β - SPREAD OF SETS IN METRIC SPACES AND CRITICAL VALUES OF SMOOTH FUNCTIONS

Y. Yomdin

83 30

Sonderforschungsbereich 40
Theoretische Mathematik
Beringstr. 4
D-5300 Bonn 1

Max-Planck-Institut
für Mathematik
Gottfried-Claren-Straße 26
D-5300 Bonn 3

MPI/SFB 83-30

β - SPREAD OF SETS IN METRIC SPACES AND CRITICAL VALUES OF SMOOTH FUNCTIONS

Y. Yomdin

1. Introduction.

The aim of this paper is to define and to study a new invariant of metric spaces, which we call the β - spread and which turns out to be closely related to the structure of critical values of differentiable mappings.

Roughly, the β - spread of a metric space X, for $\beta > 0$, is the supremum with respect to all finite subsets of X of the sum of β-th degrees of the lengths of edges in the shortest polygonal line, connecting the points of a given finite subset of X. This invariant turns out to be a close "neighbor" of another one: an ε - entropy (see [5], [6], [7], [4], [1], [11], [12]), which is the logarithm of the minimal number of balls of radius ε, covering the space X.

The main reason to study β - spread is that this invariant is responsible for the property of a given set to be the set of critical values of some function of given smoothness on some compact manifold of given dimension. To illustrate the importance of this property and its relation to known restrictions, let us remind that the classical and widely used condition on critical values of differentiable mappings is given by the Morse-Sard theorem ([9], [10]): if the mapping is C^k - smooth, with k sufficiently big, then the set of its critical values has the Lebesgue measure (or, more precisely, the Hausdorff measure of an appropriate dimension) zero.

However, in applications the Morse-Sard theorem mostly appears as the theorem of existance of regular values, and not as the contensive restriction on critical ones. The reason is that the property to be of measure zero is too weak: in most of concrete situations the apriori information on critical values is much stronger. Thus, for differentiable mappings under minimal genericity assumptions, or for analytic mappings, critical values sets can be stratified by smooth submanifolds of "right" dimensions. In many variational problems the countability or the finiteness of the set of critical values can be shown. Of course, the Morse-Sard theorem sais nothing in all these cases.

But the important point is that in fact the critical values of any differentiable mapping satisfy geometric restrictions much stronger than the property to be of measure zero. These restrictions have been obtained in [12], and they are nontrivial in all the situations above.

The main result of [12] is the upper bound for the ε - entropy of the set of "near-critical" values of a mapping with the bounded domain and with bounded partial derivatives of some order k. This result implies, in a special case of critical values, a strengthening of the Morse-Sard theorem, and gives therefore a necessary condition for a given set to be the set of critical values, which is much stronger than the condition to be of measure zero. The following example from [12] illustrate the character of new restrictions, found there:

Corollary 5.5, [12]. The set $\{1, 1/2^a, 1/3^a, \dots, 1/n^a, \dots, 0\}$ cannot be the set of critical values of a k times continuously differentiable function on an n - dimensional compact smooth manifold, if $k > n(a+1)$.

Thus the results of [12] show that actual properties of critical values of general differentiable mappings are strong enough to imply non-trivial consequences, for instance, for geodesics on compact Riemannian manifold, for critical values of a complex analytic function, etc.

The following important question then arises: to find a necessary and sufficient condition for a given compact set to be the set of critical values of some C^k - smooth function on an n - dimensional compact smooth manifold.

One of main results of this paper is theorem 4.1 , which gives the required condition in terms of β - spread for functions of one variable. We state also a conjectured necessary and sufficient condition in a general case (also in terms of β - spread), and prove its necessity.

Another central result of this paper is theorem 2.8 , relating the β - spread with the ε - entropy. It follows, in particular, that the dimensions, defined by these invariants, coincide.

Finitely, we study detaily the β - spread of subsets of R^n . Theorem 3.1 gives an upper bound for an n - spread of any bounded set in R^n , and its proof surprisingly involves rather long elementary-geometrical constructions.

The β - spread of bounded subsets of the real line can be described completely: since the set and its closure have the same β - spread, we can assume our set to be closed, and then theorem 3.10 gives the answer in terms of length of intervals, forming the complement. Because of theorem 2.8 this gives also the complete description of the entropy dimension of subsets of the real line.

The notion of the β - spread is also useful in such questions, as the possibility to cover a given set by the curve with the given properties (generalized Peano curves; see e.g. [8]). Some results in this direction will appear separately.

The author would like to thank Y.Kanai ., H.Furstenberg and Y.Katznelson for useful discussions and SFB-40, Universität Bonn, for its kind hospitality.

2. Definition and some properties of the β - spread.

Let Γ_p be the set of all connected nonoriented trees with p vertices. We write $(i,j) \in \gamma$, for $\gamma \in \Gamma_p$, if the vertices i and j are connected by the edge in γ.

Definition 2.1. Let X be a metric space, $\beta > 0$. For each $x_1,\dots,x_p \in X$ and $\gamma \in \Gamma_p$ let $\rho_\beta(\gamma,x_1,\dots,x_p) = \sum_{(i,j)\in\gamma} d(x_i,x_j)^\beta$,

where d is a distance in X. Define $\rho_\beta(x_1,\dots,x_p)$ as $\inf_{\gamma\in\Gamma_p} \rho_\beta(\gamma,x_1,\dots,x_p)$.

Now let $A \subseteq X$. We define the β - spread of A, $V_\beta(A)$, by

$$V_\beta(A) = \sup_{p,x_1,\dots,x_p\in A} \rho_\beta(x_1,\dots,x_p) .$$

Now we formulate some simple properties of the β - spread:

Let X be a metric space, $A, A_1, A_2,\dots \subseteq X$, $\beta > 0$.

Theorem 2.2.

1. If $A_1 \subseteq A_2$ then $V_\beta(A_1) \leq V_\beta(A_2)$.

2. If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$,

and $A = \cup A_i$, then $V_\beta(A) = \lim_{i\to\infty} V_\beta(A_i)$.

3. $V_\beta(A) = V_\beta(\bar{A})$, where $\bar{A}$ is the closure of A .

4. $V_\beta(A_1 \cup A_2) \leq V_\beta(A_1) + V_\beta(A_2) + [d(A_1,A_2)]^\beta$.

If $d(A_1,A_2) \geq \max(\operatorname{diam} A_1, \operatorname{diam} A_2)$ then $V_\beta(A_1 \cup A_2) =$

$$= V_\beta(A_1) + V_\beta(A_2) + d(A_1,A_2)^\beta .$$

Here $d(A_1,A_2) = \inf_{x\in A_1, y\in A_2} d(x,y)$, $\operatorname{diam}(A) = \sup_{x,y\in A} d(x,y)$.

5. Let $\beta_1 > \beta_2 > 0$. Then

$$V_{\beta_1}(A) \leq [\operatorname{diam} A]^{\beta_1-\beta_2} V_{\beta_2}(A) .$$

6. Let $A \subset X$, $B \subset Y$. If there exists an epimorphism $\psi : A \to B$ with $d(\psi(x),\psi(y)) \leqq K \cdot d(x,y)^N$, for any $x,y \in A$, $N > 0$, then for each $\beta > 0$, $V_\beta(B) \leqq K^\beta V_{N\beta}(A)$.

Proof. Properties 1,2,3 follow immediately from definition. To prove 4, take $\varepsilon > 0$ and let $x \in A_1$, $y \in A_2$ be such that $d(x,y) \leq d(A_1,A_2) + \varepsilon$. We call the tree $\gamma \in \Gamma_p$, for which the minimum of $\rho_\beta(\gamma,x_1,\ldots,x_p)$ is attained, the β-minimal tree of $x_1,\ldots,x_p$. Now let $x_1,\ldots,x_p \in A_1 \cup A_2$, and assume that, say, $x_1,\ldots,x_r \in A_1$, and $x_{r+1},\ldots,x_p \in A_2$. Then $\rho_\beta(x_1,\ldots,x_p) \leq \rho_\beta(x_1,\ldots,x_p,x,y) \leq \rho_\beta(x_1,\ldots,x_r,x) + \rho_\beta(x_{r+1},\ldots,x_p,y) + d(x,y)^\beta$.

The last inequality we obtain, considering the tree, built from minimal trees of $(x_1,\ldots,x_r,x),(x_{r+1},\ldots,x_p,y)$ and the edge (x,y). Hence, by definition of V_β and by choice of x and y, $V_\beta(A_1 \cup A_2) = \sup \rho_\beta(x_1,\ldots,x_p) \leq V_\beta(A_1) + V_\beta(A_2) + [d(A_1,A_2) + \varepsilon]^\beta$, and the first inequality of 4 follows, since ε is arbitrary small. To prove the second part of 4, we use the following lemma:

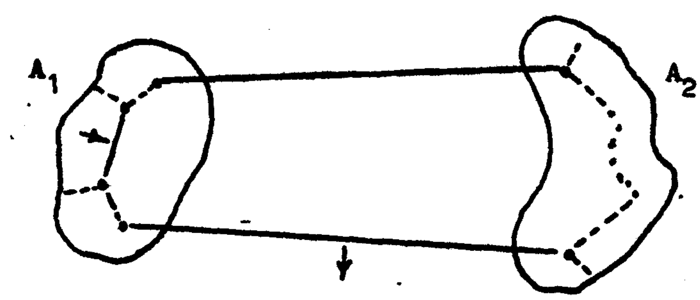
Lemma 2.3. Let $d(A_1,A_2) \geq \max(\operatorname{diam} A_1, \operatorname{diam} A_2)$.

Then for each $x_1,\ldots,x_p \in A_1 \cup A_2$ there is a minimal tree, containing exactly one edge, connecting A_1 and A_2.

Proof. We prove, that if there is a minimal tree γ with m edges, connecting A_1 and A_2, $m > 1$, then there is also a minimal tree with $m-1$

such edges. Indeed, delete one of these edges. We obtain not more than two connected components, at least one of which contains points both from A_1 and A_2 (since $m > 1$).

Let the second component contain point in A_1. We add the edge, connecting these two components inside A_1. We obtain a connected tree γ', which is also minimal, since the length of the added edge is not greater, than the length of the removed one by condition $d(A_1,A_2) \geq \max(\operatorname{diam} A_1, \operatorname{diam} A_2)$. Clearly, γ' contains exactly $m-1$ edges, connecting A_1 and A_2 (see fig. 1). □



Here and below $\to|$ and $|\to$ denote the inserted and the deleted edge, respectively.

Fig. 1.

Now let $x_1,\ldots,x_r \in A_1$, $x_{r+1},\ldots,x_p \in A_2$. By lemma 2.3 there is a minimal tree γ for $x_1,\ldots,x_p$, containing exactly one edge, connecting A_1 and A_2. Hence γ consists of a connected subtree on $x_1,\ldots,x_r$, a connected subtree on $x_{r+1},\ldots,x_p$, and an edge, connecting these two subtrees. Therefore, $V_\beta(A_1 \cup A_2) \geq \rho_\beta(x_1,\ldots,x_p) \geq \rho_\beta(x_1,\ldots,x_r) + \rho_\beta(x_{r+1},\ldots,x_p) + [d(A_1,A_2)]^\beta$, and taking supremum in a right hand side, we obtain the required inequality.

To prove statement 5 of theorem 2.2, note, that for each $x_1, x_2 \in A$, $d(x_1,x_2)^{\beta_1} = d(x_1,x_2)^{\beta_1-\beta_2} \cdot d(x_1,x_2)^{\beta_2} \leq [\mathrm{diam}\ A]^{\beta_1-\beta_2}\ d(x_1,x_2)^{\beta_2}$, and the required inequality follows from definition of V_β.

To prove 6, for each $y_1, \ldots, y_p \in B$ choose some $x_1, \ldots, x_p \in A$, $y_i = \psi(x_i)$. Then we obtain, using the condition, $\rho_\beta(\gamma, y_1, \ldots, y_p) \leq K^\beta \rho_{N\beta}(\gamma, x_1, \ldots, x_p)$ for any $\gamma \in \Gamma_p$, and the required inequality follows. □

Note, that because of properties 1 and 2, V_β is a capacity in the sense of [2] .

To clarify the geometric meaning of β - spread, we compare it with some other geometric characteristics of subsets in a metric space.

Definition 2.4. ([6]). Let $M(\varepsilon,A)$ be the minimal number of sets of a diameter $\leq 2\varepsilon$, covering A. Let $N(\varepsilon,A)$ be the maximal number of points in A, for which any two distinct are at a distance, greater than ε from each other. The functions

$$H_\varepsilon(A) = \log_2 M(\varepsilon,A) \quad \text{and} \quad C_\varepsilon(A) = \log_2 N(\varepsilon,A)$$

are called the ε - entropy and the ε - capacity of A , respectively.

The functions $M(\varepsilon,A)$, $N(\varepsilon,A)$, $H_\varepsilon(A)$ and $C_\varepsilon(A)$ are related by the following inequalities ([6], theorem IV):

$$N(2\varepsilon,A) \leq M(\varepsilon,A) \leq N(\varepsilon,A),$$

$$C_{2\varepsilon}(A) \leq H_\varepsilon(A) \leq C_\varepsilon(A) .$$

Comparing the definition of β - spread with that of $N(\varepsilon,A)$, we obtain immediately:

Proposition 2.5. For each $\beta > 0$,

$$V_\beta(A) \geq \sup_{\varepsilon>0} \varepsilon^\beta(N(\varepsilon,A) - 1) .$$

Proof. By definition, for any given $\varepsilon > 0$, there are $s = N(\varepsilon,A)$ points $x_1, \dots, x_s$ in A, with $d(x_i,x_j) > \varepsilon$ for $i \neq j$. Hence $V_\beta(A) \geq \rho_\beta(x_1, \dots, x_s) \geq (N(\varepsilon,A) - 1)\varepsilon^\beta$. □

For some A we have in fact an equality, and for others the expressions in the inequality of proposition 2.5 are far from one another. Thus for $A = [0, 1]$,

$N(\varepsilon,A) = [\frac{1}{\varepsilon}]$ for $\varepsilon \neq \frac{1}{n}$, and $N(\varepsilon,A) = \frac{1}{\varepsilon} - 1$ for $\varepsilon = \frac{1}{n}$, and hence

$$\sup_{\varepsilon>0} \varepsilon^\beta(N(\varepsilon,[0, 1]) - 1) = \begin{cases} \infty , & \beta < 1 \\ 1 , & \beta \geq 1 \end{cases} .$$

Easy computation shows that also $V_\beta([0, 1]) = \infty$ for $\beta < 1$, and $V_\beta([0, 1]) = 1$ for $\beta \geq 1$.

On the other hand, for $A = \{1, \frac{1}{2}, \dots , \frac{1}{n}, \dots\}$ one has

$$\frac{1}{\sqrt{\varepsilon}} \leq N(\varepsilon,A) \leq \frac{2}{\sqrt{\varepsilon}} , \text{ and hence}$$

$$\sup_{\varepsilon>0} \varepsilon^\beta(N(\varepsilon,A) - 1) \text{ is } \begin{cases} \text{finite}, & \beta \geq \frac{1}{2} \\ \infty , & \beta < \frac{1}{2} \end{cases}$$

For $V_\beta(A)$ one has (see theorem 3. 10 below): $V_\beta(A)$ is finite for $\beta > \frac{1}{2}$, and $V_\beta(A) = \infty$ for $\beta \leq \frac{1}{2}$. Thus for $\beta = \frac{1}{2}$ the right hand side of the inequality is finite, while the left hand side is equal to infinity.

To obtain an upper estimate for the β - spread in terms of ε - entropy is more difficult. First we consider the following function, which is, in some sense, the invers function of $N(\varepsilon, A)$:

Definition 2.6. For $x_1, \dots, x_p \in X$, let $\nu(x_1, \dots, x_p) = \min_{i \neq j} d(x_i, x_j)$. For $A \subset X$ define $\eta_A(p)$ for any natural $p \geqq 2$ by

$$\eta_A(p) = \sup_{x_1, \dots, x_p \in A} \nu(x_1, \dots, x_p) .$$

Proposition 2.7. For any $\varepsilon > 0$, $p \geqq 2$

$$\text{i.} \quad \eta_A(N(\varepsilon,A)) > \varepsilon \geqq \eta_A(N(\varepsilon,A) + 1) .$$

$$\text{ii.} \quad N(\eta_A(p),A) < p \leqq N(\eta_A(p) - \xi, A), \text{ for any } \xi > 0.$$

Proof. By definition of $N(\varepsilon,A)$, there are $s = N(\varepsilon,A)$ points $x_1, \dots, x_s$ in A, with $\nu(x_1, \dots, x_s) > \varepsilon$, and for any $s + 1$ points $x_1, \dots, x_{s+1}$ in A, $\nu(x_1, \dots, x_{s+1}) \leqq \varepsilon$. This proves i.

Since by definition of η_A, for each p points $x_1, \dots, x_p$ in A, $\nu(x_1, \dots, x_p) \leqq \eta_A(p)$, $N(\eta_A(p),A) < p$. On the other hand, for any $\xi > 0$ there are points $x_1, \dots, x_p$ in A with $\nu(x_1, \dots, x_p) > \eta_A(p) - \xi$, which proves the second inequality in ii.□

Theorem 2.8. For any $\beta > 0$

$$\sup_{p \geqq 2} (p - 1)\eta_A^{\beta}(p) \leqq V_{\beta}(A) \leqq \sum_{j=2}^{\infty} \eta_A^{\beta}(j) .$$

Proof. Let $x_1,\dots,x_p \in A$. By definition, $d(x_i,x_j) \geq \nu(x_1,\dots,x_p)$ and hence $\rho_\beta(x_1,\dots,x_p) \geq (p-1)\nu(x_1,\dots,x_p)^\beta$. Therefore $V_\beta(A) \geq (p-1)\nu(x_1,\dots,x_p)^\beta$, and taking supremum in the right hand side, we obtain

$$V_\beta(A) \geq (p-1)[\eta_A(p)]^\beta , \quad \text{for any } p .$$

This proves the lower estimate for V .

Remark. This inequality is, in fact, equivalent to the one of proposition 2.5.

To prove an upper estimate we need to study more in detail the structure of minimal trees. Let $x_1,\dots,x_p \in A$, and let $\gamma \in \Gamma_p$ be a minimal tree of $x_1,\dots,x_p$. Let $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{p-1}$ be lengths of the edges of γ (i.e., the distances $d(x_i,x_j)$, $(i,j) \in \gamma$) in decreasing order.

Proposition 2.9. $\alpha_i \leq \eta_A(i+1)$, $i=1,\dots,p-1$.

In fact, the following more precise statement implies this proposition:

Lemma 2.10. Let $1 \leq q \leq p-1$, let $e_1,\dots,e_q$ be some q edges of γ, and let α be the minimal length of e_i.

Then among the vertices of the edges $e_1,\dots,e_q$ there are at least $q+1$ different points $y_1,\dots,y_{q+1}$, for which

$$\nu(y_1,\dots,y_{q+1}) \geq \alpha .$$

Remark. One has immediately for all the vertices $z_1, \dots ,z_{2q}$ of $e_1, \dots ,e_q$, that $\nu(z_1, \dots ,z_{2q}) \leq \alpha$, but clearly one can have here the strict inequality. Proposition 2.9 claims that the opposite inequality can be obtained for still sufficiently big number $s \geq q + 1$ of distinct points among z_i.

Proof. Induction by q. For $q = 1$ the statement of the lemma is immediate.

Now let $q \geq 2$. Assume, that α is the length of e_q, and let z_1 and z_2 be the vertices of e_q. We obtain two subtrees γ' and γ'' of γ, consisting of all the vertices connected to z_1 (z_2 respectively) in $\gamma \setminus e_q$ (see fig. 2).

Lemma 2.11. For each two vertices $z' \in \gamma'$ and $z'' \in \gamma''$, $d(z',z'') \geq \alpha$.

Proof. If $d(z',z'') < \alpha$, we can add the edge (z',z'') and delete the edge e_q in γ. Clearly, we obtain a new connected tree $\tilde{\gamma}$ with $\rho_\beta(\gamma) > \rho_\beta(\tilde{\gamma})$, which contradicts to the minimality of γ . □

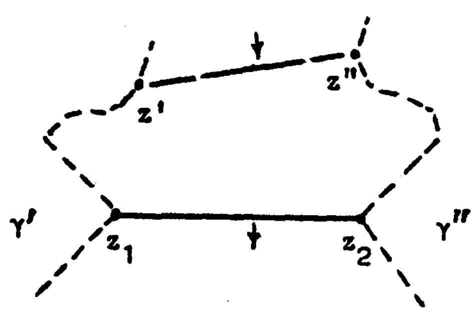


Fig.2

Lemma 2.12. Each subtree of a minimal tree is also minimal.

Proof. If the subtree γ_1 of γ is not minimal, we can find γ_2, connecting vertices of γ_1, with $\rho(\gamma_2) < \rho(\gamma_1)$. Now, replace in γ subtree γ_1 by γ_2. We obtain a new connected tree $\bar{\gamma}$ with $\rho(\bar{\gamma}) < \rho(\gamma)$, which contradicts to the minimality of γ . □

We continue the proof of lemma 2.10. Let, as above, γ' and γ'' be two subtrees, into which e_q subdivides γ .

Let among the edges $e_1, \dots, e_{q-1}$, the edges $e_1, \dots, e_k$ belong to γ', $0 \leqq k < q-1$, and $e_{k+1}, \dots, e_{q-1}$ belong to γ'' . We consider separately two cases:

a. $k > 0$. Thus among $e_1, \dots, e_{q-1}$ there are edges belonging to both γ' and γ''.

Consider subtrees of γ, $\bar{\gamma}' = \gamma' \cup e_q$ and $\bar{\gamma}'' = \gamma'' \cup e_q$. We can apply the induction assumption separately to $\bar{\gamma}'$ with the edges $e_1, \dots, e_k$, e_q and to $\bar{\gamma}''$ with the edges $e_{k+1}, \dots, e_{q-1}, e_q$. Thus we find among the vertices of $e_1, \dots, e_q$ the points $x'_1, \dots, x'_s \in \bar{\gamma}'$ with $s \geqq k+2$ and $\nu(x'_1, \dots, x'_s) \geqq \alpha$, and $x''_1, \dots, x''_r \in \bar{\gamma}''$ with $\nu(x''_1, \dots, x''_r) \geqq \alpha$ and $r \geqq q-k+1$.

Denote by $y_1, \dots, y_m$ all the different points among $x'_1, \dots, x'_s$, $x''_1, \dots, x''_r$. Since twice among these points can appear only z_1 and z_2, we have $m \geqq s+r-2 \geqq k+2+(q-k+1)-2 = q+1$. We have also $\nu(y_1, \dots, y_m) \geqq \alpha$. Indeed, for any two points y_i , y_j , $i \neq j$, if $y_i, y_j \in \{x'_1, \dots, x'_s\}$, then $d(y_i, y_j) \geqq \nu(x'_1, \dots, x'_s) \geqq \alpha$; if $y_i, y_j \in \{x''_1, \dots, x''_r\}$, then $d(y_i, y_j) \geqq \nu(x''_1, \dots, x''_r) \geqq \alpha$. Finitely, if $y_i \in \bar{\gamma}' \setminus \bar{\gamma}'' \subset \gamma'$, and $y_j \in \bar{\gamma}'' \setminus \bar{\gamma}' \subset \gamma''$, then $d(y_i, y_j) \geqq \alpha$ by lemma 2.11. In the only reminding c
where, say, $y_i, y_j \in \gamma'$ but $y_j \notin \{x'_1, \dots, x'_s\}$, we have $y_j = z_1$, and this point (
b. $k = 0$. Then we apply the induction assumption to the tree γ'' and dr
the edges $e_1, \dots, e_{q-1} \in \gamma''$. We obtain among the vertices of $e_1, \dots$, e_{q-1} the points $x_1, \dots, x_s$ with $\nu(x_1, \dots, x_s) \geqq \min_{1 \geqq i \geqq q-1} (\text{length } e_i) \geqq \alpha$, and $s \geqq q$.

Now we add to these points the vertex z_1 of e_q. By lemma 2.11, $d(z_1, x_i) \geqq \alpha$, $1 \geqq i \geqq s$, and hence $\nu(x_1, \dots, x_s, z_1) \geqq \alpha$. □

Proof of proposition 2.9. Let $e_1, \dots, e_i$ be edges in γ whose lengths are $\alpha_1, \dots, \alpha_i$. Since $\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_i$, we obtain, by lemma 2.10 , that there are points $y_1, \dots, y_{i+1}$ among the vertices of $e_1, \dots, e_i$, for which $\alpha_i \leqslant \nu(y_1, \dots, y_{i+1}) \leqslant \eta_A(i+1)$. □

To complete the proof of theorem 2.8, we consider, as above the points $x_1,\dots,x_p \in A$, the minimal tree γ of $x_1,\dots,x_p$, and the lengths $\alpha_1 \le \alpha_2 \le \dots \le \alpha_{p-1}$ of the edges of γ. By definition and by proposition 2.9,

$$\rho_\beta(x_1,\dots,x_p) = \sum_{i=1}^{p-1} \alpha_i^\beta \le \sum_{i=1}^{p-1} [\eta_A(i+1)]^\beta \leqq \sum_{j=2}^{\infty} [\eta_A(j)]^\beta .$$

Taking supremum in the left hand side, we obtain the required inequality. Theorem 2.8 is proved. □

For different A, $V_\beta(A)$ is better approximated by the one or the other side of the inequality of theorem 2.8. E.g. for $A = [0,1]$, $\eta_{[0,1]}(p) = \frac{1}{p-1}$, and

$$V_\beta([0,1]) = \sup (p-1)\eta^\beta_{[0,1]}(p) = \begin{cases} 1, & \beta \geqq 1 \\ \infty, & \beta \geqq 1 \end{cases} ,$$

while for $\beta = 1$, $\sum_{j=2}^{\infty} \eta_{[0,1]}(j) = \sum_{j=2}^{\infty} \frac{1}{j-1} = \infty$.

For $A = \{1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$ one has easily $\frac{1}{p(p-1)}) \leqq \eta_A(p) \leqq \frac{4}{p^2}$, while $V_\beta(A) = \sum_{j=2} [\frac{1}{j(j-1)}]^\beta$ (see theorem 3.10 below). Hence for each $\beta > 0$, $V_\beta(A)$ is of the same order as the right hand side of the inequality; for $\beta = \frac{1}{2}$ both $V_\beta(A)$ and the sum in the right hand side are equal to the infinity, while the left hand term is at most 4.

For $A = \{a_0, a_1, \dots, a_n, \dots\}$, where $a_i = 1 + \frac{1}{4} + \dots + \frac{1}{4^i}$, one has $\eta_A(p) = \frac{1}{4^{p-1}}$, $p \geqq 2$, and hence

$$V_\beta(A) = \sum_{j=2}^{\infty} \eta_A^\beta(j) = \sum_{i=1}^{\infty} (\tfrac{1}{4})^{\beta i} = \frac{4^\beta}{4^\beta - 1} \quad \text{for all } \beta > 0; \text{ for } \beta \to 0$$

this expression is of order $\frac{1}{\beta \cdot \ln 4}$. The lower bound in this example is

$$\sup_p \ (p-1)(\tfrac{1}{4})^{(p-1)\beta} \sim \frac{1}{e \cdot \beta \cdot \ln 4} \quad .$$

We can compare V_β also with the β - Hausdorff measure, which is defined as

$$\Lambda_\beta(A) = \lim_{\alpha \to 0} \Lambda_\beta^\alpha(A),$$

where $\Lambda_\beta^\alpha(A)$ is the lower bound of all sums of the form $\sum_{i=1}^{\infty} r_i^\beta$, $r_i \leqq \alpha$ and $A \subset \bigcup_{i=1}^{\infty} A_i$, with the diam $A_i \leqq r_i$. (see e. g. [3]).

Proposition 2.13. For all $\beta > 0$, $\Lambda_\beta(A) \leqq V_\beta(A)$.

Proof. $\Lambda_\beta^\alpha(A) \leqq \alpha^\beta \cdot M(\alpha,A) \leqq \alpha^\beta \cdot N(\alpha,A) = \alpha^\beta (N(\alpha,A)-1) \frac{N(\alpha,A)}{N(\alpha,A)-1} \leqq$ $V_\beta(A) \cdot \frac{N(\alpha,A)}{N(\alpha,A)-1}$, by theorem IV, [6] and propsition 2.5.

Taking limit as $\alpha \to 0$, we obtain the required inequality. □

Examples above show that the lower bound of β for which the expressions in the inequality of theorem 2.8 are finite, is the same.

To make this remark more precise, we remind the notions of the entropy and the Hausdorff dimensions, and define in the same way the v - dimension:

Definition 2.14. (See [1], [3], [6]). Let $A \subset X$ be a bounded subset .

1. $\dim_e A = \inf\{\beta \mid \exists K, \forall \varepsilon > 0, \ N(\varepsilon,A) \leqq K(\tfrac{1}{\varepsilon})^\beta\}$ is called the entropy dimension of A.
2. $\dim_h A = \inf\{\beta \mid \ \Lambda_\beta(A) < \infty\}$ is called the Hausdorff dimension of A.

3. $\dim_V A$ define as $\inf\{\beta | V_\beta(A) < \infty\}$.

The following result was conjectured by H.Furstenberg, who also gave another proof of it:

Theorem 2.15. For any bounded $A \subset X$, $\dim_V A = \dim_e A$.

Proof. Introduce an auxiliary dimension

$$\dim_\eta A \overset{\text{def}}{=} \inf\{\beta | \exists K, \forall p \geq 2, \ \eta_A(p) \leq K(\tfrac{1}{p})^{\frac{1}{\beta}}\} .$$

If $V_\beta(A) < \infty$, then, by theorem 2.8, $\eta_A(p) \leq K\left(\frac{1}{p-1}\right)^{1/\beta}$, and hence $\dim_\eta(A) \leq \beta$. On the other hand, if $\eta_A(p) \leq K\left(\frac{1}{p}\right)^{1/\beta}$, then for each $\beta' > \beta$, by theorem 2.8,

$$V_{\beta'}(A) \leq K^{\beta'} \sum_{p=2}^{\infty} \left(\frac{1}{p}\right)^{\beta'/\beta} < \infty .$$

Hence $\dim_V(A) = \dim_\eta(A)$.

Now, assume, that $\eta_A(p) \leq K\left(\frac{1}{p}\right)^{1/\beta}$. By proposition 2.7, i,

$$\alpha < \eta_A(N(\alpha,A)) \leq K\left(\frac{1}{N(\alpha,A)}\right)^{1/\beta}, \text{ or}$$

$N(\alpha,A) \leq K^\beta\left(\frac{1}{\alpha}\right)^\beta$, ie. $\dim_e(A) \leq \dim_\eta(A)$. On the other hand, if $N(\alpha,A) \leq K\left(\frac{1}{\alpha}\right)^\beta$, by proposition 2.7, ii,

$$p \leq N(\eta_A(p)-\xi, A) \leq K\left(\frac{1}{\eta_A(p)-\xi}\right)^\beta, \quad \text{for any } \xi > 0,$$

and we have

$$p \leq K\frac{1}{\eta_A(p)^\beta} \quad \text{or} \quad \eta_A(p) \leq K^{1/\beta} \cdot \left(\frac{1}{p}\right)^{1/\beta} .$$

Hence $\dim_\eta(A) \leq \dim_e(A)$ and therefore $\dim_\eta(A) = \dim_e(A)$. □

Remark. One has also $\dim_e A \gneq \dim_h A$ (see e.g. [1] , where a slightly different notion of the entropy dimension is used; this follows also from proposition 2.13.). This inequality can be strict. Thus, $\dim_h A = 0$ for any countable A, while, e.g.

$$\dim_e \{1, \tfrac{1}{2^a}, \dots, \tfrac{1}{n^a}, \dots\} = \frac{1}{a+1}$$ (see corollary 3.11 below).

3. V_β for subsets of R^n.

From now on we assume that our metric space X is the Eucledian space R^n.

Coincidence of dimensions $\dim_v$ and $\dim_e$, proved in theorem 2.15, shows that for any bounded $A \subseteq R^n$, $\dim_v(A) \leq n$. We shall obtain the following more precise result:

Theorem 3.1. There is a constant K_n, depending only on dimension n, such that for any $A \subseteq R^n$,

$$V_n(A) \leq K_n\,[\mathrm{diam}(A)]^n .$$

Proof. We obtain this inequality as a consequence of some result, concerning the structure of minimal trees in Euclidean space.

Definition 3.2. For any φ, $0 \leq \varphi < \frac{\pi}{2}$ and for $x_1, x_2 \in R^n$ denote by $W(\varphi, x_1, x_2)$ the union of two closed pyramids with vertices x_1 and x_2 respectively, with vertix angle 2φ, with common axis $[x_1, x_2]$, and with the common base, lying in a hyperplane, passing through the centre of $[x_1, x_2]$ and orthogonal to $[x_1, x_2]$ (see fig. 3).

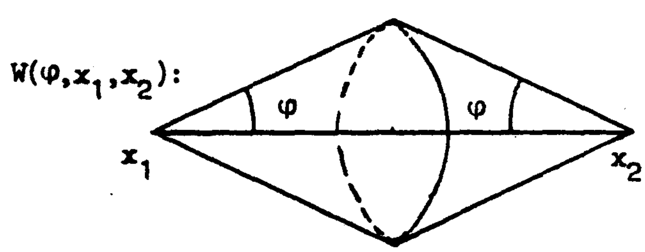


Fig.3.

For our purpose it is sufficient to take $\varphi = \frac{\pi}{100}$, so we denote $W(\frac{\pi}{100}, x_1, x_2)$ by $W(x_1, x_2)$.

Theorem 3.3. Let $x_1, \ldots, x_p \in R^n$ and let $\gamma \in \Gamma_p$ be a minimal tree of $x_1, \ldots, x_p$. Then for each $(i,j), (i',j') \in \gamma$, $(i,j) \neq (i',j')$, $\dot{W}(x_i, x_j) \cap \dot{W}(x_{i'}, x_{j'}) = \emptyset$. (Here and below minimal means β-minimal for some $\beta > 0$.

Proof. We consider two cases:

1. The edges $[x_i, x_j]$ and $[x_{i'}, x_{j'}]$ have a common vertex, say $x_i = x_{i'}$. If $\dot{W}(x_i, x_j) \cap \dot{W}(x_i, x_j') \neq \emptyset$, then the angle between $\overrightarrow{x_i x_j}$ and

$\overrightarrow{x_i x_{j'}}$ is less than $\frac{\pi}{50} < \frac{\pi}{3}$. Hence in the triangle $x_i, x_j, x_{j'}$, $[x_j, x_{j'}]$ cannot be the biggest edge. Let, say, $|x_j - x_{j'}| < |x_i - x_{j'}|$. Then we replace in γ the edge (i,j') by (j,j'). We obtain a connected tree γ' with $\rho_\beta(\gamma') < \rho_\beta(\gamma)$, which contradicts to the minimality of γ.

2. Now assume, that all the vertices i, j, i', j' of γ are distinct

In $\gamma \setminus (i,j) \cup (i',j')$ at least two vertices among i, j, i', j' can be still joined. Let these vertices be i and i'.

Lemma 3.4. The following inequalities are satisfied.

1. $|x_i - x_j| \le |x_{i'} - x_j|$
2. $|x_i - x_j| \le |x_j - x_{j'}|$
3. $|x_{i'} - x_{j'}| \le |x_i - x_{j'}|$
4. $|x_{i'} - x_{j'}| \le |x_j - x_{j'}|$

Proof. If one of these inequalities is not valid we can obtain from γ a new tree γ', as shown on fig. 4, with $\rho_\beta(\gamma') < \rho_\beta(\gamma)$. □

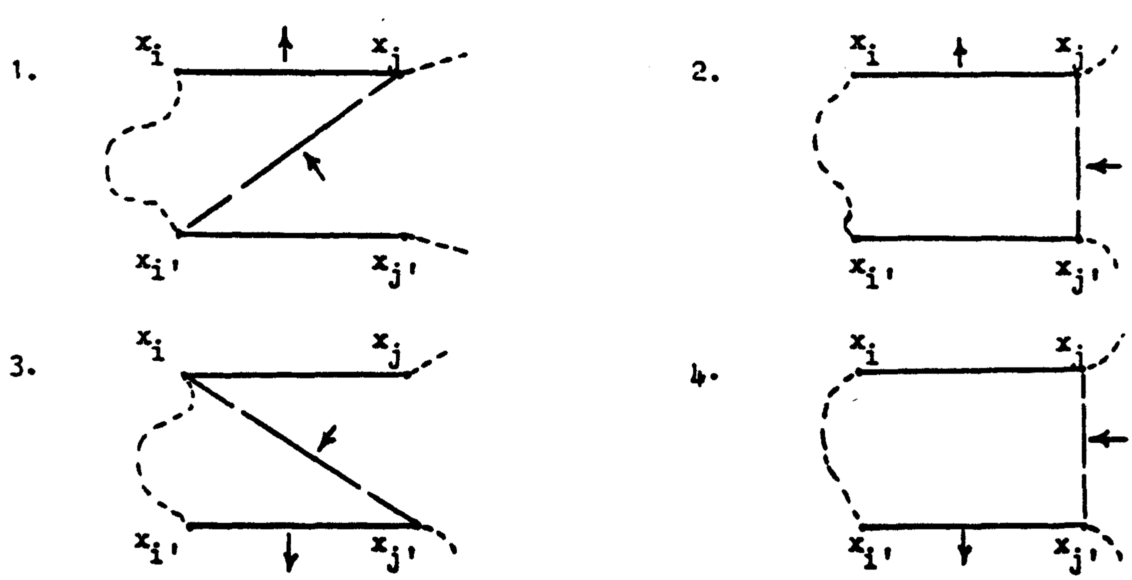


Fig.4.

Corollary 3.5. Let L be a hyperplane, orthogonal to the segment $[x_i, x_{i'}]$ and passing through its center. Denote R^n_+ the closed semispace, containing x_i and R^n_- the closed semispace, containing $x_{i'}$. Then $x_j \in R^n_+$, $x_{j'} \in R^n_-$.

Proof. This follows immediately from inequalities 1 and 3 of lemma 3.4. □

Lemma 3.6. Let L be a hyperplane in R^n, R^n_+ and R^n_- - two corresponding closed semispaces. Let $x_1, x_2 \in R^n_+$ and $W(x_1,x_2) \cap R^n_- \neq \emptyset$. Then

$$d_i \leq \frac{1}{20} d_i' , \quad i = 1,2, \quad \text{where}$$

$$d_i = d(x_i, L), \quad d_i' = d(x_i, L \cap W(x_1,x_2)).$$

Proof. We can restrict consideration to the plane passing through x_1, x_2 and orthogonal to L. (see fig. 4). Assume also that $d_1 \geq d_2$.

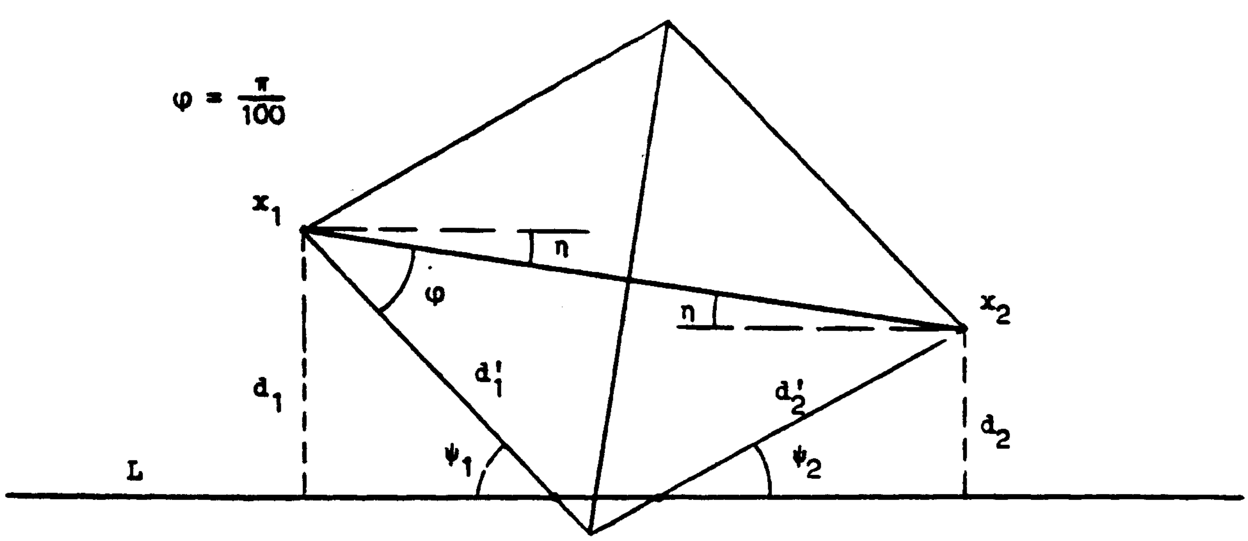


Fig.4.

We have: $d_1 = d_1' \sin\psi_1$, $d_2 = d_2' \sin\psi_2$. But $\psi_1 = \varphi + \eta$, and since, clearly, $\eta \le \varphi$, $\psi_1 \le 2\varphi$, and $\sin\psi_1 \le \sin 2\varphi < \frac{1}{20}$. Similarly, $\psi_2 = \varphi - \eta \le \varphi$, hence

$$\sin\psi_2 < \sin\varphi < \frac{1}{20} \,.\quad (\text{ We fixed } \varphi = \frac{\pi}{100}) \qquad \square$$

Lemma 3.7. Let L be a hyperplane, passing through the middle of $[x_1,x_1']$ and orthogonal to it. Assume, that x_2 (x_2') belongs to the same semispace as x_1 (x_1', respectively).

If $W(x_1,x_2) \cap W(x_1,x_2') \ne \emptyset$, then

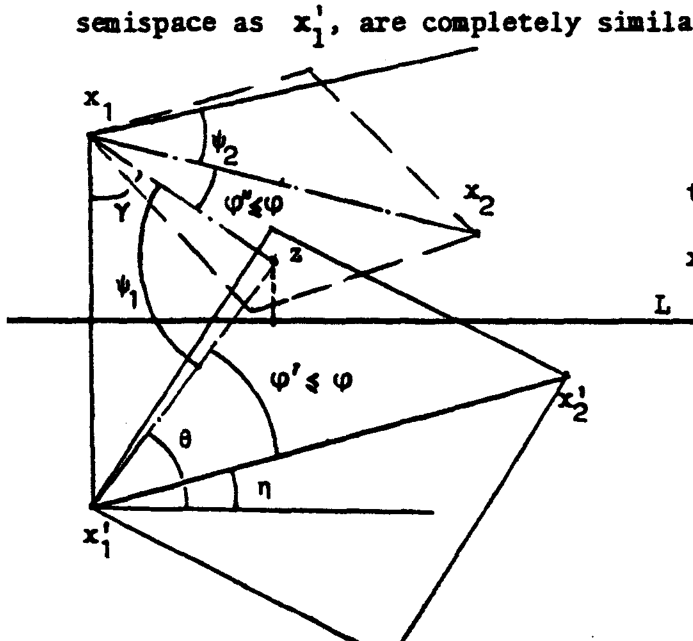
1. $|x_1-x_1'| \le \frac{1}{9}\min(|x_1-x_2|, |x_1'-x_2'|)$.

2. The angle ψ_2 between $\overrightarrow{x_1x_2}$ and $\overrightarrow{x_1'x_2'}$ is smaller than $\frac{\pi}{10}$.

3. Assume, that, for instance, $|x_1-x_2| \le |x_1'-x_2'|$ and let $y \in [x_1',x_2']$ be such that $|x_1'-y| = |x_1-x_2|$.

Then $|x_2-y| \le \frac{2}{3}\min(|x_1-x_2|, |x_1'-x_2'|)$ and $|x_2-x_2'| < |x_2'-x_1'|$.

Proof. Take some $z \in W(x_1,x_2) \cap W(x_1',x_2')$, and let z belong to the same semispace as x_1 (considerations in the case where z belongs to the same semispace as x_1', are completely similar).



The plane of the picture passes through x_1, x_1' and x_2'. The points x_2 and z generally do not belong to this plane.

Fig.5.

Hence $W(x_1',x_2')$ intersects the "second" semispace, and, by lemma 3.6, $d(x_1',L) = \frac{1}{2}|x_1-x_1'| \leq \frac{1}{20} d(x_1', L \cap W(x_1',x_2')) \leq \frac{1}{20}|z-x_1'| \leq \frac{1}{20}|x_2'-x_1'|$. On the other hand, $|x_1-z| \geq \frac{19}{20}|x_1'-z|$. Indeed, $|x_1-z| \geq \mathrm{proj}_L(x_1,z) = \mathrm{proj}_L(x_1',z) = |x_1'-z|\cos\theta \geq |x_1'-z|\cos(\varphi+\eta) \geq |x_1'-z|\cos 2\varphi \geq \frac{19}{20}|x_1'-z|$.

Hence $\frac{1}{2}|x_1-x_1'| \leq \frac{1}{20}|x_1'-z| \leq \frac{1}{19}|x_1-z|$, and in each case we have $|x_1-x_1'| \leq \frac{1}{9}\min(|x_1,x_2|, |x_1'-x_2'|)$. To prove 2, denote ψ_1 the angle between $\overrightarrow{zx_1}$ and $\overrightarrow{zx_1'}$. Then $\frac{\sin\psi_1}{|x_1-x_1'|} = \frac{\sin\gamma}{|x_1'-z|} \leq \frac{1}{|x_1'-z|}$ and therefore $\sin\psi_1 \leq \frac{|x_1-x_1'|}{|x_1'-z|} \leq \frac{1}{9}$ by part 1. Hence $\psi_1 < \frac{\pi}{20}$. Clearly, the angle ψ_2 between $\overrightarrow{x_1x_2}$ and $\overrightarrow{x_1'x_2'}$ is not greater than $\varphi' + \psi_1 + \varphi'' \leq \psi_1 + 2\varphi < \frac{\pi}{20} + \frac{\pi}{50} < \frac{\pi}{10}$.

To prove 3 we have (taking y' such that $\overline{x_1\,y'}$ is parallel to $\overline{x_1'\,y}$ and $|x_1 - y'| = |x_1' - y| = |x_1 - x_2|$; see fig. 6):

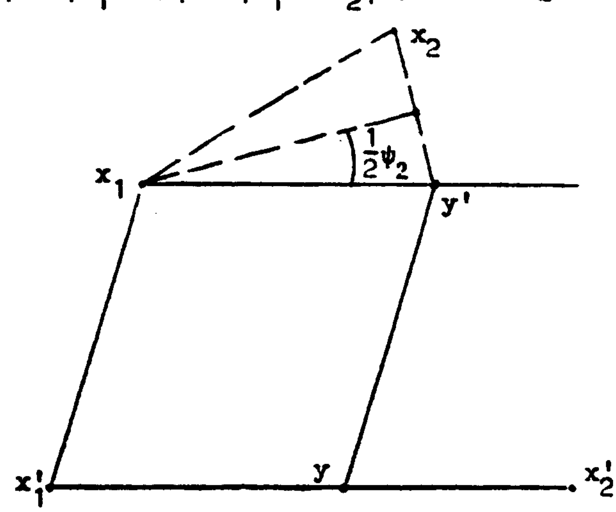


Fig.6.

$$|x_2-y| \le |x_2-y'| + |y'-y| = |x_2-y'| + |x_1-x_1'| \le$$

$$\le |x_1-x_2|\cdot 2\sin\frac{\psi_2}{2} + \frac{1}{9}|x_1-x_2| \le 2|x_1-x_2|\sin\frac{\pi}{20} + \frac{1}{9}|x_1-x_2| <$$

$$< \frac{2}{3}|x_1-x_2| = \frac{2}{3}\min(|x_1-x_2|, |x_1'-x_2'|) .$$

Finitely, to prove that $|x_2-x_2'| < |x_1'-x_2'|$, we have:
$|x_2-x_2'| \le |x_2-y| + |y-x_2'| \le \frac{2}{3}|x_1-x_2| + |y-x_2'| =$
$\frac{2}{3}|y-x_1'| + |x_2'-y| < |x_2'-x_1'|$, since y belongs to $[x_1',x_2']$, and $|y-x_1'| = |x_1-x_2| \ne 0$. □

Now we turn back to proof of theorem 3.3. In situation of lemma 3.4 because of corollary 3.5, we can apply lemma 3.7, 3. We obtain, that if $W(x_i,x_j) \cap W(x_{i'},x_{j'}) \ne \emptyset$, then $|x_j-x_{j'}| < |x_{i'}-x_{j'}|$ (assuming that $|x_i-x_j| \le |x_{i'}-x_{j'}'|$). But this contradicts to inequality 4 of lemma 3.4. Theorem is proved. □

Proof of theorem 3.1. For $x_1,x_2 \in R^n$ we have: $m(W(x_1,x_2) = D_n|x_1-x_2|^n$, where m is the usual Lebesque measure and D_n depends only on n. Clearly, the ball B of radius $2\mathrm{diam}(A)$, centered at any point of A, contains $W(x_1,x_2)$ for each $x_1,x_2 \in A$. Therefore for each $x_1,\dots,x_p \in A$ and for each minimal $\gamma \in \Gamma_p$,

$$m(B) \ge \sum_{(i,j)\in\gamma} m(W(x_i,x_j)) = D_n \sum_{(i,j)\in\gamma}|x_i-x_j|^n = D_n\, \rho_n(x_1,\dots,x_p)$$

(By theorem 3.3 all $W(x_i,x_j)$ are disjoint). Hence $V_n(A) = \sup \rho_n(x_1,\dots,x_p) \le K_n[\mathrm{diam}(A)]^n$,

where $K_n = \dfrac{m(B_1^n)}{D_n}$ depends only on n . □

Theorem 3.3 implies immediately also the follwoing corollary:

Proposition 3.8. Let $x_1,\dots,x_p \in R^n$ and let γ be a minimal tree of $x_1,\dots,x_p$. Then the multiplicity of each vertex of γ (i.e. the number of edges of γ, joining at the vertex) is not greater than some number L_n, depending only on n. □

For open subsets in R^n we have also the following inequality:

Theorem 3.9. Let $A \subseteq R^n$ be open.
Then $V_n(A) \geq m(A)$.

Proof. Subdivide R^n into equal cubes with the edge δ, and let N_δ be the number of those with the center in A. Clearly for open A, $\lim_{\delta\to 0} \delta^n \cdot N_\delta = m(A)$. On the other hand the distance between any two centers x_i, x_j is not less than δ, and hence

$$V_n(A) \geq \rho_n(x_1,\dots,x_{N_\delta}) \geq \delta^n N_\delta . \qquad \square$$

For subsets of a real line V_β can be easily computed. Since $V_\beta(A) = V_\beta(\bar{A})$ we can assume that A is closed.

Theorem 3.10. Let $A \subset R$ be a closed bounded set, $a = \inf A$, $b = \sup A$, and $A = [a,b] \setminus \bigcup_{i=1}^{\infty} V_i$, where V_i are disjoint open intervals. Denote the length of V_i by α_i. Then

1. $V_\beta(A) = (b-a)^\beta$ for $\beta \geq 1$.

2. For $\beta < 1$, $V_\beta(A) = \infty$ if $\sum_{i=1}^{\infty} \alpha_i < b-a$.

3. If $\sum_{i=1}^{\infty} \alpha_i = b-a$, then $V_\beta(A) = \sum_{i=1}^{\infty} \alpha_i^\beta$ for $\beta < 1$.

Proof. We have the following inequalities: for $\alpha_i \geq 0$,

$$(3.1) \qquad \left(\sum_{i=1}^{\infty} \alpha_i\right)^\beta \leq \sum_{i=1}^{\infty} \alpha_i^\beta, \qquad \beta \leq 1$$

$$(3.2) \qquad \left(\sum_{i=1}^{\infty} \alpha_i\right)^\beta \geq \sum_{i=1}^{\infty} \alpha_i^\beta, \qquad \beta \geq 1.$$

Since, clearly, for any $x_1 \leq x_2 \leq \ldots \leq x_p \in R$ the minimal tree of $x_1,\ldots,x_p$ is a chain $(1,2), (2,3),\ldots,(p-1,p)$, we obtain from (3.2): for $\beta \geq 1$, $\rho_\beta(x_1,\ldots,x_p) \leq |x_p-x_1|^\beta$, and hence $V_\beta(A) = |b-a|^\beta$.

To prove 2 note, that if $\sum_{i=1}^{\infty} \alpha_i < b-a$, then $m(A) > 0$ and hence $\Lambda_1(A) > 0$. Hence by properties of a Hausdorff measure (see e.g. [3]), $\Lambda_\beta(A) = \infty$ for $\beta < 1$ and hence, by proposition 2.13, $V_\beta(A) \geq \Lambda_\beta(A) = \infty$
To prove 3 note, that it is sufficient to consider the set A' consisting of all the ends of intervals V_i (since $\sum_{i=1}^{\infty} \alpha_i = b-a$, $A = \bar{A}'$). Then $V_\beta(A') \geq \sum_{i=1}^{\infty} \alpha_i^\beta$, since we can take as $x_1,\ldots,x_{2p}$ the ends of the first p intervals, and $\rho_\beta(x_1,\ldots,x_{2p}) \geq \sum_{i=1}^{p} \alpha_i^\beta$. On the other hand, for each $x_1 \leq \ldots \leq x_p \in A'$ if we denote by I_q, $q = 1,\ldots,p-1$, the set of indices i for which $V_i \subseteq [x_q,x_{q+1}]$, we have:

$$|x_q-x_{q+1}| = \sum_{i\in I_q} \alpha_i \qquad \left(\text{since } \sum_{i=1}^{\infty} \alpha_i = b-a\right).$$

By inequality (3.1), $\rho_\beta(x_1,\ldots,x_p) = \sum_{q=1}^{p-1} |x_{q+1}-x_p|^\beta \leq$

$\leq \sum_{q=1}^{p-1} \sum_{i\in I_q} \alpha_i^\beta = \sum_{i=1}^{\infty} \alpha_i^\beta$. Hence $V_\beta(A') \leq \sum_{i=1}^{\infty} \alpha_i^\beta$ and we obtain

$V_\beta(A) = \sum_{i=1}^{\infty} \alpha_i^\beta$. □

Corollary 3.11. Let $A = [a, b] \setminus \bigcup_{i=1} V_i$, α_i – the length of V_i. Then

1. $\dim_e A = 1$, if $\sum_{i=1}^{\infty} \alpha_i < b - a$.

2. $\dim_e A = \inf\{\beta | \sum_{i=1}^{\infty} \alpha_i^\beta < \infty\}$, if $\sum_{i=1}^{\infty} \alpha_i = b - a$. □

We need also the following results, concernings the mappings of subsets in R with the bounded β – spread:

Proposition 3.12. Let $A \subset R$ be a bounded subset, $0 < \beta < 1$, and let $V_\beta(A) < \infty$. Then there exists a homeomorphism $\psi : R \to R$, such that for any $x_1, x_2 \in A$

$$|\psi(x_1) - \psi(x_2)| \geqq \omega(|x_1 - x_2|)\cdot|x_1 - x_2|^\beta ,$$

where $\omega(\xi) \to \infty$ as $\xi \to 0$.

Proof. Without loss of generality we can assume that A is closed. Since $V_\beta(A) < \infty$, we have by theorem 3.10:

$$A = [a, b] \setminus \bigcup_{i=1}^{\infty} V_i \quad \text{with} \quad \sum_{i=1}^{\infty} \alpha_i = b - a , \quad \sum_{i=1}^{\infty} \alpha_i^\beta < \infty ,$$

where V_i are disjoint open intervals and α_i denotes the length of V_i.

There exists some decreasing function $\omega : (0, \infty) \to (0, \infty)$, such that $\omega(\xi) \geq 1$, $\omega(\xi) \to \infty$, as $\xi \to 0$, but still

$\sum_{i=1}^{\infty} \omega(\alpha_i)\, \alpha_i^{\beta} < \infty$. Now define $\psi : R \to R$ by

$$\psi(x) = \begin{cases} x & , x \leq a \\ a + \sum_{V_i \subset (\infty, x]} \omega(\alpha_i)\alpha_i^{\beta} & , x \in A \\ \psi(a_j) + \frac{x-a_j}{\alpha_j} \cdot \omega(\alpha_j)\alpha_j^{\beta} & , x \in V_j \\ \psi(b) + x - b & , x \geq b \end{cases}$$

where $a_j \in A$ is the left end point of the closed interval $\bar{V}_j$.

One checks easily, that ψ is a homeomorphism. Also for any $x_1, x_2 \in A$, $x_1 < x_2$,

$$\psi(x_2) - \psi(x_1) = \sum_{V_i \in [x_1, x_2]} \omega(\alpha_i)\alpha_i^{\beta} \geq \omega(x_2 - x_1) \sum_{V_i \in [x_1, x_2]} \alpha_i^{\beta} \geq$$

$$\omega(x_2 - x_1)\Big[\sum_{V_i \in [x_1, x_2]} \alpha_i\Big]^{\beta} = \omega(x_2 - x_1)\cdot(x_2 - x_1)^{\beta} . \quad \square$$

Proposition 3.13. Let $A \subset R$ be a bounded set, $0 < \beta < 1$, and let $V_{\beta}(A) < \infty$. Then there exists a bounded set $A' \subset R$ and a homeomorphism $\phi : R \to R$, $\phi(A') = A$, such that for any $x_1, x_2 \in A'$

$$|\phi(x_1) - \phi(x_2)| \leq \gamma(|x_1 - x_2|)|x_1 - x_2|^{\frac{1}{\beta}},$$

where $\gamma(\xi) \to 0$ as $\xi \to 0$.

Proof. We apply proposition 3.12 and take $A' = \psi(A)$, $\phi = \psi^{-1} : R \to R$ and $\gamma(\xi) = [\frac{1}{\omega(\xi^{\frac{1}{\beta}})}]^{\frac{1}{\beta}}$. $\square$

In a similar way one proves the following:

Proposition 3.14. Let $A \subset R$ be a bounded subset, such that $V_\beta(A) < \infty$ for all $\beta > 0$. Then there exists a bounded subset $A' \subset R$ and a homeomorphism $\phi : R \to R$, $\phi(A') = A$, such that for each $N > 0$, $|\phi(x_1) - \phi(x_2)| \leqq |x_1 - x_2|^N$, for any $x_1, x_2 \in A'$ with $|x_1 - x_2|$ sufficiently small. □

4. β - spread of critical values.

Let $f : M^n \to R$ be a C^k - smooth function on a smooth n-dimensional manifold M, $k \geqq 1$. We denote by $\Delta(f)$ the set of critical values of f, $\Delta(f) = f(\Sigma(f)) \subset R$, where $\Sigma(f) = \{x \in M |\ df(x) = 0\}$.

The Morse-Sard theorem (see [9], [10], also H.Federer [3], theorem 3.4.3), gives an upper bound for the Hausdorff dimension of $\Delta(f)$:

$$(*) \qquad \dim_h \Delta(f) \leqq \frac{n}{k} .$$

In [12] it is shown, that the entropy dimension is a more adequate and stronger notion in study of critical values. In particular, the following strengthening of (*) is true ([12], theorem 5.4):

$$(**) \qquad \dim_e \Delta(f) \leqq \frac{n}{k} , \quad \text{for } M \text{ compact} .$$

Moreover, the necessary condition (**) for a given set to be of the form $\Delta(f)$, turns out to be "almost sufficient ":

Let us say that a bounded set $A \subset R$ has a property $P(n,k)$ if there is a compact n-dimensional manifold M and a C^k - smooth function $f : M \to R$, such that $A \subset \Delta(f)$. Then we have ([12], theorem 5.6):

(***) If a bounded set $A \subset R$ has a property $P(n,k)$, then $\dim_e A \leqq \frac{n}{k}$. If $\dim_e A < \frac{n}{k}$, then A has a property $P(n,k)$.

However, to give a necessary and sufficient condition for a given compact set to be the set of critical values of a function of a given smoothness on a compact manifold of a given dimension, one must consider more precisely the metric properties of this set.

Conjecture. A compact set $A \subset R$ is of a form $A = \Delta(f)$ for some C^k - smooth $f : M^n \to R$, $k > n$, M compact, if and only if $V_{\frac{n}{k}}(A) < \infty$.

In this section we check the conjecture for functions of one variable and also prove the necessety of the condition $V_{\frac{n}{k}}(A) < \infty$ for arbitrary n (and also for mappings to R^p , under the restriction $k \leq 3$)

Theorem 4.1. A compact set $A \subset R$ is the set of all the critical values of a k times continuously differentiable function $f : [0, 1] \to R$, $k > 1$, if and only if $V_{\frac{1}{k}}(A) < \infty$.

A is the set of all the critical values of an infinitely differentiable function $f : [0, 1] \to R$ if and only if $\dim_e A = 0$.

Proof. 1. Necessity. We use the following lemma, which can be proved easily by successive applications of the mean value theorem:

Lemma 4.2. Let $f : [a, b] \to R$ be a k times differentiable function . If f has in $[a, b]$ more than $k - 2$ critical points, then

$$\max_{x\in[a,b]} f(x) - \min_{x\in[a,b]} f(x) \leq |f^{(k)}(c)|(b - a)^k ,$$

for some $c \in [a, b]$. □

Remark. Generalization of this lemma to functions of several variables is obtained in [12] (theorem 3.11 [12]). For $k \leq 3$ this generalization follows easily from lemma, and it is given below (lemma 4.4).

Now let $f : [0, 1] \to R$ be a k times continuously differentiable function. Denote by M the $\max_{c\in[a,b]} |f^{(k)}(c)|$.

For any $y_1, \dots, y_p \in \Delta(f)$, $p \geq k-1$, let $x_1, \dots, x_p \in [0, 1]$ be the critical points of f, such that $f(x_i) = y_i$, $i = 1, \dots, p$. Reordering this set, if necessary, we can assume that $x_1 < x_2 < \dots < x_p$.

Consider intervals $e_1 = [a, x_{k-1}]$, $e_2 = [x_{k-1}, x_{2k-3}], \dots,$

$e_s = [x_r, b]$, where $s = [\frac{p-1}{k-2}]$, $r = (k-2)(s-1) + 1$, for $k \geq 3$, and $s = r = p-1$ for $k = 2$. Each of these intervals contains exactly $k-1$ points x_i ($2 = k$ points for $k = 2$), except the last one, which contains not less than $k-1$ and not more than $2k-4$ points x_i.

Denote by d_j the length of e_j, $j = 1, \dots, s$. For any x_m, $x_n \in e_j$,

$|y_m - y_n| = |f(x_m) - f(x_n)| \leq Md_j^k$, by lemma 4.2.

Now the chain $\gamma = \{(1,2), (2,3), \dots, (p-1,p)\}$ is a tree on $y_1, \dots, y_p$ (possibly, non-minimal), and thus we have:

$$\rho_{\frac{1}{k}}(y_1, \dots, y_p) \leq \rho_{\frac{1}{k}}(\gamma, y_1, \dots, y_p) = \sum_{i=1}^{p-1} |y_{i+1} - y_i|^{\frac{1}{k}} =$$

$$\sum_{j=1}^{s} \sum_{[x_i, x_{i+1}]\in e_j} |y_{i+1} - y_i|^{\frac{1}{k}} \leq (2k-4)M^{\frac{1}{k}} \sum_{j=1}^{s} d_j = M^{\frac{1}{k}}(2k-4)(b-a).$$

Taking supremum in the left hand side, we obtain

$$V_{\frac{1}{k}}(\Delta(f)) \leq M^{\frac{1}{k}}(2k-4)(b-a) < \infty .$$

For infinitely differentiable f we have $V_{\frac{1}{k}}(\Delta(f)) < \infty$ for all k, and hence $\dim_e \Delta(f) = 0$.

2. Sufficiency. Let us fix some infinitely differentiable function $u : [0, 1] \to R$ with the following properties:

1. $u(t) > 0$ for $t \in (0,1)$.

2. $\int_0^1 u(t)dt = 1$.

3. $u(0) = u'(0) = \ldots = u(1) = u'(1) = \ldots = 0$.

Denote by N_q the $\max_{t\in[0,1]} |u^{(q-1)}(t)|$.

Now let $A \subset R$ be a compact set with $V_{\frac{1}{k}}(A) < \infty$. By proposition 3.13 there exists a compact set $A' \subset R$ and a homeomorphism $\phi: R \to R$ with $\phi(A') = A$ and $|\phi(x_1) - \phi(x_2)| \leq \gamma(|x_1 - x_2|)|x_1 - x_2|^k$, where $\gamma(\xi) \to 0$ as $\xi \to 0$.

Let $a = \inf A'$, $b = \sup A'$, and let $A' = [a, b] \setminus \bigcup_{i=1} U_i$, where $U_i = (a_i, b_i)$ are disjoint intervals. Let β_i be the length of U_i.

Denote also by α_i the length of $V_i = \phi(U_i) = (\phi(a_i), \phi(b_i))$. We have $\alpha_i \leq \gamma(\beta_i)\beta_i^k$ for all i.

Define h on [a, b] by

$$\begin{cases} h(x) = 0, & x \in A', \\ h(x) = \dfrac{\alpha_i}{\beta_i} u(\dfrac{x-a_i}{\beta_i}), & x \in U_i \end{cases},$$

and let $f(x) = \phi(a) + \int_a^x h(t)dt$.

We shall prove that $f \in C^k$ on [a, b] and $\Delta(f) = A$.

Clearly, $f \in C^\infty$ on $[a, b] \setminus A'$. To prove that $f \in C^k$ on all [a, b] it is sufficient to show that $\lim_{y\to x, y\in[a,b]\setminus A'} f^{(q)}(y) = 0$

for any q, $1 \leq q \leq k$, and $x \in A'$.

Now if $x \in A'$ is not a point of a condensation of intervals U_i, this follows from the property 3 of u. If $x \in A'$ is a condensation point of U_i, we use the following estimate:

$$|f^{(q)}(y)| = |h^{(q-1)}(y)| \leq \left(\frac{1}{\beta_i}\right)^q N_q \gamma(\beta_i)\beta_i^k \leq N_q \gamma(\beta_i)\beta_i^{k-q} \quad \text{for } y \in U_i .$$

Since for intervals, converging to $x \in A'$, their lengths tend to zero and since $\gamma(\xi) \to 0$ as $\xi \to 0$, we obtain that $\lim\limits_{y \to x} f^{(q)}(y) = 0$, $1 \leq q \leq k$.

Thus we proved that $f \in C^k[a, b]$ and also that all the derivatives of f up to order k vanish on A'. Because of the property 1 of u we have in fact $\Sigma(f) = A'$.

To prove that $\Delta(f) = f(\Sigma(f)) = A$, we show that $f(x) = \phi(x)$ for each $x \in A'$. Indeed, $A = \phi(A') = [\phi(a), \phi(b)] \setminus \bigcup\limits_{i=1}^{\infty} V_i$

and $\sum\limits_{i=1}^{\infty} \alpha_i = \phi(b) - \phi(a)$ (by corollary 3.11, since $\dim_e A \leq \frac{1}{k} < 1$).

Hence for each $x \in A'$, $\phi(x) = \phi(a) + \sum\limits_{U_i \subset [a,x]} \alpha_i$. But

$$f(x) = \phi(a) + \int_a^x h(t)dt = \phi(a) + \sum_{U_i \subset [a,x]} \frac{\alpha_i}{\beta_i} \int_{U_i} u\left(\frac{t-a_i}{\beta_i}\right)dt =$$

$$= \phi(a) + \sum_{U_i \subset [a,x]} \alpha_i \int_0^1 u(t)dt = \phi(a) + \sum_{U_i \subset [a,x]} \alpha_i = \phi(x),$$

by property 2 of u.

The proof of the sufficiency for the case of infinitely differentiable funstions is the same, but instead of proposition 3.13 we use proposition 3.14. □

In the case of several variables we consider mappings $f : M^n \to R^p$ and the critical points of a rank zero : $\Sigma(f) \overset{def}{=} \{x \in M \mid df(x) = 0\}$ and, as above, $\Delta(f) = f(\Sigma(f)) \subset R^p$.

Theorem 4.3. Let $f : M^n \to R^p$ be a k times differentiable mapping, $k = 1, 2, 3$, M compact. Then $V_{\frac{n}{k}}(\Delta(f)) < \infty$. Also for p=1 and any k, $V_{\frac{n}{k}}(\Delta(f)$

Proof. Without loss of generality we can assume that $M = B_1^n$,the ball of a radius 1 in R^n . Denote by D_k the $\max_{y \in B_1^n} ||d^k f(y)||$.

Lemma 4.4. For k = 1, 2, 3, and for any two critical points $x_1, x_2 \in \Sigma(f)$, $||f(x_1) - f(x_2)|| \leq p \cdot D_k ||x_1 - x_2||^k$.

Proof. We apply lemma 4.2 to the restriction of f to the straight line, passing through x_1 and x_2 in R^n . The two critical points of this restriction, which are required in lemma 4.2 (for $k \leq 3$), are x_1 and x_2 themselves. □

Now by lemma 4.4 the mapping $f : \Sigma(f) \to \Delta(f)$ satisfies the conditions of p.6, theorem 2.2. Since $\Sigma(f) \subset B_1^n$, $V_n(\Sigma(f)) < \infty$ by theorem 3.1. Hence, by therem 2.2, 6,

$$V_{\frac{n}{k}}(\Delta(f)) \leq (pD_k)^{\frac{n}{k}} V_{\frac{n}{k} \cdot k}(\Sigma(f)) < \infty \quad .$$

For p = 1, in a special case n = 1, the result was proved in theorem 4.1. For several variables it follows from a deep theorem 2 of chapter VI, §3, [4] , according to therem 3.10 above. □

Remark. If we define $\Sigma_{k-1}(f)$ to be the set of all $x \in M$, where all the derivatives of f up to the order k-1 vanish, and $\Delta_{k-1}(f) = f(\Sigma_{k-1}(f))$, then, using instead of lemma 4.4 the Taylor formula, we obtain that for any k (and not only for $k \leq 3$, as above)

$$V_{\frac{n}{k}}(\Delta_{k-1}(f)) < \infty \quad .$$

REFERENCES

1. V.I.Bakhtin, Properties of the entropy and the Hausdorff dimensions, Vestnik Mosk. Gos. Univ., Ser.1, no.5, 1982, 25 - 29.

2. L.Carleson, Selected Problems on Exceptional Sets, Van Nostrand Math. Studies 13, 1967.

3. H.Federer, Geometric Measure Theory, Die Grundlehren der Math. Wissenschafften, Vol. 153, Springer - Verlag, 1969.

4. L.D.Ivanov, Variazii množestv i funkzii, Moskva , Nauka 1975 .

5. A.N.Kolmogorov, Asymptotic characteristics of some completely bounded metric spaces, Dokl. Akad. Nauk SSSR 108, 1956, 585 - 589.

6. A.N.Kolmogorov, V.M.Tihomirov, ε - entropy and ε - capacity of sets in functional spaces, Uspehi Mat. Nauk 2 (86), 14, 1959, 3 - 86. English transl.,Amer. Math. Soc. Transl. (2) 17, 1961, 277 - 364.

7. G.G.Lorentz, Metric entropy and approximation, Bull. Amer. Math. Soc. 72, 1966, 903 - 937.

8. S.C.Milne, Peano curves and smoothness of functions, Adv. in Math. 35. no. 2, 1980, 129 - 157.

9. A.Sard, The measure of the critical values of differentiable maps, Bull. Amer. Math. Soc. 48, 1942, 883 - 890.

10. A.Sard, Hausdorff measure of critical images on Banach manifolds, Amer. J. of Math., 87, 1965, 158 - 174.

11. A.G,Vitushkin, On representation of functions by means of superpositions and related topics, L'Enseignement Math. XXIII, 1977, 255 - 320.

12. Y.Yomdin, The geometry of critical and near-critical values of differentiable mappings, to appear .

Max-Planck Institut für Mathematik, Gottfried-Claren Str.26, 5300 Bonn 3, BRD and Dept. of Math., Ben-Gurion University of the Negev, Beer-Sheva 84120, Israel .