenerate hypersurface in an affine subspace R^{r+1} transversal to R^s, where $r + s = n$. Take $\overline{\xi}$ to be the projection of ξ as in [N-O] and consider its restriction to M^r as transversal vector field to M^r in R^{r+1}. By virtue of Lemma 2, we see from the proposition in [N-O] that the connection $\overline{\nabla}$ induced on M^r by $\overline{\xi}$ is locally symmetric. Now by Theorem 1 we conclude the proof of Theorem 2.

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