HAMILTONIAN STRUCTURE OF NONABELIAN TODA LATTICE

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1. Introduction

In this paper we answer affirmatively the question posed in [RS] on the Liouville complete integrability of the finite non-periodic nonabelian Toda lattice. The nonabelian Toda lattice

$$\begin{aligned} \dot{A}_k &= B_{k+1}A_k - A_kB_k \\ \dot{B}_k &= A_k - A_{k-1} \end{aligned} \tag{1.1}$$

where A_k, B_k are $n\times n$ matrices and A_k are invertible, was introduced by Polyakov as a discretization of the principal chiral field equation. In the double-infinite case for a suitable class of initial data it was solved via the inverse scattering method in [Br MLR]. Krichever [Kr] found the solution in theta-functions for the periodic nonabelian Toda lattice. In this case the complete integrability in the Liouville sense was also established (see [RS]). The direct and inverse spectral problems developed in [B1] for second-order self-adjoint difference expressions with operator coefficients were used in [BGSh], [BG] to extend the method proposed by Berezansky in [B2] for solving semi-infinite scalar Toda lattice to the case of nonabelian equations similar to (1.1) associated with semi-infinite self-adjoint Jacobi matrices with operator coefficients. A generalization of the inverse spectral problem to the case of non-symmetric difference operators with operator coefficients was applied in [G1-G3] to integrate the nonabelian Toda lattice and some of its reductions in the semi-infinite and finite non-periodic cases. Another approach, based on a factorization of ordinary differential operators with noncommuting coefficients, was applied in [EGR] to integrate both the finite nonabelian Toda lattice and its two-dimensional generalization.

Let $\mathbf{1}$ be the $n\times n$ identity matrix. The treatment of (1.1) is based on its Lax representation

$$\dot{L} = [L, L_-], \tag{1.2}$$

where in the finite non-periodic case we shall be concerned with, L is a block tri-diagonal matrix

$$L = \begin{pmatrix} B_0 & \mathbf{1} & & 0 \\ A_0 & \ddots & \ddots & \\ & \ddots & \ddots & \mathbf{1} \\ 0 & & A_{N-1} & B_N \end{pmatrix} \tag{1.3}$$

and L_- is its block lower triangular part. In (1.1) this corresponds to letting k run from 0 to N and setting boundary conditions $A_{-1} = A_N = 0$. In sections 2 and 3 we review, following [G1, G2], essential steps of the inverse problem method for solving (1.2). In particular, we define the $n \times n$ matrix rational function

$$M(\lambda) = P^{-1}(\lambda)Q(\lambda) = \tilde{Q}(\lambda)\tilde{P}^{-1}(\lambda) \tag{1.4}$$

that determines L uniquely and allows one to linearize (1.2). Here $P(\lambda) = \lambda^{N+1}\mathbf{1} + \dots, Q(\lambda) = \lambda^N\mathbf{1} + \dots, \tilde{P}(\lambda) = \lambda^{N+1}\mathbf{1} + \dots, \tilde{Q}(\lambda) = \lambda^N\mathbf{1} + \dots$ are $n \times n$ matrix polynomials.

The Hamiltonian description of (1.2) was first given by Reiman in [R] in terms of the Hamiltonian reduction. In the beginning of section 4 we derive this description following the Adler-Kostant-Symes formalism and describe generic symplectic leaves of the affine space of matrices of the form (1.3) w.r.t. the corresponding Poisson structure. Then we compute the induced Poisson structure on rational matrix functions (1.4) and use it to construct a maximal family of Poisson-commuting independent integrals of the nonabelian Toda lattice.

Notations. In what follows a block matrix means a matrix whose entries are $n \times n$ blocks. C_{jk} denotes a (j,k) block entry of a block matrix C, while $e_{\alpha\beta}$ denotes the $n \times n$ matrix $(\delta_{\mu\alpha}\delta_{\nu\beta})_{\mu,\nu=1}^n$.

The gradient ∇f of a function on matrices is always defined via the trace-form. For a rational matrix function $W(\lambda), (W(\lambda))_+$ is its polynomial part and $(W(\lambda))_- = W(\lambda) - (W(\lambda))_+$. For a function $W(\lambda,\mu)$, $(W(\lambda,\mu))_{+,\lambda}$ denotes its polynomial part in λ. We denote by ϵ the block strictly upper triangular part of L:

$$\epsilon = \begin{pmatrix} 0 & \mathbf{1} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \mathbf{1} \\ 0 & & & 0 \end{pmatrix} \tag{1.5}$$

2. Inverse problem data

With each block-tridiagonal matrix (1.3) associated is the so-called *Weyl function* (cf. [B1], chap. VII)

$$M(\lambda) = \Delta_0^T R_\lambda(L)\Delta_0, \tag{2.1}$$

where $R_\lambda(L) = (L - \lambda\mathbf{1})^{-1}$ and Δ_0 is the $n(N+1) \times n$ matrix $\Delta_0 = \mathrm{col}[\mathbf{1}, 0, \dots, 0]$. In other words, $M(\lambda)$ is the upper left block entry of the resolvent $R_\lambda(L)$. Clearly, $M(\lambda)$ is a rational matrix-valued function whose Laurent expansion has a form

$$M(\lambda) = -\sum_{k=0}^{\infty} \frac{1}{\lambda^{k+1}} h_k, h_0 = \mathbf{1}. \tag{2.2}$$

Denote by H_k a block-Hankel matrix $(h_{i+j})_{i,j=0}^k$. The following theorem along with its semi-infinite version can be found in [G1, G2, Sh].

Theorem 2.1. *There is a 1-to-1 correspondence between matrices (1.3) with invertible $A_k, k = 0, \dots, N-1$ and $n \times n$ matrix-valued functions (2.2) satisfying conditions*

(i)

$$\det H_k \neq 0, k = 0, \dots, N \tag{2.3}$$

(ii) *there exist $F_0, \dots, F_N$ such that*

$$h_{k+N+1} = \sum_{j=0}^{n} F_j h_{k+j}, k = 0, 1, \dots \tag{2.4}$$

Effective procedures for solving the inverse problem of restoring L from its Weyl function $M(\lambda)$ or, equivalently from the *moment sequence* $(h_j)_{j=0}^{\infty}$ were proposed in [G1, G2, Sh].

Note that (2.4) is equivalent to the statement that $Q(\lambda) = P(\lambda)M(\lambda)$, where $P(\lambda) = \lambda^{N+1}\mathbf{1} - \sum_{j=0}^{N} F_; \lambda^j$, is a polynomial in λ of degree N. Given L with all A_k invertible, polynomials $P(\lambda), Q(\lambda)$ can be constructed as follows. Consider a truncated eigenvalue problem

$$(L - \lambda\mathbf{1}) \operatorname{col}[P_0(\lambda), \dots P_N(\lambda)] = \operatorname{col}[0, \dots, 0, x], \tag{2.5}$$

where $P_0(\lambda) = \mathbf{1}$. This problem has a solution if and only if $x = -P(\lambda)$. It follows that

$$\det(L - \lambda\mathbf{1}) = \det(P(\lambda)). \tag{2.6}$$

Coefficients of $P(\lambda)$ can be found from (2.5) recursively. In particular, $F_N = B_0 + \dots + B_N$.

$Q(\lambda)$ can be constructed in the same way as $P(\lambda)$ from the matrix L, obtained by deleting the first row and the first column from L. Thus, one has a factorization

$$M(\lambda) = P^{-1}(\lambda)Q(\lambda) \tag{2.7}$$

Similarly, $M(\lambda)$ can be factorized into

$$M(\lambda) = \tilde{Q}(\lambda)\tilde{P}^{-1}(\lambda), \tag{2.8}$$

where $\tilde{P}, \tilde{Q}$ are monic matrix polynomials of degree $N+1$ and N resp. To obtain (2.8), one can consider $\tilde{L} = D^{-1}LD$, where

$$D = \operatorname{diag}(\mathbf{1}, A_0, A_1A_0, \dots, A_{n-1} \cdots A_0) \ .$$

Then $\tilde{L}$ is a block tri-diagonal matrix with unit blocks under the diagonal and $\Delta_0^T R_\lambda(\tilde{L})\Delta_0 = \Delta_0^T R_\lambda(\lambda)\Delta_0 = M(\lambda)$. Similar to (2.5), $\tilde{P}(\lambda)$ can be found from

$$[\tilde{P}_0(\lambda), \dots, \tilde{P}_N(\lambda)](\tilde{L} - \lambda\mathbf{1}) = [0, \dots, 0, -\tilde{P}(\lambda)], \tag{2.9}$$

where $\tilde{P}_0(\lambda) = \mathbf{1}$. It is important to note, that

$$\tilde{P}_0(\lambda) = \lambda^{N+1}\mathbf{1} + \left(B_0 + A_0^{-1}B_1A_0 + \ldots + (A_0\cdots A_{N-1})^{-1}B_NA_0\cdots A_{N-1}\right)\lambda^N + \ldots \tag{2.10}$$

Polynomials $P(\lambda), Q(\lambda); \tilde{P}(\lambda), \tilde{Q}(\lambda)$ in (2.7), (2.8) are defined uniquely and thus, any pair $(P(\lambda), Q(\lambda))$, (resp. $(\tilde{P}(\lambda), \tilde{Q}(\lambda))$ such that $M(\lambda)$ defined by (2.7) (resp. (2.8)) satisfies the nondegeneracy condition (2.3), determines unambiguously block-tridiagonal matrix L.

To conclude this section we formulate the following corollary of conditions (2.30), (2.40) that will be useful later.

Lemma 2.2. *Let $S(\lambda)$ be a polynomial of degree m such that $(S(\lambda)M(\lambda))_- = 0$. Then there exists a polynomial $S_1(\lambda)$. such that $S(\lambda) = S_1(\lambda)P(\lambda)$. In particular, if $m < N+1$ then $S(\lambda) = 0$.*

Proof. Since $P(\lambda)$ is monic, we can write $S(\lambda) = r(\lambda) + S_1(\lambda)P(\lambda)$, where $r(\lambda)$ is a polynomial of degree less than $N+1$. Then $(r(\lambda)M(\lambda))_- = 0$, which implies $[r_0, \ldots, r_k]H_k = 0$, where we assumed $r(\lambda) = r_k\lambda^k + \ldots + r_0$. Since $k < N+1$, this contradicts (2.3). ■

3. Evolution equations

As it was mentioned in the Introduction the nonabelian Toda lattice has a Lax representation (1.2). Consider a general Lax equation

$$\dot{L} = [L, C], \tag{3.1}$$

that preserves the tridiagonal form (1.3) of L. This condition imposes strong restrictions on the auxiliary matrix A. Namely, it was shown in [G1], that C is completely determined by its block entries $C_{0i}, i = 0, \ldots, N$ and $C_{j0}, j = 1, \ldots, N$. More precisely, we have the following reccurence relations for C_{jk}:

$$\begin{aligned} C_{j+1,k+j+1} = C_{j,k+j} &+ C_{j,k+1+j}B_{j+k+1} - B_jC_{j,k+1+j} \\ &+ C_{j,k+2+j}A_{j+k+1} - A_{j-1}C_{j-1,k+1+j} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} C_{j,j-k} = (A_{j-1}C_{j-1,j-k-1} &+ B_jC_{j,j-k-1} + C_{j+1,j-1-k} - C_{j,j-k-2} \\ &- C_{j,j-k-1}B_{j-k-1})A_{j-k-1}^{-1}. \end{aligned} \tag{3.3}$$

In (3.2) k runs from N to 0 and j from 0 to $n-k$, while in (3.3) k runs from N to 1 and j from k to N.

In many instances equation (3.1) can be explicitly integrated via the inverse problem outlined in the previous section. Indeed, (2.1) and (2.2) imply that

$$h_k = (L^k)_{00}\ ,$$

which justifies the term *moments* for h_k. Then, one can show ([G1-G3]) that the Lax equation (3.1) forces the following evolution of h_k.

Proposition 3.1. *Let*

$$\begin{aligned}\psi_+(\lambda) &= \sum_{j=0}^{N} C_{0,j} P_j(\lambda) := \sum_{j=0}^{N} \psi_j \lambda^j, \\ \psi_-(\lambda) &= \sum_{j=0}^{N} \tilde{P}_k(\lambda)(A_{k-1}\dots A_0)^{-1} C_{k,0} := \sum_{j=0}^{N} \psi_{-j}\lambda^j,\end{aligned} \tag{3.4}$$

where $P_j(\lambda), \tilde{P}_k(\lambda)$ are defined by (2.5), (2.9). If L obeys (3.1) then the evolution of h_k is given by

$$\dot{h}_k = \sum_{j=0}^{N} (h_{k+j}\psi_{-j} - \psi_j h_{k+j}). \tag{3.5}$$

In the particular case of the nonabelian Toda lattice the only non-zero element among A_{0j}, A_{k0} is $A_{10} = A_0$. Then $\psi_+(\lambda) = 0, \psi_-(\lambda) = \tilde{P}_1(\lambda) = \lambda\mathbf{1} - B_0$ and we have

$$\dot{h}_k = h_{k+1} - h_k h_1 \tag{3.6}$$

Here we used the equality $B_0 = (L)_{00} = h_1$.

Proposition 3.2 ([G2, G3]). *Let h_k involves according to (3.60) and let $F_0, \dots F_N$ be defined as in (2.4). Then*

(i) $\dot{F}_k = 0, k = 0, \dots, N$
(ii) *solution to (3.6) is given by $h_k = \tilde{h}_k \tilde{h}_0^{-1}$, where the matrix $\tilde{H}_N = (\tilde{h}_{i+j})_{i,j=0}^N$ solves the linear equation with constant coefficients*

$$\dot{\tilde{H}}_N = \begin{pmatrix} 0 & \mathbf{1} & & 0 \\ \vdots & & \ddots & \\ 0 & & & \mathbf{1} \\ F_0 & & \dots & F_N \end{pmatrix} \tilde{H}_N.$$

Proposition 3.2 shows how to linearize the nonabelian Toda lattice. But what is more important in the context of the present paper, it provides us with a large number of first integrals given by matrix elements of the polynomial $P(\lambda)$ defined in (2.7). However, as it will be seen below, these integrals are not sufficient to prove the complete integrability of the nonabelian Toda lattice in the Liouville sense.

Proposition 3.3. *Coefficients of the polynomial* $\det(\tilde{P}(\lambda) + z\mathbf{1})$ *are constant along trajectories of the nonabelian Toda flow.*

Proof. Let $\tilde{P}(\lambda) = \lambda^{N+1} - \sum_{j=0}^{N} \tilde{F}_j \lambda^j$. Then, similar to (2.4) we have

$$h_{k+N+1} = \sum_{j=0}^{N} h_{k+j}\tilde{F}_j, k = 0, 1, \dots$$

Then (3.6) implies

$$\sum_{j=0}^{N} h_{k+j}\dot{\tilde{F}}_j = h_{k+N+2} - h_{k+N+1}h_1 - \sum_{j=0}^{N}(h_{k+j+1} - h_{k+j}h_1)\tilde{F}_j$$
$$= \sum_{j=0}^{N} h_{k+j}[h_1, \tilde{F}_j].$$

Considering $k = 0, \dots, N$ and using the invertibility of $H_N = (h_{i+j})_{i,j=0}^{N}$, we obtain $\dot{\tilde{F}}_j = [h_1, \tilde{F}_j]$ or, equivalently, $\dot{\tilde{P}}(\lambda) = [h_1, \tilde{P}(\lambda)]$ and the statement follows. ■

4. Hamiltonian structure and compete integrability of the nonabelian Toda Lattice

Proposition 3.1 allows in many cases to linearize the Lax equation (3.1). However, (3.1) with an arbitrary matrix C satisfying (3.2), (3.3) is too general if we want to study the integrability of (1.2) in the Liouville sense. In particular, (3.1) does not even have to be Hamiltonian. In this section we shall derive the Poisson structure induced on the inverse problem data $\{M(\lambda); P(\lambda), Q(\lambda); \tilde{P}(\lambda), \tilde{Q}(\lambda)\}$ by that of the nonabelian Toda lattice and use it to find Hamiltonians generating a maximal commutative family of isospectral flows that commute with (1.2). This flows will necessarily have the Lax form.

In what follows it will be convenient to use the tensor notations for Poisson brackets of matrix elements of matrix-valued functions (see [FT, ch. 3, § 1]): for $n \times n$ matrices A and B , $\{A \overset{\otimes}{,} B\}$ is defined by

$$\{A \overset{\otimes}{,} B\}_{ij,i'j'} = \{A_{ii'}, B_{jj'}\}. \tag{4.1}$$

Following [FT], we list properties of (4.1) that will be used later:

$$\{A \overset{\otimes}{,} B\} = -\mathcal{P}\{B \overset{\otimes}{,} A\}\mathcal{P}, \tag{4.2}$$
$$\{A \overset{\otimes}{,} BC\} = \{A \overset{\otimes}{,} B\}(\mathbf{1} \otimes C) + (\mathbf{1} \otimes B)\{A \overset{\otimes}{,} C\}, \tag{4.3}$$
$$\{A^{-1} \overset{\otimes}{,} B\} = -(A^{-1} \otimes \mathbf{1})\{A \overset{\otimes}{,} B\}(A^{-1} \otimes \mathbf{1}). \tag{4.4}$$

Here $\mathcal{P}$ is the permutation operator in $\mathbb{C}^n \otimes \mathbb{C}^n : \mathcal{P}(x \otimes y) = y \otimes x$. Note also that $\mathcal{P}$ satisfies the identity

$$\mathcal{P}(A \otimes B) = (B \otimes A)\mathcal{P} \tag{4.5}$$

Proposition 4.1. *The nonabelian Toda lattice (1.1) is generated by the Hamiltonian*

$$H = \frac{1}{2}Tr(L^2) = Tr(\sum_{j=0}^{N-1} A_j + \frac{1}{2}\sum_{j=0}^{N} B_j^2)$$

in the Poisson structure given by

$$\begin{aligned} &\{A_j \overset{\otimes}{,} A_k\} = 0, \{B_j \overset{\otimes}{,} B_k\} = \delta_{jk}[\mathbf{1} \otimes B_j, \mathcal{P}], \\ &\{A_j \overset{\otimes}{,} B_k\} = \mathcal{P}(\delta_{j+1,k}(A_j \otimes \mathbf{1}) - \delta_{jk}(\mathbf{1} \otimes A_j)) \end{aligned} \tag{4.6}$$

Proof. The Lax form (1.2) of the nonabelian Toda lattice suggests that it fits into the Adler-Kostant-Symes scheme (see [K, Sy]). Indeed, if we consider the decomposition

$$s\ell(n(N+1)) = g_+ + g_-, \tag{4.7}$$

where g_+ is a subalgebra of block-upper triangular matrices and g_- is a subalgebra of block-strictly lower triangular matrices with $n \times n$ blocks, then the identification

$$\epsilon + g_-^\perp \cong g_+^*, \tag{4.8}$$

where ϵ is as in (1.5) and an orthogonal complement is taken w.r.t. the trace-from, equips the affine space $\epsilon + g_-^\perp$ of block-lower Hessenberg matrices with a Poisson bracket obtained as a pull-back of the Kirillov-Kostant one on g_+^*:

$$\{f_1, f_2\}(X) = Tr(X, [\pi_+ \nabla f_1(X), \pi_+ \nabla f_2(X)]), \tag{4.9}$$

where $X \in \epsilon + g_-^\perp$ and π_+ is a projection onto g_+ parallel to g_-.

The Poisson brackets (4.6) follow from (4.9) and the fact that set of block-tridiagonal matrices is a Poisson submanifold in $\epsilon + g_-^\perp$. Then the statement in Proposition 1 is a direct consequence of the AKS theorem ([K,Sy]). ■

Remark. When $j = k$, the second equation in (4.6) coincides with the definition of the Lie-Poisson bracket on $g\ell(n)$.

Denote by J the set of block-tridiagonal elements in $\epsilon + g_-^\perp$. As it was mentioned above, J is a Poisson submanifold of $\epsilon + g_-^\perp$ equipped with the bracket (4.9). However, it is not symplectic: the Poisson structure (4.6) is degenerate on J. Symplectic leaves of (4.6) are orbits in J of the co-adjoint action of the group G_+ of invertible block-upper triangular matrices in $SL(n(N+1))$. We shall consider orbits through elements L in J satisfying the following two genericity conditions:

1.
$$A_j, j = 0, \ldots, N-1 \quad \text{are invertible.} \tag{4.10}$$

2. Matrix
$$\Lambda_L = B_0 + A_0^{-1} B_1 A_0 + \ldots + A_0^{-1} \cdots A_{N-1}^{-1} B_N A_{N-1} \cdots A_0 \tag{4.11}$$

   has distinct eigenvalues.

Proposition 4.2. *The orbit $\mathcal{O}_L$ of the co-adjoint action of G_+ through the element $L \in J$ satisfying conditions 1 and 2 has a dimension*

$$\dim \mathcal{O}_L = 2Nn^2 + n(n-1) \tag{4.12}$$

and is completely determined by eigenvalues of Λ_L.

Proof. Under the identification (4.8) the co-adjoint action of G_+ on $\epsilon + g_-^\perp$ takes the form

$$Ad^*_\beta X = \pi_{g_-^\perp}(Ad_\beta^{-1} X) + \epsilon. \tag{4.13}$$

Let $L \in J$ satisfies condition 1. We define a sequence of elements $\beta_0, \dots, \beta_{N-1} \in G_+$ as follows: $\beta_k = \mathbf{1} + E_{k,k+1} \otimes (\gamma_k A_k^{-1}), k = 0, \dots, N-1$, where $\gamma_k \in g\ell_n$ are given recursively by

$$\gamma_0 = B_0, \gamma_{k+1} = (B_{k+1} + A_k \gamma_k A_k^{-1}), k = 0, \dots, N-2.$$

Then one computes

$$Ad^*_{\beta_0 \dots \beta_{N-1}} L = \begin{pmatrix} 0 & \mathbf{1} & & 0 \\ A_0 & 0 & \ddots & \\ & \ddots & \ddots & \mathbf{1} \\ 0 & & A_{N-1} & \gamma_N \end{pmatrix} \tag{4.14}$$

where

$$\begin{aligned} \gamma_N &= B_N + A_{N-1}\gamma_{N-1}A_{N-1}^{-1} \\ &= A_{N-1} \dots A_0 B_0 A_0^{-1} \cdots A_{N-1}^{-1} + \dots + A_{N-1} B_{N-1} A_{N-1}^{-1} + B_N \\ &= (A_{N-1} \cdots A_0) \Lambda_L (A_{N-1} \cdots A_0)^{-1} \ . \end{aligned}$$

Acting further by

$$D = \mathrm{diag}(\mathbf{1}, A_0, A_1 A_0, \dots, A_{N1} \cdots A_0) \ ,$$

we see that

$$L_0 := Ad^*_{\beta_0 \dots \beta_{N-1} D} L = \begin{pmatrix} 0 & \mathbf{1} & & 0 \\ \mathbf{1} & 0 & \ddots & \\ & \ddots & \ddots & \mathbf{1} \\ 0 & & \mathbf{1} & \Lambda_L \end{pmatrix} \tag{4.15}$$

A stabilizer G_{L_0} of the element L_0 is a subgroup of block-upper triangular matrices with zero second diagonal and with a diagonal part of the form diag $(Y, Y, \dots, Y)$, where Y belongs to the subgroup $G_{\Lambda_L} \subset GL(n)$ stabilizing Λ_L under the adjoint action of $SL(n)$.

Furthermore, since the property (4.10) is preserved under the co-adjoint action of G_+, every element of $\mathcal{O}_L$ can be brought to the form (4.15). This, along with the fact that

$$\dim G_{L_0} = n^2 \cdot \frac{N(N-1)}{2} + \dim G_{\Lambda_L},$$

implies

$$\dim \mathcal{O}_L = 2N\, n^2 + \dim \mathcal{O}_{\Lambda_L},$$

where $\mathcal{O}_{\Lambda_L}$ is the orbit through Λ_L of the adjoint action of $SL(n)$. Then (4.12) follows from the condition 2. ∎

Remark. Though under certain conditions solutions of the nonabelian Toda lattice can be conjugated to a symmetric form [BG], we should emphasize the difference in geometric features of the nonabelian Toda lattice and symmetric Toda flows on block tri-diagonal matrices considered as a particular case of full symmetric Toda flows [DLNT]. In the latter case, symplectic leaves are orbits of the co-adjoint action of the upper Borel subgroup, condition (4.11) is not preserved by this action and the set of block-tridiagonal symmetric matrices satisfying (4.10) is stratified into many symplectic leaves whose type and dimension depend on types of Bruhat cells containing $A_k, k = 0, \dots, N-1$. On the other hand, *adjoint* invariants of the Borel subgroup are not necessarily constant along the flows of (1.2), so the parabolic chain construction of commuting integrals proposed in [DLNT] does not work for the nonabelian Toda lattice.

Moreover, Propositions 3.2, 3.3 suggest that one can use the inverse problem data defined by (2.1), (2.7), (2.8) to construct a maximal Poisson-commutative family of integrals.

Proposition 4.3. *The Poisson structure (4.6) induces the following Poisson brackets for matrix elements of $M(\lambda)$:*

$$\begin{aligned}\{M(\lambda) \overset{\otimes}{,} M(\mu)\} = &\frac{1}{\lambda-\mu}\left(M(\lambda)-M(\mu)\right)\otimes\left(M(\lambda)-M(\mu)\right)\mathcal{P}\\ &+\left(M(\lambda)M(\mu)\otimes M(\mu)-M(\lambda)\otimes M(\mu)M(\lambda)\right)\mathcal{P}\end{aligned} \tag{4.16}$$

Proof. By (2.1),

$$\delta M(\lambda) = -\Delta_0^T R_\lambda(L)\delta L R_\lambda(L)\Delta_0.$$

Therefore,

$$\nabla M_{jk}(\lambda) = -R_\lambda(L)\Delta_0 e_{kj}\Delta_0^T R_\lambda(L).$$

Recall that (4.6) is equivalent to (4.9), which we rewrite in the form

$$\begin{aligned}\{f_1,f_2\}(L) = Tr(L([\nabla f_1(L),\nabla f_2(L)] &- [\nabla F_1(L),\pi_-\nabla f_2(L)]\\ &-[\pi_-\nabla f_2(L),\nabla f_1(L)])).\end{aligned} \tag{4.17}$$

Firstly, we compute

$$\begin{aligned}Tr(L[\nabla M_{jk}(\lambda),\nabla M_{\alpha\beta}(\mu)]) &= Tr([L-\lambda\mathbf{1},\nabla M_{jk}(\lambda)]\nabla M_{\alpha\beta}(\mu))\\ &= -Tr\left(\left(\Delta_0 e_{kj}\Delta_0^T R_\lambda(L) - R_\lambda(L)\Delta_0 e_{kj}\Delta_0^T\right)\nabla M_{\alpha\beta}(M)\right)\end{aligned}$$

By virtue of the identity $R_\lambda(L)R_\mu(L) = \frac{1}{\lambda-\mu}(R_\lambda(L)-R_\mu(L))$, the last expression is equal to

$$\begin{aligned}&\frac{1}{\lambda-\mu}\left(M(\lambda)_{j\beta}M(\mu)_{\alpha k} - M(\lambda)_{\alpha k}M(\mu)_{j\beta}\right)\\ &\quad= \left[\frac{\mathcal{P}}{\lambda-\mu}, M(\lambda)\otimes M(\mu)\right]_{jk,\alpha\beta}.\end{aligned} \tag{4.18}$$

Secondly,

$$Tr(L[\nabla M_{jk}(\lambda), \pi_-\nabla M_{\alpha\beta}(M)]) =$$

$$Tr\left((\Delta_0 e_{kj}\Delta_0^T R_\lambda(L) - R_\lambda(L)\Delta_0 e_{kj}\Delta_0^T)\pi_-(R_\mu^{(L)}\Delta_0 e_{\beta\alpha}\Delta_0^T R_\mu(L))\right)$$

$$= Tr\left(e_{kj}\sum_{\nu=1}^{N}(R_\lambda(L))_{0\nu}(R_\mu(L))_{\nu 0}e_{\beta\alpha}M(\mu)\right)$$

$$= Tr\left(e_{kj}(\Delta_0^T R_\lambda(L)R_\mu(L)\Delta_0 - M(\lambda)M(\mu))e_{\beta\alpha}M(\mu)\right)$$

$$= Tr\left(\left(\frac{1}{\lambda-\mu}(M(\lambda)-M(\mu)) - M(\lambda)M(\mu)\right)e_{\beta\alpha}M(\mu)e_{kj}\right)$$

Similarly,

$$Tr\left([L, \nabla M_{\alpha\beta}(L)], \pi_-\nabla M_{jk}(\lambda)\right) =$$

$$Tr\left(\left(\frac{1}{\lambda-\mu}(M(\lambda)-M(\mu)) - M(\mu)M(\lambda)\right)e_{kj}M(\lambda)e_{\beta\alpha}\right).$$

Substituting back into (4.17) we obtain the desired formula (4.16). ■

Now we can use the uniqueness of the factorization $M(\lambda) = P^{-1}(\lambda)Q(\lambda) = \tilde{Q}(\lambda)\tilde{P}^{-1}(\lambda)$ in order to derive the following

Proposition 4.4. *The Poisson brackets for matrix elements of $P(\lambda), Q(\lambda), \tilde{Q}(\lambda)$, $\tilde{P}(\lambda)$ are given by equations*

$$\{P(\lambda) \overset{\otimes}{,} P(\mu)\} = \left[\frac{1}{\lambda-\mu}\mathcal{P}, P(\lambda)\otimes P(\mu)\right] \tag{4.19}$$

$$\{Q(\lambda) \overset{\otimes}{,} Q(\mu)\} = \left[\frac{1}{\lambda-\mu}\mathcal{P}, Q(\lambda)\otimes Q(\mu)\right] \tag{4.20}$$

$$\{P(\lambda) \overset{\otimes}{,} Q(\mu)\} = \frac{1}{\lambda-\mu}(Q(\mu)\otimes P(\lambda) - Q(\lambda)\otimes P(\mu))\mathcal{P} - Q(\lambda)\otimes Q(\mu)\mathcal{P} \tag{4.21}$$

$$\begin{aligned}\{\tilde{P}(\lambda) \overset{\otimes}{,} \tilde{P}(\mu)\} = &\left[\tilde{P}(\lambda)\otimes\tilde{P}(\mu), \frac{1}{\lambda-\mu}\mathcal{P}\right] + [\tilde{P}(\lambda)\otimes 1, (\tilde{Q}(\mu)\otimes 1)\mathcal{P}]\\ &- [1\otimes\tilde{P}(\mu), (1\otimes\tilde{Q}(\lambda))\mathcal{P}]\end{aligned} \tag{4.22}$$

$$\{\tilde{Q}(\lambda) \overset{\otimes}{,} \tilde{Q}(\mu)\} = \left[\tilde{Q}(\lambda)\otimes\tilde{Q}(\mu), \frac{\mathcal{P}}{\lambda-\mu}\right] \tag{4.23}$$

$$\{\tilde{P}(\lambda) \overset{\otimes}{,} \tilde{Q}(\mu)\} = \frac{1}{\lambda-\mu}(\tilde{P}(\lambda)\otimes\tilde{Q}(\mu) - \tilde{P}(\mu)\otimes\tilde{Q}(\lambda))\mathcal{P} - \mathbf{1}\otimes\tilde{Q}(\mu)\tilde{Q}(\lambda)\mathcal{P} \tag{4.24}$$

$$\{P(\lambda) \overset{\otimes}{,} \tilde{P}(\mu)\} = [(Q(\lambda)\otimes\mathbf{1})\mathcal{P}, \mathbf{1}\otimes\tilde{P}(\mu)] \tag{4.25}$$

$$\{P(\lambda) \overset{\otimes}{,} \tilde{Q}(\mu)\} = \left(\frac{1}{\lambda-\mu}(P(\lambda)\tilde{Q}(\mu) - Q(\lambda)\tilde{P}(\mu))\otimes\mathbf{1} - Q(\lambda)\otimes\tilde{Q}(\mu)\right)\mathcal{P} \tag{4.26}$$

$$\{Q(\lambda) \overset{\otimes}{,} \tilde{P}(\mu)\} = \left(\frac{1}{\lambda-\mu}(Q(\lambda)\tilde{P}(\mu) - P(\lambda)\tilde{Q}(\mu))\otimes 1 + Q(\lambda)\tilde{Q}(\mu)\otimes\mathbf{1}\right)\mathcal{P} \tag{4.27}$$

$$\{Q(\lambda) \overset{\otimes}{,} \tilde{Q}(\mu)\} = 0 \tag{4.28}$$

Proof. Consider $\{Q(\lambda) \overset{\otimes}{,} M(\mu)\} = -\{P(\lambda)M(\lambda) \overset{\otimes}{,} M(\mu)\}$. This is a polynomial in λ, therefore, by (4.3), one can see that

$$
\begin{aligned}
&((P(\lambda)\otimes \mathbf{1})\{M(\lambda) \overset{\otimes}{,} M(\mu)\} + \{P(\lambda)\otimes M(\mu)\} \\
&\quad + \{P(\lambda) \overset{\otimes}{,} M(\mu)\}(M(\lambda)\otimes \mathbf{1}))_{-,\lambda} = 0.
\end{aligned}
\tag{4.29}
$$

Due to (4.16),

$$
\begin{gathered}
((P(\lambda)\otimes \mathbf{1})\{M(\lambda) \overset{\otimes}{,} M(\mu)\})_{-,\lambda} = \\
\left((P(\lambda)\otimes \mathbf{1})\left(\frac{1}{\lambda-\mu}(M(\lambda)-M(\mu))\otimes \mathbf{1} - M(\lambda)\otimes M(\mu)\right)\mathcal{P}(M(\lambda)\otimes 1)\right)_{-,\lambda}.
\end{gathered}
$$

Thus, (4.29) can be rewritten as $(X(\lambda,\mu)\cdot(M(\lambda)\otimes\mathbf{1}))_{-,\lambda} = 0$, where

$$
\begin{aligned}
X(\lambda,\mu) &= \{P(\lambda) \overset{\otimes}{,} M(\mu)\} \\
&\quad + (P(\lambda)\otimes \mathbf{1})\left(\frac{1}{\lambda-\mu}(M(\lambda)-M(\mu))\otimes \mathbf{1} - M(\lambda)\otimes M(\mu)\right)\mathcal{P}
\end{aligned}
$$

is a polynomial in λ of degree N. But then, due to Lemma 2.2, $X(\lambda,\mu) = 0$ and we have

$$
\{P(\lambda) \overset{\otimes}{,} M(\mu)\} = -(P(\lambda)\otimes \mathbf{1})\left(\frac{1}{\lambda-\mu}(M(\lambda)-M(\mu))\otimes \mathbf{1} - M(\lambda)\otimes M(\mu)\right)\mathcal{P}.
\tag{4.30}
$$

Similarly, $\{P(\lambda) \overset{\otimes}{,} P(\mu)M(\mu)\} = -\{P(\lambda) \overset{\otimes}{,} Q(\mu)\}$ is polynomial in both λ and μ, so

$$
\begin{aligned}
0 &= (\{P(\lambda) \overset{\otimes}{,} P(\mu)\}(1\otimes M(\mu)) + (1\otimes P(\mu))\{P(\lambda) \overset{\otimes}{,} M(\mu)\})_{-,\mu} \\
&= \left(\{P(\lambda) \overset{\otimes}{,} P(\mu)\}(1\otimes M(\mu)) + \frac{1}{\lambda-\mu}(P(\lambda)\otimes P(\mu))(M(\mu)\otimes \mathbf{1})\mathcal{P}\right)_{-,\mu} \\
&= \left((\{P(\lambda) \overset{\otimes}{,} P(\mu)\} + \frac{1}{\lambda-\mu})P(\lambda)\otimes P(\mu) - P(\mu)\otimes P(\lambda))\mathcal{P})(\mathbf{1}\otimes M(\mu))\right)_{-,\mu}
\end{aligned}
$$

and, again by Lemma 2.2,

$$
\{P(\lambda) \overset{\otimes}{,} P(\mu)\} = \frac{1}{\lambda-\mu}(P(\lambda)\otimes P(\mu) - P(\mu)\otimes P(\lambda))\mathcal{P} = \left[\frac{1}{\lambda-\mu}\mathcal{P}, P(\lambda)\otimes P(\mu)\right],
$$

which proves (4.19).

Equation (4.21) is obtained by substituting (4.30), (4.19) into

$$
\{P(\lambda) \overset{\otimes}{,} Q(\mu)\} = -\{P(\lambda) \overset{\otimes}{,} P(\mu)\}(1\otimes M(\mu)) - (1\otimes P(\mu))\{P(\lambda) \overset{\otimes}{,} M(\mu)\},
$$

whereas (4.20) can be derived from the identity

$$\{Q(\lambda) \stackrel{\otimes}{,} Q(\mu)\} = (P(\lambda)\otimes P(\mu))(\{M(\lambda) \stackrel{\otimes}{,} M(\mu)\} - \{P^{-1}(\lambda) \stackrel{\otimes}{,} Q(\mu)\}(Q(\lambda)\otimes 1)$$
$$-(1\otimes Q(\mu))\{Q(\lambda) \stackrel{\otimes}{,} P^{-1}(\mu)\})(P(\lambda)\otimes P(\mu)),$$

which follows from (2.7), (4.2)-(4.4). The rest of the Poisson relations (4.19)-(4.28) can be proven analogously. ■

Remark. The *Sklyanin bracket* (4.19) is one of the basic Poisson structures in the theory of integrable systems associated with loop algebras (see [FT], [RS]). We see that the Poisson structure (4.6) is equivalent to Sklyanin brackets for matrix polynomials $P(\lambda), Q(\lambda)$ coupled by equation (4.21). Note also, that as one sees from (4.18), the standard Lie-Poisson structure on $s\ell(n(N+1))$ would have resulted in the Sklyanin bracket for matrix elements of $M(\lambda)$. Instead, we have obtained, to the best of our knowledge, new Poisson bracket (4.16) for matrix rational function of the form (2.7).

Poisson relations from Proposition 4.4 can be used to prove a complete integrability of the nonabelian Toda lattice (1.2).

In Propositions 3.2, 3.3 we have found a large family of first integrals for (1.2) given by matrix elements of $ZP(\lambda)$, where Z is an arbitrary constant matrix, and by the coefficients of the polynomial $\det(\tilde{P}(\lambda) + z\mathbf{1})$. Note, that matrix elements of $ZP(\lambda)$ satisfy Poisson relations of the form (4.19):

$$\{ZP(\lambda) \stackrel{\otimes}{,} ZP(\mu)\} = \left[\frac{1}{\lambda-\mu}\mathcal{P}, ZP(\lambda)\otimes ZP(\mu)\right], \tag{4.31}$$

therefore the family of integrals that we have is not Poisson-commutative. On the other hand, not all of the functions in this family are independent. In particular,

$$\det(P(\lambda)) = \det(\tilde{P}(\lambda)) = \det(L - \lambda\mathbf{1}).$$

Note also, that $\det(P(\lambda))$ generated a center of the Poisson structure (4.31) considered independently from relations (4.20), (4.21) (see, e.g. [RS]). We can claim even more:

Proposition 4.5.

$$\{\det(\tilde{P}(\lambda) + z\mathbf{1}), \det(\tilde{P}(\mu) + w\mathbf{1})\} = 0, \tag{4.32}$$
$$\{P(\lambda), \det(\tilde{P}(\mu) + z\mathbf{1})\} = 0. \tag{4.33}$$

Proof. It suffices to prove that $Tr(\tilde{P}^k(\mu)), Tr(\tilde{P}^\ell(\nu))$ and matrix elements of $P(\lambda)$ Poisson-commute for any $k, \ell, \mu, \nu, \lambda$.

Due to (4.22) and (4.3),

$$\{Tr(\tilde{P}^k(\mu)), Tr(\tilde{P}^l(\nu))\} = kl\, Tr\left((\tilde{P}^k(\mu)\otimes\tilde{P}^l(\nu))\{\tilde{P}(\mu) \stackrel{\otimes}{,} \tilde{P}(\nu)\}\right) = 0,$$

which proves (4.32).

Furthermore, (4.25) implies

$$\{P(\lambda) \overset{\otimes}{,} \tilde{P}^k(\mu)\} = [(Q(\lambda) \otimes \mathbf{1})\mathcal{P}, 1 \otimes \tilde{P}^k(\mu)].$$

Then (4.33) follows from the identity

$$\sum_{i=1}^{n} [X, \mathbf{1} \otimes Y]_{\alpha i, \beta i} = 0$$

for any $X \in g\ell(n) \otimes g\ell(n), Y \in g\ell(n), \alpha$ and β. ∎

As an immediate consequence of (4.33), we have

Corollary 4.6. *Any function $f(P(\lambda))$ that generates a nontrivial Hamiltonian flow w.r.t. the Poisson structure (4.19) is independent with coefficients of* $\det(\tilde{P}(\lambda) + z\mathbf{1})$.

Let us now evaluate the number of independent on $\mathcal{O}_L$ functions that appear as coefficients of $\det(\tilde{P}(\lambda) + z\mathbf{1})$. It follows from (2.10) that $\tilde{P}(\lambda)$ that corresponds to L has a form

$$\tilde{P}(\lambda) = \lambda^{N+1}\mathbf{1} + \Lambda_L \lambda^N + \ldots \tag{4.34}$$

On the other hand, given a pair $S(\lambda) = \lambda^{N+1}\mathbf{1} + S_N\lambda^N + \ldots, T(\lambda) = \lambda^N\mathbf{1} + \ldots$, such that $T(\lambda)S^{-1}(\lambda)$ satisfies nondegeneracy conditions (2.3) one can construct a block-tridiagonal matrix $\mathcal{L} \in \epsilon + g_-^\perp$ such that $-T(\lambda)S^{-1}(\lambda)$ is its Weyl function. Moreover, due to Proposition 4.2, $\mathcal{L}$ belongs to $\mathcal{O}_L$ if and only if S_N belongs to the adjoint orbit of $SL(n)$ through Λ_L. Employing condition (4.11), we can state now that the number we are looking for is equal to the number of coefficients of the polynomial $\det(S(\lambda) + z\mathbf{1})$ independent as functions on the affine space $\{S(\lambda) = \lambda^{N+1}\mathbf{1} + S_N\lambda^N + \ldots\}$, where S_N is a fixed diagonal matrix with distinct eigenvalues. The latter number is known to be equal to $N\frac{n(n+1)}{2}$ (see, e.g., [RS, AHH1]).

Now, due to Corollary 4.6, any Poisson-commutative family of $(N+1)\frac{n(n-1)}{2}$ independent functions of $P(\lambda)$ that generate nontrivial Hamiltonian flows w.r.t. the Sklyanin's bracket (4.19) (or, equivalently, (4.31)) will yield the complete integrability of the nonabelian Toda lattice. Let us choose Z in (4.31) to be invertible matrix with distinct eigenvalues. Then ([RS]) non-leading coefficients of polynomials $p_1(\lambda), \ldots, p_{n-1}(\lambda)$ in the expansion

$$\det(ZP(\lambda) + z\mathbf{1}) = z^n + p_{n-1}(\lambda)z^{n-1} + \ldots + p_0(\lambda)$$

provide the requisite number of integrals in involution.

Alternatively, one can use the following construction proposed independently in [G4, S] (see also [AHH2]). In these papers the generating functions were found for the Darboux coordinates on the manifold $\{P(\lambda) = Z\lambda^{N+1} + \ldots; \det P(\lambda)$ has distinct fixed zeroes $\}$, equipped with the Sklyanin's bracket (4.31). Let $v \in \mathbb{C}^n$ be a cyclic vector for Z, i.e. vectors $v, Zv, \ldots, Z^{n-1}v$ are linearly independent. Then $(N+1)\frac{n(n-1)}{2}$ nonleading coefficients of the polynomial

$$\mathcal{B}(P(\lambda)) = \det[v, P(\lambda)v, \ldots, P^{n-1}(\lambda)v]$$

are in involution and independent. Thus, we have proved

Theorem 4.7. *The nonabelian Toda lattice is completely integrable on generic symplectic leaves $\mathcal{O}_L$.*

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