The Gauss-Manin connexion of an isolated singularity and logarithmic Picard-Fuchs systems

A.G.Aleksandrov*

Abstract: This paper deals with the notion of generalized Picard-Fuchs systems with logarithmic singularities along a divisor on a complex manifold. Using properties of logarithmic differential forms and vector fields, we prove that such a system is characterized by the property to be regular singular along the divisor. In the case when the system in question has singularities along a free Saito divisor, a purely algebraic method of computing the integrability condition in terms of the commutation relations between its coefficient matrices is described. We calculate all Picard-Fuchs systems logarithmic along the discriminant of the minimal versal deformation of a plane curve singularity of type A_2 and of type A_3, as well as of a space curve singularity which is a wedge of a line and of an A_2-singularity. As a consequence, the Gauss-Manin connexion associated with isolated singularities of the above types is written down explicitly.

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1. Introduction

In the works of B.Malgrange, P.Deligne and F.Pham (see [11], [28]), a general notion of the Gauss-Manin system has been defined. E.Brieskorn seems, however, to be the first to adopt these ideas in the study of hypersurface *singularities* and some related questions. In his famous work [10], he has proved that the Gauss-Manin connexion, associated with the 1-parameter principal deformation of an isolated hypersurface singularity, can be described by a system of ordinary differential equations with *regular* singularities. Furthermore, it follows from his results that, after a meromorphic change of variables all coefficients of the system are transformed to the ones having poles at worst of the first order. Brieskorn's approach has been slightly modified, and extended to the case of complete intersections with isolated singularities by G.-M.Greuel [16], and to the general case of complex space germs with non-isolated singularities by H.Hamm [17].

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Meanwhile K.Saito [29], while calculating the connexion associated with the 3-parameter versal deformation of the hypersurface A_3-singularity, has obtained an elegant representation of the corresponding Gauss-Manin system of differential equations, making use of logarithmic differential forms. His ideas have not only led to the new important results in the theory of connexions, but also have given a strong impetus to the development of a quite intriguing theory of logarithmic differential forms (see [30], [1]).

In a different manner, the study of the Gauss-Manin systems for versal integrals of types A, D, E (closely related with the study of generalized functions) has been developed by V.P.Palamodov with H.Malikov [22], [26]. In particular, the corresponding systems of differential equations of the first order have been described as local Picard-Fuchs systems.

The concept of the Gauss-Manin connexion has been also investigated by S.Ishiura and M.Noumi [19], from another point of view. They describe the Gauss-Manin systems for A_μ-singularities with the help of the Hamiltonian representation. The Gauss-Manin systems appear naturally also in works of I.M. Gelfand, K. Aomoto, A.N. Varchenko and their followers, in connection with the study of the integrals of the type

$$J(t) = \int F_0^{\lambda_0}(z,t) F_1^{\lambda_1}(z,t) \dots F_k^{\lambda_k}(z,t)\, dz.$$

For instance, if $F_j(z,t)$, $1 \le j \le k$, are linear functions in variables $z = (z_1, \dots, z_m)$, these integrals are studied in [8], [35]; some special cases, in which $F_0(z,t)$ is a linear deformation of the Pham singularities, have been studied in [25], the case when all the $F_j(z,t)$ are quasihomogeneous is investigated in [13], and so on.

To conclude our brief excursus in the history of the problem, it should be noted that the famous proof by Yu.I.Manin of the Mordell conjecture for algebraic curves over functional fields [23] depends on the properties of the differential equations satisfied by the *period integrals* in an essential way. At the time when these works had impressed many a mathematician, it was A. Grothendieck who named the whole class of these systems of differential equations, considered in Manin's papers, *Gauss-Manin systems* or the *Gauss-Manin connexions.*

In fact, in all the cases when explicit solutions of the Gauss-Manin systems have been computed, there is also an appropriate representation of the solutions in terms of the generalized hypergeometric functions of known types. On the other hand, it has been discovered by K.Saito that in the case of an A_3-singularity, there is a system of linear differential equations with logarithmic singularities along the discriminant, whose solutions do not have an integral representation of Euler type. Consequently, it gives functions satisfying systems of differential equations rather close to the hypergeometric type.

In view of this phenomenon, we hope that the following *question* might be a subject of much current interest: does there exist a Gauss-Manin system associated with the versal deformation of an isolated singularity whose solutions can not be expressed in terms of hypergeometric functions of the known types and their generalizations ?

In recent papers (see [4], [5]), we have calculated the Gauss-Manin connexions associated with the 2-parameter principal deformations of complete intersection quasihomogeneous singularities $S_\mu,\ \mu \geq 5$, and with the versal deformations of simple hypersurface singularities $A_\mu,\ \mu \geq 2$. We also develop (see [6]) a computational approach to the study of the Gauss-Manin connexion for deformations of bouquets of curve singularities which are *non-complete* intersections. As an example, the Gauss-Manin system associated to the versal deformation of a bouquet of a line and an A_2-singularity is described as a connexion with logarithmic poles along the reduced part of the discriminant set. In all these cases, the corresponding Gauss-Manin systems have been presented as these ones, whose coefficients have logarithmic poles along the reduced part of the discriminant set. This enables one to present their solutions in the form of generalized hypergeometric functions of Pochhammer's type.

Analysis of the obtained results leads to a natural definition of a class of systems of linear differential equations with logarithmic singularities along a divisor, which we call logarithmic Picard-Fuchs systems or generalized Picard-Fuchs systems with logarithmic singularities.

It should be remarked that this class contains all the classical Fuchsian systems. For instance, such a system whose singular locus is a divisor with normal crossings is reduced to the system of Fuchsian type in the sense of [14]. Another famous example is the classical Picard-Fuchs system which describes Gauss-Manin connexion of a flat mapping (see [11], [31]). Furthermore, Feynman integrals, Pearcey and Connor functions as well as Cauchy- and Airy-type integrals (see [27]), and many other special functions may be regarded as solutions of appropriate generalized Picard-Fuchs systems with logarithmic singularities. The present notes are devoted to the study of the basic properties of such systems.

We start by reminding the basic notions and the definition of a generalized Picard-Fuchs system with logarithmic singularities. Then a theorem characterizing such a system as regular singular along a divisor is proved. This proof is based essentially on the basic properties of logarithmic differential forms and vector fields. In the case when the system in question has singularities along a free Saito divisor, a purely algebraic method for computing the integrability condition in terms of the commutation relations between its coefficient matrices is described. All the Picard-Fuchs systems logarithmic along the discriminant of the minimal versal deformation of plane curve singularities of type A_2 and A_3, and of a space curve, which is a bouquet of a line and an A_2-singularity are computed. As a consequence, the Gauss-Manin connexion associated with isolated singularities of the above types is written down explicitly.

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2. Logarithmic differential forms and vector fields

Let $S = (\mathbf{C}^m, 0)$, $h(z) = h(z_1, \dots, z_m)$ be a holomorphic function on S, and let $D \subset S$ be the hypersurface defined by the equation $h(z) = 0$. Suppose that D is *reduced,* that is, $h(z)$ has no multiple factors.

Definition 2.1. A meromorphic q-form ω, $q \geq 1$, on S is said to be a *logarithmic* (along a divisor D) differential form if and only if ω and its differential $d\omega$ have poles along D at worst of the first order. Thus both $h\omega$ and $hd\omega$ are *holomorphic* differential forms on S.

The $\mathcal{O}_S$-module of logarithmic differential q-forms is denoted by $\Omega_S^q(\log D)$ (see [29]). It should be remarked that the logarithmic differential forms have many remarkable analytic and algebraic properties (see [30], [1]).

Following [29], we denote the $\mathcal{O}_S$-module of vector fields *logarithmic* along $D \subset S$ by $\mathrm{Der}_S(\log D)$. This module consists of the germs of holomorphic vector fields η on S such that $\eta(h)$ belongs to the principal ideal $(h){\cdot}\mathcal{O}_S$ (in particular, the vector field η is tangent to D at its smooth points). The inner multiplication of vector fields and differential forms induces a natural pairing of $\mathcal{O}_S$-modules

$$\mathrm{Der}_S(\log D) \times \Omega_S^q(\log D) \longrightarrow \Omega_S^{q-1}(\log D). \tag{1}$$

Moreover, for $q = 1$ this $\mathcal{O}_S$-bilinear mapping is a *non-degenerate* (perfect) pairing so that $\mathrm{Der}_S(\log D)$ and $\Omega_S^1(\log D)$ become $\mathcal{O}_S$-dual. It is known [1] that there exists the following exact sequence of $\mathcal{O}_S$-modules:

$$0 \to \frac{dh}{h}\mathcal{O}_S + \Omega_S^1 \to \Omega_S^1(\log D) \xrightarrow{\cdot h} \mathrm{Tors}\,\Omega_D^1 \to 0, \tag{2}$$

where Ω_S^1 is the module of holomorphic differential 1-forms on S generated by the differentials $dz_1, \dots, dz_m$ over $\mathcal{O}_S$,

$$\Omega_D^1 = \Omega_S^1/(h\,\Omega_S^1 + \mathcal{O}_S dh) \tag{3}$$

is the module of regular holomorphic differential 1-forms on the divisor D, and $\mathrm{Tors}\,\Omega_D^1$ is the torsion submodule of Ω_D^1. The support of $\mathrm{Tors}\,\Omega_D^1$ is contained in the singular locus $\mathrm{Sing}\,D$ of the hypersurface D. The torsion $\mathcal{O}_D$-module $\mathrm{Tors}\,\Omega_D^1$ has a system of generators containing at least $m-1$ elements. A divisor D for which $\Omega_S^1(\log D)$ is a free $\mathcal{O}_S$-module, is usually called a *free* or *Saito-divisor.*

Proposition 2.2 (see [30]). *Suppose that there exist m vector fields $v_1, \dots, v_m \in$* $\mathrm{Der}_S(\log D)$ *so that for the square $m \times m$-matrix $\mathcal{V}$ whose entries are the coefficients of v_i, $i = 1, \dots, m$, one has* $\det(\mathcal{V}) = ch$, *where c is a unit. Then $v_1, \dots, v_m$ is a basis of the $\mathcal{O}_S$-module* $\mathrm{Der}_S(\log D)$. *In particular, $\Omega_S^1(\log D)$ is a free $\mathcal{O}_S$-module.*

For example, $\Omega_S^1(\log D)$ is free when D is the discriminant of the versal deformation of an isolated hypersurface or complete intersection singularity.

3. Logarithmic Picard-Fuchs systems

Let us consider a system of linear differential equations on S with *meromorphic* coefficients contained in $\Omega^1_S(\log D)$:

$$d\,\mathbf{I} = \left(A_0 \frac{dh}{h} + \sum_{i=1}^{\ell} A_i \frac{\vartheta_i}{h} \right) \mathbf{I}, \tag{4}$$

where $\mathbf{I} = {}^{\mathsf{T}}(f_1, \dots, f_k)$ is a vector-column of unknown functions. The differential 1-forms $\vartheta_i \in \Omega^1_S,\ i = 1, \dots, \ell$, correspond via (2) to the non-zero elements of the torsion module $\operatorname{Tors} \Omega^1_D$, and $A_i \in \operatorname{End}(\mathbf{C}^k) \otimes \mathcal{O}_S,\ i = 0, 1, \dots, \ell$, are the coefficient matrices with holomorphic entries satisfying the integrability condition:

$$d\,\Theta = \Theta \wedge \Theta,$$

where

$$\Theta = A_0 \frac{dh}{h} + \sum_{i=1}^{\ell} A_i \frac{\vartheta_i}{h}, \tag{5}$$

or, equivalently,

$$\sum_{i=0}^{\ell} dA_i \wedge \frac{\vartheta_i}{h} + A_i d(\frac{\vartheta_i}{h}) = \sum_{0 \le i < j \le \ell} [A_i,\ A_j] \frac{\vartheta_i}{h} \wedge \frac{\vartheta_j}{h}. \tag{6}$$

Here the total differential dh is denoted by ϑ_0, for brevity.

Definition 3.1 (see [2]). The system of linear differential equations (4) is called a (local) Fuchsian system with logarithmic singularities or a logarithmic Picard-Fuchs system.

Assume now that $D = \bigcup_{i=1}^n D_i$, where $D_i = \{z \in \mathbf{C}^m \colon h_i(z) = 0\}$, is the local decomposition of the divisor D at a point $o \in D$, so that if the hypersurface D is defined by the germ of a holomorphic function $h(z)$ then $h(z) = h_1(z) \cdots h_n(z),\ n \ge 2$, is the primary decomposition of the germ $h(z)$. Suppose that D is *reduced,* that is, $h(z)$ has no multiple factors. It is evident that the meromorphic forms

$$\frac{dh_1}{h_1}, \dots, \frac{dh_n}{h_n}$$

are contained in $\Omega^1_S(\log D)$, and one may consider a linear Pfaffian system of the above type (4) :

$$d\,\mathbf{I} = (\sum_{i=1}^{n} A_i \frac{dh_i}{h_i})\,\mathbf{I}. \tag{7}$$

In the case when all D_i are *nonsingular* the system of differential equations (7) was investigated in a series of works (see [34] and references there). One of the most important results of the theory is the K.Takano theorem (loc. cit.) which can be stated as following. Suppose that the total sum of the coefficient matrices A_i at the distinguished point $o \in S$ has eigenvalues which do not differ by positive integers. Then, by means of a *holomorphic* change of variables in a neighbourhood of the point o, the system (7) can be reduced to a similar one with constant coefficient matrices $A_i(o)$. In the case, when the *nonsingular* hypersurfaces D_i are in a *general position* a fundamental solution of the latter system has been described by Gérard and Levelt [14], Yoshida and Takano [37], and A.Bolibruch [9].

Remark 3.2. Suppose $h_i(z)$ be *linear* or, more generally, *homogeneous* functions, so that the corresponding $D_i,\ i = 1, \dots, n$, are *hyperplanes* or, *non-singular* hypersurfaces of degrees d_i, respectively. Assume that $\sum_{i=1}^{n} d_i A_i = 0$. Then the system (7) has a canonical *extension* to the whole $\mathbf{P}^m$. Such systems have appeared in [14] and are usually called *Pfaffian systems of Fuchsian type* (cf. [7]).

In this work, we shall discuss another interesting example which arises naturally in the theory of singularities (see, e.g. [31], [5]). Let X_0 be an isolated singularity of positive dimension $n \geq 1$. Consider its minimal versal deformation $f\colon X \to S$. We will suppose that the following conditions are fulfilled:

1) the base space of the deformation f is *smooth,* that is $S \cong \mathbf{C}^k$;

2) the reduced part $D_{\text{red}} \subset S$ of the discriminant D of the deformation f (or, equivalently, of the set of *critical values* of the mapping f) is a *hypersurface.*

Let $\Omega^\bullet_{X/S}$ be the relative de Rham complex of X over S with the usual differential d. Then the sheaves of cohomology groups $\mathcal{H}^p = \mathcal{H}^p(f_*\Omega^\bullet_{X/S}, d)$ are well-defined. The coherence theorem implies that $\mathcal{H}^p = \mathcal{H}^p(f_*\Omega^\bullet_{X/S}, d)$ are *Noetherian* $\mathcal{O}_S$-modules for every $0 < p \leq n$, and $\mathcal{H}^p = 0$ for all $p > n$ (see [21], (8.7)). Moreover, if f is a *smoothing*, the rank of $\mathcal{H}^p(\Omega^\bullet_{X/S})$ is equal to the p^{th} Betti number of the Milnor fibre. It should be also remarked that the stalks of the restriction of this sheaf to the complement $S' = S \setminus D_{\text{red}}$ are isomorphic to the p-th cohomology group $H^p(X_s, \mathbf{C})$ for all $s \in S'$.

Using the action on the de Rham complex $\Omega^\bullet_X$ of the Lie derivative associated with *liftable* along f vector fields on the base S, it is possible to define an integrable connexion (see [21], §8)

$$\nabla_{X/S} : \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \ \longrightarrow\ \mathcal{H}^p(f_*\Omega^\bullet_{X/S}) \otimes \Omega^1_S(*D) \tag{8}$$

where $\Omega^1_S(*D)$ denotes the $\mathcal{O}_S$-module of meromorphic 1-forms with polar singularities along D. The connexion ∇ coincides over the complement $S' = S \setminus D_{\text{red}}$ with the natural topological connexion on $R^p f_* \mathbf{C}_X \otimes_{\mathbf{C}} \mathcal{O}_{S'}$ which can be realized as the transfer of the cohomology classes $H^p(X_s, \mathbf{C}),\ s \in S'$, along the tangent directions on the complement S' (see [21], (8.9)). It should be underlined that in this construction we do not need any assumptions concerning the freeness of $\mathcal{O}_S$-module $\mathcal{H}^p(f_*\Omega^\bullet_{X/S})$ although its restriction on S' is a free $\mathcal{O}_{S'}$-module.

On the other hand, according to [29], a connexion ∇ on $\mathcal{O}_S$-module $\mathcal{H}$ with *logarithmic* poles along the divisor D

$$\nabla_{X/S}\colon\ \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{O}_S} \Omega^1_S(\log D) \tag{9}$$

is a morphism satisfying the following conditions:

1) $\nabla(\omega+\omega') = \nabla(\omega) + \nabla(\omega')$,

2) $\nabla(f\omega) = \omega \otimes df + f\nabla(\omega)$,

where $\omega,\ \omega' \in \mathcal{H}$ and $f \in \mathcal{O}_S$.

Assume that D is a *free Saito-divisor*. Let $\omega_1, \ldots, \omega_m$ be free generators of $\Omega^1_S(\log D)$. Then the connexion ∇ on $\mathcal{H} = \Omega^1_S(\log D)$:

$$\nabla\colon\ \Omega^1_S(\log D) \longrightarrow \Omega^1_S(\log D) \otimes_{\mathcal{O}_S} \Omega^1_S(\log D) \tag{10}$$

can be expressed with the help of the Christoffel symbols in the following way:

$$\nabla\omega_i = \sum_{j=1}^{m} \omega_j \otimes \omega_i^j, \quad \omega_i^j = \sum_{k=1}^{m} \Gamma_i^{jk}\omega_k.$$

The connexion ∇ is called *torsion free* if

$$d\,\omega_i = \sum_{j=1}^{m} \omega_i^j \wedge \omega_j = \sum_{k,j=1}^{m} \Gamma_i^{jk}\omega_k \wedge \omega_j,$$

and ∇ is called *integrable* if

$$d\omega_i^j = \sum_{k=1}^{m} \omega_i^k \wedge \omega_k^j,$$

that is,

$$d\,\nabla = \nabla \wedge \nabla.$$

Hence one can associate to any *integrable* and *torsion free* connexion ∇ on the module $\Omega^1_S(\log D)$ the logarithmic Picard-Fuchs system (4).

In fact, this construction helps one to compute the Gauss-Manin connexion (8) in some special cases involving the case of a plane (or a space) curve singularity. Since in these cases all the above assumptions hold, the discriminant (or its reduced part) is a free Saito divisor, and the $\mathcal{O}_S$-module $\mathcal{H} = \mathcal{H}^1(f_*\Omega^\bullet_{X/S})$ has the *same rank* as the $\mathcal{O}_S$-module $\Omega^1_S(\log D)$.

4. Fuchsian systems and regular singularities

Let Φ be a square k-matrix whose entries are meromorphic differential 1-forms on S. Suppose Φ satisfy the integrability condition

$$d\,\Phi = \Phi \wedge \Phi,$$

and have the following presentation:

$$\Phi = B_1(z)\frac{dz_1}{z_1} + \sum_{i=2}^{m} B_i(z)dz_i, \tag{11}$$

where $(z_1, \dots, z_m)$ is a local coordinate system at the distinguished point $o \in S$, and $B_i(z) \in \mathrm{End}(\mathbf{C}^k) \otimes \mathcal{O}_S|_U$, $i = 0, 1, \dots, m$, are matrices with holomorphic entries on S. According to [11] (cf. [24]), the system of linear meromorphic differential equations

$$d\mathbf{I} = \Phi \cdot \mathbf{I}, \tag{12}$$

in an unknown vector-valued function $\mathbf{I} = {}^{\mathsf{T}}(f_1, \dots, f_k)$ is said to have a *regular singularity* along the subset $\{z_1 = 0\}$.

Definition 4.1. The system (12) has *regular singularities* along the hypersurface $D \subset S$ if and only if

1) all the entries of the matrix Φ are holomorphic differential 1-forms on $S \setminus D$;

2) for any nonsingular point $x \in D$ there exists a neighbourhood $U \subset S$ of the point x such that Φ has a regular singularity along the set $D \cap U$ defined by the local equation $z_1 = 0$.

In the terminology of [18], when D is a *nonsingular* hypersurface the matrix-valued differential 1-form Φ defines a meromorphic connection with *logarithmic pole* over D.

Theorem 4.2. *Let Φ be a square k-matrix as above whose entries are differential 1-forms with poles along the hypersurface D given by the equation $h = 0$. Then the system* (12) *has regular singularity along D if and only if it is Fuchsian of the type* (4) *with logarithmic singularities along the divisor D.*

The proof is a direct consequence of elementary properties of logarithmic differential forms and vector fields. Let us consider a Fuchsian system (4) with logarithmic singularities. Then any entry of the matrix Φ is a meromorphic differential form $\omega \in \Omega^1_S(\log D)$. Fix a *non-singular* point $x \in D$. One can choose a neighbourhood U of x in S and a local coordinate system in such a way that the local equation of D in $x \in U$ looks as follows: $z_1 = 0$. The differential form ω is transformed to the form $\tilde{\omega}$ logarithmic along the hyperplane $z_1 = 0$ in the neighbourhood U of $x \in S$. It is not difficult to see that

$$\tilde{\omega} \in \mathcal{O}_U\langle dz_1/z_1\rangle = \Omega^1_S(\log D)|_U.$$

For, by definition, any logarithmic vector field η is tangent to D at its smooth points, that is $\eta = g(z)z_1\partial/\partial z_1$, where $g(z)$ is holomorphic in U. The perfect duality (1) between logarithmic 1-forms and vector fields implies that $\tilde{\omega} \in \Omega^1_S(\log D)|_U$ has the desired representation:

$$\tilde{\omega} = g_1(z)\frac{dz_1}{z_1} + \sum_{i=2}^{m} g_i(z)dz_i.$$

Consequently, the total coefficient matrix is transformed analogously:

$$\widetilde{\Phi} = B_1(z)\frac{dz_1}{z_1} + \sum_{i=2}^{m} B_i(z)dz_i,$$

that is, the system in question has regular singularities along the hypersurface D in the sense of definition (4.1). The converse is proved in a similar manner. In fact, in a local coordinate system any entry of the matrix Φ has a representation similar to (11). In virtue of the perfect duality (1), there are a vector field η and a function g, both holomorphic over $S \setminus \operatorname{Sing} D$ and such that

$$\eta(h) = g \cdot h.$$

Since D is reduced, the codimension of the analytic subset $\operatorname{Sing} D$ in S is at least two. Hence all the coefficients of the vector fields η, g and the latter identity, can be prolonged holomorphically to the whole of S. Thus $\eta \in \operatorname{Der}_S(\log D)$, and, in virtue of the perfect duality, it defines again a *logarithmic* differential 1-form $\omega \in \Omega^1_S(\log D)$, i.e. the corresponding element of the coefficient matrix Φ.

Remark 4.3. As usual, we define regular singularities for a class of meromorphically equivalent coherent sheaves on S (cf. [11], [31]). That is, the corresponding systems of differential equations are considered up to meromorphic changes of independent variables. Moreover, from a such point of view, the notions of a system with regular singularities and with logarithmic poles are equivalent. This fact is proved [11] by a reducing the general case to the case of a divisor having only normal crossing singularities. It should be noted that K.Saito [31] introduced a slightly different notion of a regular singularity. In his terminology, a system with regular singularities along a divisor D, has logarithmic poles along D. However, the converse is not true, at least in the case of the Picard-Fuchs system corresponding to the Gauss-Manin connexion associated with a flat family of isolated singularities.

5. The integrability condition and commutation relations

Let us now pass to the question of how one can express the integrability condition for the Fuchsian system (4) in terms of the coefficient matrices $A_i,\ i = 1, \dots, \ell$.

First consider a special case of such systems (7) studied by Gérard and Levelt, and their followers:

$$d\,\mathbf{I} = (\sum_{i=1}^{n} A_i \frac{dh_i}{h_i})\,\mathbf{I}.$$

Thus $D = \bigcup_{i=1}^{n} D_i$, and all the divisors D_i are non-singular. As it has been mentioned above, in view of the results in [34], one can suppose the coefficient matrices

$A_i,\ i = 1, \dots, n$, be constant. Assume additionally that the divisors D_i are in a general position, in the sense of [14]. It means that for every pair i, j, there is a point $x \in D_i \cap D_j$ such that $x \notin D_k$ for all $k \neq i, j$. It is well-known that in this case the integrability condition (6) is equivalent to the permutability of the matrices $A_i,\ i = 1, \dots, n$, that is, the following commutation relations hold (loc. cit.):

$$[A_i,\, A_j] = 0, \qquad 1 \leq i \neq j \leq n. \tag{13}$$

In particular, this is true when the divisors D_i have *normal crossings.* One can prove that similar relations hold without the assumption "D_i are in a general position" (see [36]).

It is not difficult to see that the system (4) is well defined in the case when D is a union of *singular* hypersurfaces. In general, the problem of description of commutation relations is highly non-trivial and very interesting.

In this section, we show that the theory of logarithmic forms is a rather effective tool in computing commutation relations for the systems (7), as well as in the general case of the Fuchsian system (4) with logarithmic singularities.

In what follows two examples $m = 2$, and the divisor D is a plane curve are considered. In these cases, D is always a *free* divisor [29]. First we consider the case when D is *irreducible.* Set $x = z_1,\ y = z_2$ and $h = 27x^2 + 4y^3 = 0$. Thus D is the plane cusp coinciding with the discriminant of the minimal versal deformation of an A_2-singularity. Consider the generalized Picard-Fuchs system with logarithmic singularities:

$$d\,\mathbf{I} = \left(A_0 \frac{dh}{h} + A_1 \frac{\vartheta}{h} + A_2 \frac{y\vartheta}{h} \right) \mathbf{I}, \tag{14}$$

where the differential form $\vartheta = 2ydx - 3xdy$ corresponds to the element of the torsion submodule $\mathrm{Tors}\,(\Omega^1_D)$. Assume additionally that all the entries of the matrices $A_i,\ i = 0, 1, 2$, are *complex numbers.* Set $\omega_1 = dh/h,\ \omega_2 = \vartheta/h$. One can verify that the logarithmic differential forms $\omega_1,\ \omega_2$ are *free* generators of $\Omega^1_S(\log D)$, and that the following identities hold:

$$d\,\omega_1 = 0,\ d\,\omega_2 = -\frac{1}{6}\omega_1 \wedge \omega_2,$$

$$\omega_1 \wedge \omega_2 = -6\,\frac{dx \wedge dy}{h},\ dy = \frac{1}{3} y\omega_1 - 9x\omega_2.$$

On substituting this in the integrability condition (6), making use of the fact that the module

$$\Omega^2_S(\log D) = \bigwedge^2 \Omega^1_S(\log D)$$

is free, one can obtain, by an easy computation, the following commutation relations

$$-\frac{1}{6} A_1 + [A_0,\, A_1] = 0,\ \frac{1}{6} A_2 + [A_0,\, A_2] = 0.$$

In the next section, all the systems (14) will be described explicitly. In the same manner, using elementary properties of the logarithmic differential forms one can obtain commutation relations for the generalized Picard-Fuchs system (4) associated with a *reducible* free Saito-divisor D. As an example, let D have two irreducible components with *non-normal* crossing:

$$h = x(x^{n-1} - y^k), \quad h_1 = x, \quad h_2 = (x^{n-1} - y^k),$$

where $n \geq 2$, $k \geq 1$, and $D = D_1 \cup D_2 \subset S = \mathbf{C}^2$.

Then the free $\mathcal{O}_S$-module $\Omega^1_S(\log D)$ is generated by the two logarithmic differential 1-forms $\omega_1 = dh/h$, $\omega_2 = \vartheta/h$, where

$$\vartheta = -\frac{n-1}{kn} y dx + \frac{1}{n} x dy.$$

It is easy to verify that

$$\frac{dh_1}{h_1} = \frac{1}{n}\omega_1 + ky^{k-1}\omega_2, \quad \frac{dh_2}{h_2} = (1 - \frac{1}{n})\omega_1 - ky^{k-1}\omega_2.$$

Hence the system (7)

$$d\,\mathbf{I} = (A_1 \frac{dh_1}{h_1} + A_2 \frac{dh_2}{h_2})\,\mathbf{I},$$

is equivalent to the following Fuchsian system with logarithmic singularities

$$d\,\mathbf{I} = \left((\frac{1}{n}A_1 + (1 - \frac{1}{n})A_2)\frac{dh}{h} + k(A_1 - A_2)\frac{y^{k-1}\vartheta}{h} \right) \mathbf{I}. \tag{15}$$

For simplicity, assume that all the entries of the matrices A_1 and A_2 are constant, that is, $dA_1 = dA_2 = 0$. Again, since the module $\Omega^2_S(\log D)$ is free, one then gets that the standard integrability condition is equivalent to the following commutation relation:

$$0 = [\frac{1}{n}A_1 + (1 - \frac{1}{n})A_2,\, k(A_1 - A_2)] = -k[A_1,\, A_2] = 0.$$

Remark 5.1. Another approach to computing the commutation relations for the system (7) has been developed in [15], [20], and in a series of subsequent papers. It turns out that, under certain assumptions on the divisor D, the commutation relations between the coefficient matrices A_i of the system (7) reflect some topological properties of the fundamental group $\pi_1(\mathbf{P}^m \setminus D)$ of the complement of the union $D = \bigcup_{i=1}^n D_i$, and vice versa. To be more precise, this method works in the case when the fundamental group is abelian, as well as in the case when all the relations between the generators of the fundamental group are expressed in terms of the commutators of the generators. In the latter example, the fundamental group of the complement is *not abelian* (cf. e.g. [29]). However, in the case $n = 2$

it is generated by the two elements g_1 and g_2, corresponding to the irreducible components D_1 and D_2, respectively, with the basic relation $(g_1g_2)^k = (g_2g_1)^k$, and the above approach still works. For completeness, it should be also noted that in 1975 K.Saito conjectured (see [29]) that there is a kind of "duality" between the module of logarithmic differential 1-forms $\Omega^1_S(\log D)$ and the local fundamental group $\pi_1(S \setminus D)$ of the complement of the divisor $D = \bigcup_{i=1}^n D_i$.

6. Logarithmic Picard-Fuchs systems for A_μ-singularities

At first let us describe logarithmic Picard-Fuchs systems (4) whose singular locus is the discriminant D of the versal deformation of a simple A_2-singularity given by the equation $h = 27x^2 + 4y^3 = 0$. Thus, using the natural grading on $\mathcal{O}_S$, $\Omega^1_S(\log D)$ and Ω^1_D induced by the weights of x and y (equals to 3 and 2, respectively), one can show that the matrix ∇ from (10) looks as follows:

$$\nabla = \begin{pmatrix} g\omega_1 & a\omega_2 \\ by\omega_2 & c\omega_1 \end{pmatrix},$$

where a, b, c, g are $\mathbf{C}$-valued parameters, $\omega_1 = dh/h$, $\omega_2 = \vartheta/h$, and $\vartheta = 2ydx - 3xdy$. Note that

$$dy = \frac{1}{3}y\omega_1 - 9x\omega_2.$$

In addition, the integrability condition $d\nabla = \nabla \wedge \nabla$ yields $c = g + \frac{1}{6}$. Thus, one gets the 3-parameter family depending on a, b and g which defines the connexion (10). The characteristic polynomial of the corresponding system of differential equations is equal to

$$\lambda^2 - 2(g + \frac{1}{12})\lambda + (\frac{ab}{4\cdot 27} + g(g + \frac{1}{6})).$$

On the other hand, one can consider the versal integrals of an A_2-singularity:

$$I_0(s) = \int \frac{dz}{\sqrt{z^3 + s_1z + s_0}}, \qquad I_1(s) = \int \frac{zdz}{\sqrt{z^3 + s_1z + s_0}},$$

where the contour of the integration is a homology cycle of the elliptic curve given by the equation $w^2 = z^3 + s_1z + s_0$ (see [26]). Using the presentation from (loc. cit.), it is easy to verify by a direct computation that the Gauss-Manin system for versal integrals is of the form (4), with

$$\Theta = \begin{pmatrix} \left(\frac{1}{2}q - \frac{1}{3}\right)\frac{dh}{h} & -3(3q-1)\frac{\vartheta}{h} \\ (3q-2)s_1\frac{\vartheta}{h} & \left(\frac{1}{2}q - \frac{1}{6}\right)\frac{dh}{h} \end{pmatrix},$$

where q is a $\mathbf{C}$-valued parameter, $\vartheta = 2s_1 ds_0 - 3s_0 ds_1$, so that $\omega = \vartheta/h$ is a *logarithmic* differential form and $d\Theta = \Theta \wedge \Theta$. The characteristic equation is

$$\lambda(\lambda + q - \frac{1}{2}) = 0.$$

Setting $q = 1/2$, one obtains the presentation [29]:

$$\Theta = \begin{pmatrix} -\frac{1}{12}\frac{dh}{h} & -\frac{3}{2}\frac{\vartheta}{h} \\ -\frac{1}{2}s_1\frac{\vartheta}{h} & \frac{1}{12}\frac{dh}{h} \end{pmatrix}.$$

The corresponding Picard-Fuchs system is reduced to the equation of the second order for the integral $u = I_0$, which is nothing else but the classical Legendre equation:

$$(t^2 - \frac{1}{4})u'' + 2tu' + \frac{5}{36}u = 0, \tag{16}$$

where the parameter t is equal to $s_0 s_1^{-3/2}$ up to a constant multiple.

In fact, the equation (16) is a special case of the hypergeometric differential equation. To be more precise, it is a *Fuchsian* equation with 3 singular points $a_1 = 1/2$, $a_2 = -1/2$, $a_3 = \infty$ on Riemann sphere $\overline{\mathbf{C}}$. On making a substitution $t := -t + 1/2$, one proves with the help of formula (7.3.6) in [7] that the corresponding *Fuchsian system* can be given as follows:

$$\frac{d\mathbf{x}}{dt} = \left(\begin{pmatrix} 0 & 0 \\ \frac{5}{36} & -1 \end{pmatrix} \frac{1}{t - 1/2} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{t + 1/2} \right) \mathbf{x},$$

where $\mathbf{x}$ denotes the vector-column ${}^{\mathsf{T}}(x_1, x_2)$ of unknown functions. It is not difficult to compute that the characteristic (or initial, or indical) polynomial of the equation (16) has the double root: $\lambda^2 = 0$. However, all the connexions ∇ from the above 3-parameter family with $ab = 3/4$ have initial polynomials of the same kind. After all, the general theory implies (cf. [12]) that all the corresponding Fuchsian systems or Fuchsian equations are reduced to the equation (16).

In fact, one can write down explicitly all the Pfaffian systems of the Fuchsian type on the projective plane $\mathbf{P}^2$, whose singular locus is the plane cusp (for the details, see [24]).

Let us now consider the case of an A_3-singularity. First we recall and correct slightly some computational results due to K.Saito [29], where he studied the connexion (10):

$$\nabla_{X/S}\colon \Omega^1_S(\log D) \longrightarrow \Omega^1_S(\log D) \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)$$

for the versal deformation of an A_3-singularity.

Let $F(z,s) = z^4 + t_2 z^2 + t_3 z + t_4 = 0$ be the equation of the minimal versal deformation of an A_3-singularity. Then its discriminant $D \subset S = \mathbf{C}^3$ is defined by the equation

$$h = 256t_4^3 - 128t_2^2t_4^2 + 16t_2^4t_4 + 144t_2t_3^2t_4 - 4t_2^3t_3^2 - 27t_3^4 = 0.$$

Let $\mathcal{V}$ be the square matrix whose entries are coefficients of logarithmic vector fields. Then the determinant of $\mathcal{V}$ is equal to h up to a unit multiple. To be more precise,

$$V^1 = \frac{1}{6}(2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 4t_4\frac{\partial}{\partial t_4}),$$

$$V^2 = 3t_3\frac{\partial}{\partial t_2} + (4t_4 - t_2^2)\frac{\partial}{\partial t_3} - \frac{1}{2}t_2t_3\frac{\partial}{\partial t_4},$$

$$V^3 = (t_2^2 - 12t_4)\frac{\partial}{\partial t_2} + 3t_2t_3\frac{\partial}{\partial t_3} + (\frac{9}{4}t_3^2 - 4t_2t_4)\frac{\partial}{\partial t_4}.$$

The corresponding matrix $\mathcal{V}$ has the following representation:

$$\mathcal{V} = \begin{pmatrix} \frac{1}{3}t_2 & \frac{1}{2}t_3 & \frac{2}{3}t_4 \\ 3t_3 & 4t_4 - t_2^2 & -\frac{1}{2}t_2t_3 \\ t_2^2 - 12t_4 & 3t_2t_3 & \frac{9}{4}t_3^2 - 4t_2t_4 \end{pmatrix}.$$

One can verify that $\det(\mathcal{V}) = 2^{-3}P$. In our case the $\mathcal{O}_S$-module of the logarithmic differential forms $\Omega_S^1(\log D)$ is free and there is the dual basis with respect to the vector fields V^1, V^2, V^3 defined by the 2×2-minors of the matrix $\mathcal{V}$:

$$\omega_1 = \frac{1}{2}\frac{dh}{h}, \quad \omega_2 = \frac{\vartheta_2}{h}, \quad \omega_1 = \frac{2}{3}\frac{\vartheta_3}{h}, \tag{17}$$

$$\vartheta_2 = (32t_2t_3t_4 - 9t_3^3)d\,t_2 + (6t_2t_3^2 + 64t_4^2 - 16t_2^2t_4)d\,t_3 - (4t_2^2t_3 + 48t_3t_4)d\,t_4\,;$$

$$\vartheta_3 = (8t_2^2t_4 - 3t_2t_3^2 - 32t_4^2)d\,t_2 + (2t_2^2t_3 + 24t_3t_4)d\,t_3 + (16t_2t_4 - 4t_2^3 - 18t_3^2))d\,t_4\,.$$

One can verify the following identities:

$$d\,\omega_1 = 0,\ d\,\omega_2 = -\frac{1}{6}\omega_1\wedge\omega_2 - t_2\omega_2\wedge\omega_3,\ d\,\omega_3 = -\frac{1}{3}\omega_1\wedge\omega_3;$$

$$\omega_1\wedge\omega_2\wedge\omega_3 = 8\frac{dt_2\wedge dt_3\wedge dt_4}{h}.$$

By definition, $\iota_{V^i}(\omega_j) = \delta_{ij}$, where we denote the contraction along the vector field V by ι_V.

Let us recall, that our goal is to describe the connexion on $\Omega^1_S(\log D)$. Because the weights of variables t_2, t_3, t_4 induce the natural grading on the modules of logarithmic differential forms and vector fields, it is easy to see that such a connexion can be described by the following connection matrix:

$$\nabla = \begin{pmatrix} g\omega_1 + at_2\omega_3 & m\omega_2 & n\omega_3 \\ dt_2\omega_2 + et_3\omega_3 & (g+\frac{1}{6})\omega_1 + kt_2\omega_3 & f\omega_2 \\ (bt_2^2 + ct_4)\omega_3 + ht_3\omega_2 & it_2\omega_2 + jt_3\omega_3 & (g+\frac{1}{3})\omega_1 + lt_2\omega_2 \end{pmatrix},$$

where a, b, c, d, e, f, g, h, i, j, k, l, m, n are the fourteen $\mathbf{C}$-valued parameters. The matrix ∇ satisfies the integrability condition

$$d\nabla = \nabla \wedge \nabla.$$

Note, that

$$\sum_{i=1}^{3} V^i(f)\omega_i = \sum_{i=1}^{3} \frac{\partial f}{\partial t_i} dt_i \quad \text{for any } f \in \mathcal{O}^3_{\mathbf{C}}.$$

Hence, the columns of the matrix $\mathcal{V}$ define the relations between the total differentials of variables t_i and the basic logarithmic forms ω_i:

$$dt_2 = \frac{1}{3}t_2\omega_1 + 3t_3\omega_2 + (t_2^2 - 12t_4)\omega_3,$$

$$dt_3 = \frac{1}{2}t_3\omega_1 + (4t_4 - t_2^2)\omega_2 + 3t_2t_3\omega_3,$$

$$dt_4 = \frac{2}{3}t_4\omega_1 - \frac{1}{2}t_2t_3\omega_2 + (\frac{9}{4}t_3^2 - 4t_2t_4)\omega_3.$$

After some calculations, we obtain the following system of 10 equations on 13 parameters, which are equivalent to the *integrability* condition $d\nabla - \nabla \wedge \nabla = 0$:

$$\begin{cases} 3a - em + hn & = 0 \\ m + mk - in - am & = 0 \\ 2d + e - kd + bf + ad & = 0 \\ 12d + 4e - cf & = 0 \\ 3k + em - fj & = 0 \\ f - fk + dn + fl & = 0 \\ 3l - hn + fj & = 0 \\ 2i + j - bm + ki - il & = 0 \\ 12i + 4j + cm & = 0 \\ 4h + 1/2c - 6b + ei - dj - hl + ah & = 0 \end{cases}$$

None of these equations depends on the parameter g. Note that the equations (1), (5), (7) imply the condition

$$a + k + l = 0.$$

Further, two triples (2), (8), (9) and (3), (4), (6) of homogeneous equations in variables i, j, m and d, e, f, respectively, have the same determinant. Replacing two equations (3), (8) by the determinant and excluding a, e, k, l, j, b we get the following standard basis of the corresponding ideal:

$$\begin{cases} cfm^2 + 4fim - 8dm^2 - \frac{4}{3}hmn + 4in - 4m & = \; 0 \\ cf^2m + 8f^2i - 4dfm + \frac{4}{3}fhn + 4dn + 4f & = \; 0 \\ f^2i^2 + 2dfim + d^2m^2 + \frac{1}{6}cfin + \frac{2}{3}fhin + \frac{1}{6}cdmn & \\ +\frac{2}{3}dhmn + \frac{5}{2}fi + \frac{5}{2}dm + \frac{1}{3}cn + \frac{4}{3}hn + 1 & = \; 0 \end{cases}$$

These 3 equations in 7 variables define a 4-dimensional space of solutions. Besides, each of the solutions depends on the *free* parameter g.

Now, look at the system (4):

$$d\mathbf{I} = \Theta \cdot \mathbf{I}.$$

This system is a maximal overdetermined system of differential equations with *regular singular* solutions. Choose a curve in the base space, which is in a general position relatively to the singular locus of the system. Then the restriction of the system to this curve is reduced to the differential equation with *regular singularities* of the type

$$\frac{d\mathbf{I}}{dt} = \frac{\overline{\Theta}_{-1}}{t} + \sum_{s \geq 0} \overline{\Theta}_s t^s.$$

Making a substitution $\mathbf{I} = A\,\mathbf{I}'$, one can transform this system to the similar one with

$$\overline{\Theta}'_{-1} = A^{-1}\overline{\Theta}_{-1}A.$$

Moreover, up to such transformations, the above system is defined completely by the *elementary divisors* of the corresponding initial polynomial $Q(\lambda)$ of the characteristic matrix $\| \lambda E - \overline{\Theta}_{-1} \|$ associated with the *residue matrix* $\overline{\Theta}_{-1}$ (see [12], Ch.XI, §10), so that

$$Q(\lambda) = \det(\lambda E - \overline{\Theta}_{-1}). \tag{18}$$

One can see that only one of the following 3 possibilities occurs:
(#1) the polynomial $Q(\lambda)$ of the third degree has one root of multiplicity three;
(#2) the polynomial $Q(\lambda)$ has 2 distinct roots, that is, one of them is multiple;
(#3) the polynomial $Q(\lambda)$ has 3 distinct roots.

It turns out that the second and third cases are actually realized, they give us *two different* types of Picard-Fuchs systems. In other words, we obtain *two irreducible components* parameterizing the family of all the uniformization equations in the sense of K.Saito [29]. More precisely, Saito found the following two sets of parameters (see [29], §3) for the connexion ∇.

Type I:

$$a=0,\ b=(2k+\frac{1}{2}),\ c=6,\ d=-\frac{4}{3}k,\ e=h=3k,\ f=-\frac{2}{3}k,$$

$$g=\frac{1}{3}k-\frac{1}{6},\ i=(k+1),\ j=-\frac{9}{4}(2k+1),\ l=-k,\ m=n=(k-\frac{1}{2}).$$

Type II:

$$a=0,\ b=-\frac{1}{4}(k+4)(k-2),\ c=3(3k^2+2k-4),\ d=-\frac{1}{3k}(4k-2),$$

$$e=h=-3(\frac{1}{2}k-1),\ f=-\frac{2}{3k},\ g=\frac{1}{3}k-\frac{1}{6},\ i=(k+1),\ j=-\frac{9}{4}k(k+2),$$
$$l=-k,\ m=n=1.$$

The system of differential equations of the first type has the *characteristic polynomial*

$$\lambda^2(\lambda-\frac{1}{2}k),$$

while the system of the second type has the characteristic polynomial

$$\lambda(\lambda-\frac{1}{4}k+\frac{1}{4})(\lambda-\frac{1}{4}k-\frac{1}{4}),$$

where in both cases $k=3g+\frac{1}{2}$ is a $\mathbf{C}$-valued parameter. For $k=\pm1$, the characteristic polynomials of both types of the systems coincide.

For completeness, it should be noted that the characteristic polynomial in the second case written down in the paper of K.Saito [29] is not correct.

It remains to observe that the system of Type I corresponds indeed to the Gauss-Manin connexion associated with an A_3-singularity. In fact, the generic fiber X_t of the minimal versal deformation of an A_3-singularity

$$w^2=z^4+t_2z^2+t_3z+t_4 \tag{19}$$

is isomorphic to a hyperelliptic curve of genus 1. Hence its first homology and cohomology groups with coefficients in $\mathbf{C}$ have the rank 2, and the three period integrals

$$I_\ell(t)=\int_{\gamma(t)}\frac{z^\ell dz}{\sqrt{z^4+t_2z^2+t_3z+t_4}},\qquad \ell=0,1,2, \tag{20}$$

where the contour of the integration $\gamma(t)$ is a homology cycle of the hyperelliptic curve given by the equation (19) satisfy the differential equation of the second order, although the Milnor number of the special fibre X_0 is equal to 3. That is why the corresponding system of three differential equations has the characteristic polynomial with a multiple root (cf. [29]).

In order to better understand the obtained results, let us write down the system of differential equations for the versal integrals (20). Using the presentation [26],it is possible again to compute the coefficient matrix Θ of the Gauss-Manin system (4) for the versal integrals:

$$\begin{pmatrix} (\frac{2}{3}q-\frac{1}{2})\omega_1+(\frac{3}{2}-4q)x\omega_3 & (4q-2)\omega_2 & (3-12q)\omega_3 \\ \frac{1}{2}x\omega_2+3(q-\frac{3}{4})y\omega_3 & (\frac{2}{3}q-\frac{1}{3})\omega_1+(2q-1)x\omega_3 & (4q-1)\omega_2 \\ -qy\omega_2+3z\omega_3 & (1-2q)x\omega_2+3(q-\frac{1}{2})y\omega_3 & (\frac{2}{3}q-\frac{1}{6})\omega_1+(2q-\frac{1}{2})x\omega_3 \end{pmatrix}$$

where q is a $\mathbf{C}$-valued parameter, $\omega_1,\ \omega_2,\ \omega_3$ are the logarithmic differential forms defined as (17). The characteristic equation is

$$(\lambda-q+1)^2(\lambda+\frac{1}{2}q-\frac{1}{4})=0,$$

so that, in the above notations, $q=(k+1)/2$.

In conclusion of this section, using the matrix Θ with $q=\lambda+1$ from the above presentation for the system of versal integrals, we shall slightly correct the following assertion from [5], (3.4), which may be considered as an analogue of K.Saito's results [29], relating to the computation of the *uniformization equations.*

In the notations of [5], let $F(z,s)=z^4+s_2z^2+s_1z+s_0=0$ be the minimal versal deformation of an A_3-singularity, and let

$$K_\ell^\lambda(s)=\int_{\gamma(t)} z^\ell(z^4+s_2z^2+s_1z+s_0)^\lambda dz, \qquad \ell=0,1,2, \quad \lambda\in\mathbf{C},$$

be the period integrals defined analogously to (20) (see [26]).

Proposition 6.1. *The period integrals*

$$\mathbf{K}_{A_3}(s)=(K_0^\lambda(s),K_1^\lambda(s),K_2^\lambda(s)), \quad s=(s_0,s_1,s_2),$$

associated with an A_3-singularity satisfy the following logarithmic Picard-Fuchs system of differential equations:

$$d\,\mathbf{K}_{A_3}(s)=\Big(\Omega_3\cdot(L+\frac{1}{8}V(s_2))-\frac{1}{4}V(ds_2)\Big)\cdot\mathbf{K}_{A_3}(s), \tag{21}$$

where

$$L \;=\; \mathrm{diag}\,(\lambda+\frac{1}{4},\lambda+\frac{1}{2},\lambda+\frac{3}{4}),$$

$$
\begin{aligned}
V(s_2) &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_2 & 0 & 0 \end{bmatrix}, \\
\Omega_3 &= \frac{1}{\Delta_3}\begin{bmatrix} \frac{1}{3}d\Delta_3 - \frac{1}{3}s_2\eta_3 & -\frac{1}{4}\eta_2 & -\eta_3 \\ \frac{1}{4}s_1\eta_3 & \frac{1}{3}d\Delta_3 + \frac{1}{6}s_2\eta_3 & -\frac{1}{4}\eta_2 \\ \frac{1}{16}s_1\eta_2 & \frac{1}{8}s_2\eta_2 + \frac{1}{4}s_1\eta_3 & \frac{1}{3}d\Delta_3 + \frac{1}{6}s_2\eta_3 \end{bmatrix},
\end{aligned}
$$

$$
\begin{aligned}
\Delta_3 &= 16s_0^3 - \frac{27}{16}s_1^4 + 9s_0s_1^2s_2 - 8s_0^2s_2^2 - \frac{1}{4}s_1^2s_2^3 + s_0s_2^4, \\
\eta_2 &= (4s_1s_2^2 + 48s_0s_1)ds_0 + (16s_0s_2^2 - 64s_0^2 - 6s_1^2s_2)ds_1 + (9s_1^3 - 32s_0s_1s_2)ds_2, \\
\eta_3 &= -(9s_1^2 - 8s_0s_2 + 2s_2^3)ds_0 + s_1(12s_0 + s_2^2)ds_1 - (16s_0^2 + \frac{3}{2}s_1^2s_2 - 4s_0s_2^2)ds_2.
\end{aligned}
$$

Remark 6.2. It is possible to verify that the characteristic polynomials of the system (21) and the system of Type I described above coincide. Hence these systems are equivalent, that is, there is a linear transformation induced by an invertible matrix $\Gamma \in \mathrm{GL}\,(3, \mathbf{C})$ (see [12]) such that

$$\widetilde{\Omega} = d\,\Gamma \cdot \Gamma^{-1} + \Gamma \cdot \Omega \cdot \Gamma^{-1}.$$

We also remark that the system (21) has been computed in [5] by differentiation of a set of the explicitly given period integrals, making use of some basic properties of logarithmic forms from the paper [1], while the matrix ∇ has been obtained as a consequence of the standard integrability condition $d\nabla = \nabla \wedge \nabla$.

A simple comparison of the formula (21) with K.Saito's result shows that the weight λ in the former and the parameter k in the latter are related in the following way:

$$\lambda = \frac{1}{2}(k-1).$$

We see that the solutions of the "uniformization equation" of Type I from the paper [29], §3, for every value of the parameter k can be obtained by means of the formula (6) from [5], making use of period integrals of type A_2 (that is, of the Legendre function). It should be noted that K.Saito has resolved the uniformization equation only in the case when $k = 0$ (or, equivalently, for $\lambda = -1/2$ in our notation). In addition, it follows from our calculations that the system of Type II (loc. cit.), whose solutions "do not have an Euler integral representation", arises in an essentially different context than the Gauss-Manin systems associated with an A_3-singularity.

7. The case of a space curve singularity

The following example is closely related to the material of the previous section. Let us consider the germ of the space curve X_0 which is a bouquet of a line and an A_2-singularity. Its defining equations look as follows:

$$\left\{\begin{array}{rcl} x^3 - y^2 & = & 0 \\ xz & = & 0 \\ yz & = & 0. \end{array}\right. \tag{22}$$

The minimal versal deformation $f\colon (X, 0) \to (S, 0)$ of this germ can be described as follows. The base of the deformation is $S \simeq (\mathbf{C}^4, 0)$, and the total deformation space $X \subset \mathbf{C}^3 \times S$ is defined by the three equations

$$\left\{\begin{array}{rcl} x^3 - y^2 + t_2x^2 + t_4x + t_3y & = & 0 \\ xz + t_1y - t_1t_3 & = & 0 \\ yz + t_1x^2 + t_1t_2x + t_1t_4 & = & 0. \end{array}\right. \tag{23}$$

One can verify (see [6]) that the discriminant D of the minimal versal deformation is *non-reduced,* and its reduced part $D_{\rm red}$ consists of two irreducible components. More precisely, $D_{\rm red}$ is given by the equation $t_1P = 0$ where

$$P = 64t_4^3 - 16t_2^2t_4^2 - 72t_2t_3^2t_4 + 16t_2^3t_3^2 + 27t_3^4.$$

An easy calculation shows that $D_{\rm red}$ is a *free* Saito divisor. In fact, it is a rather general property of space-curve singularities (see [33]). Moreover, the non-smooth irreducible component of D given by the polynomial P is also a *free* divisor. On the other hand, the discriminant of the minimal versal deformation of an A_3-singularity is defined by the famous polynomial in three variables t_2, t_3, t_4 of weights $2, 3, 4$, respectively:

$$h = 256t_4^3 - 128t_2^2t_4^2 + 16t_2^4t_4 + 144t_2t_3^2t_4 - 4t_2^3t_3^2 - 27t_3^4 = 0. \tag{24}$$

It is also a free divisor. It is often called the swallow tail. Both hypersurfaces are non-normal since their singular loci have codimension 1. One can prove that there are *only two* polynomials of degree 12 in 3 variables of weights $2, 3, 4$ that define free divisors and are not equivalent under a weighted coordinate transformation. For example, the polynomial P can be transformed to the following ones:

$$P_1 = 64t_4^3 + 48t_2^2t_4^2 - 72t_2t_3^2t_4 + 27t_3^4,$$

$$P_2 = t_4^3 - 18t_2t_3^2t_4 - 3t_2^4t_4 + 18t_2^3t_3^2 + 27t_3^4 + 2t_2^6$$

but not to the polynomial h in (24). For completeness, let us mention here that the polynomial P_2 has been found by M.Sato, in an essentially different context [32].

It is not difficult to verify that

$$\mathrm{rank}_{S'}\mathcal{H}' = \mathrm{rank}_S\mathcal{H} = \mu(X_0) = 3,$$

while $\mathrm{rank}_S\Omega^1_S(\log D_{\mathrm{red}}) = \dim S = 4$.

Suppose $t_1 \neq 0$. Choose a point $t = (t_1, \ldots, t_4) \notin D_{\mathrm{red}}$, and make the substitution $z \mapsto zt_1$:

$$\begin{cases} x^3 - y^2 + t_2x^2 + t_4x + t_3y &= 0 \\ xz + y - t_3 &= 0 \\ yz + x^2 + t_2x + t_4 &= 0. \end{cases}$$

It can be easily seen that the left hand side of the first equation coincides with the left hand side of the third equation multiplied by x, if one lets $y = t_3 - xz$ (from the second equation). Hence the fiber X_t over such a point is isomorphic to the complete intersection given by the second and the third equations:

$$\begin{cases} xz + y - t_3 &= 0 \\ x^2 - xz^2 + t_2x + t_3z + t_4 &= 0, \end{cases} \tag{25}$$

or equivalently, X_t is contained in the 3-parameter versal deformation of the quasi-homogeneous complete intersection space curve singularity:

$$\begin{cases} xz + y &= 0 \\ x^2 - xz^2 &= 0. \end{cases} \tag{26}$$

The latter germ is just the union of the z-axis and the twisted cubic:

$$\begin{cases} x &= s^2 \\ y &= -s^3 \\ z &= s. \end{cases}$$

Hence the fiber X_t given by the system (25) is isomorphic to the plane singularity:

$$z^4 - 2t_2z^2 - 4t_3z - 4t_4 - 4t_2x - 4x^2 = 0,$$

which is equivalent to the minimal versal deformation of an A_3-singularity, because after the following substitutions

$$t_2 = -2t_2,\ t_3 = -4t_3,\ t_4 = -4t_4,\ x = 2x,\ x = x + t_2/2,\ t_4 = t_4 + t_2^2/4$$

this equation reduces to the form:

$$z^4 + t_2z^2 + t_3z + t_4 - x^2 = 0.$$

Thus $\mathcal{H}^1(f_*\Omega^\bullet_{X'/S'})$ is a free module of rank 3 (although $\tau = \dim S = 4$), its generators being defined by the period integrals similar to those arising in the case of an A_3-singularity. For calculations, one can choose the same integrable differential forms while the contour of integration will differ in the case of an A_3-singularity because the corresponding discriminant sets are different. Let us now explain how to compute the Gauss-Manin connexion and how to solve the problem of the extension of the corresponding system of differential equations associated with the versal deformation of the space curve singularity (23) to the whole base S.

Consider the logarithmic connexion (10):

$$\nabla_{X/S}\colon\ \Omega^1_S(\log D) \longrightarrow \Omega^1_S(\log D) \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)$$

for the discriminant of the versal deformation. Then $S \cong \mathbf{C}^4$, the discriminant $D \subset S$ is defined by the equation $t_1 P = 0$, where

$$P = 64t_4^3 - 16t_2^2t_4^2 - 72t_2t_3^2t_4 + 16t_2^3t_3^2 + 27t_3^4.$$

A basis of the module of logarithmic along P vector fields consists of the following elements:

$$V^1 = \frac{1}{6}\Big(2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + 4t_4\frac{\partial}{\partial t_4}\Big),$$

$$V^2 = 3t_3\frac{\partial}{\partial t_2} + 2t_4\frac{\partial}{\partial t_3} + 2t_2t_3\frac{\partial}{\partial t_4},$$

$$V^3 = (\frac{4}{3}t_2^2 - 8t_4)\frac{\partial}{\partial t_2} - 2t_2t_3\frac{\partial}{\partial t_3} - (3t_3^2 + \frac{4}{3}t_2t_4)\frac{\partial}{\partial t_4}.$$

The corresponding coefficient matrix has the form:

$$\mathcal{V} = \begin{pmatrix} \frac{1}{3}t_2 & \frac{1}{2}t_3 & \frac{2}{3}t_4 \\ 3t_3 & 2t_4 & 2t_2t_3 \\ \frac{4}{3}t_2^2 - 8t_4 & -2t_2t_3 & -3t_3^2 - \frac{4}{3}t_2t_4 \end{pmatrix},$$

and $\det(\mathcal{V}) = 2^{-1}3^{-1}P$. The $\mathcal{O}_S$-module of logarithmic differential forms $\Omega^1_S(\log D)$ is also free, the dual basis, with respect to the vector fields V^1, V^2, V^3, is defined by the 2×2-minors of the matrix $\mathcal{V}$:

$$\omega_1 = \frac{1}{2}\frac{dP}{P},$$

$$\omega_2 = \frac{1}{P}\Big((9t_3^3 - 4t_2t_3t_4)d\,t_2 + (32t_4^2 - 6t_2t_3^2 - 8t_2^2t_4)d\,t_3 + (8t_2^2t_3 - 24t_3t_4)d\,t_4\Big),$$

$$\omega_3 = \frac{1}{P}\Big((6t_2t_3^2 - 8t_4^2)d\,t_2 + (12t_3t_4 - 4t_2^2t_3)d\,t_3 + (4t_2t_4 - 9t_3^2)d\,t_4\Big).$$

One can verify the following identities:

$$d\,\omega_1 = 0,\ d\,\omega_2 = -\frac{1}{6}\omega_1\wedge\omega_2 + \frac{2}{3}t_2\omega_2\wedge\omega_3,\ d\,\omega_3 = -\frac{1}{3}\omega_1\wedge\omega_3;$$

$$\omega_1\wedge\omega_2\wedge\omega_3 = 6\cdot\frac{dt_2\wedge dt_3\wedge dt_4}{P}.$$

We have $\iota_{V^i}(\omega_j) = \delta_{ij}$. Let us define a connexion on $\Omega^1_S(\log D)$. Since the weights of the variables t_2, t_3, t_4 induce the canonical grading on the modules of logarithmic differential forms and vector fields, it is easy to see that this connexion can be given by the matrix:

$$\nabla = \begin{pmatrix} g\omega_1 + at_2\omega_3 & m\omega_2 & n\omega_3 \\ dt_2\omega_2 + et_3\omega_3 & (g+\frac{1}{6})\omega_1 + kt_2\omega_3 & f\omega_2 \\ (bt_2^2 + ct_4)\omega_3 + ht_3\omega_2 & it_2\omega_2 + jt_3\omega_3 & (g+\frac{1}{3})\omega_1 + lt_2\omega_2 \end{pmatrix},$$

where $a, b, c, d, e, f, g, h, i, j, k, l, m, n$ are the fourteen $\mathbf{C}$-valued parameters. The matrix ∇ satisfies the following integrability condition:

$$d\,\nabla = \nabla\wedge\nabla.$$

As it has been already noted in the preceding section,

$$\sum_{i=1}^{3} V^i(f)\omega_i = \sum_{i=1}^{3}\frac{\partial f}{\partial t_i}dt_i \quad \text{for any } f\in\mathcal{O}^3_{\mathbf{C}}.$$

Hence the columns of the matrix $\mathcal{V}$ define the relations between the total differentials of the variables t_i and the basic logarithmic forms ω_i:

$$dt_2 = \frac{1}{3}t_2\omega_1 + 3t_3\omega_2 + (\frac{4}{3}t_2^2 - 8t_4)\omega_3,$$
$$dt_3 = \frac{1}{2}t_3\omega_1 + 2t_4\omega_2 - 2t_2t_3\omega_3,$$
$$dt_4 = \frac{2}{3}t_4\omega_1 + 2t_2t_3\omega_2 - (3t_3^2 + \frac{4}{3}t_2t_4)\omega_3.$$

The integrability condition gives us a system of 10 equations with 13 parameters:

$$\begin{cases} 3a - em + hn & = & 0 \\ -2/3m + mk - in - am & = & 0 \\ 2/3d - kd + bf + ad & = & 0 \\ 8d + 2e - cf & = & 0 \\ 3k + em - fj & = & 0 \\ -2/3f - fk + dn + fl & = & 0 \\ 3l - hn + fj & = & 0 \\ 2/3i - bm + ki - il & = & 0 \\ 8i + 2j + cm & = & 0 \\ -8/3h - 2c - 6b + ei - dj - hl + ah & = & 0. \end{cases}$$

These equations, which do not depend on g, when combined with the equations (1), (5), (7) imply the condition

$$a + k + l = 0.$$

The two pairs (2), (8) and (3), (6) of homogeneous equations in variables i, m and d, f, respectively, have the same Jacobian determinants. It is not difficult to see that one can replace the two equations (3), (8) by the determinant. Eliminating the parameters a, e, k, l, j, b we obtain the standard basis of the corresponding ideal

$$\begin{cases} cfm^2 + \frac{8}{3}fim - \frac{16}{3}dm^2 - \frac{2}{3}hmn + 2in + \frac{4}{3}m & = 0 \\ cf^2m + \frac{16}{3}f^2i - \frac{8}{3}dfm + \frac{2}{3}fhn + 2dn - \frac{4}{3}f & = 0 \\ f^2i^2 + 2dfim + d^2m^2 + \frac{3}{16}cfin + \frac{1}{2}fhin + \frac{3}{16}cdmn & \\ +\frac{1}{2}dhmn - \frac{5}{4}fi - \frac{5}{4}dm - \frac{3}{16}cn - \frac{1}{2}hn + \frac{1}{4} & = 0. \end{cases}$$

These three equations in 7 variables define a 4-dimensional space of solutions. Each of the solutions depends on the *free* parameter g.

One can analyze the system (4)

$$d\mathbf{I} = \Theta \cdot \mathbf{I}$$

as in the previous section. The calculations show, however, that in contrast to the example of an A_3-singularity considered in the previous section, only the second case (#2) in (18) occurs. Thus the characteristic polynomial $Q(\lambda)$ has 2 distinct roots, one of them of multiplicity two.

As a result, all the logarithmic connexions belong to the only connected component of the parameter space. They are equivalent, for example, to the following connexion:

$$a = 0,\ b = i = k - \frac{2}{3},\ c = 2,\ d = f = \frac{4}{3}k,\ e = h = -\frac{9}{4}k,$$

$$g = \frac{1}{2}k - \frac{1}{6},\ j = -6k + 2,\ l = -k,\ m = n = 2(k + \frac{1}{3}),$$

which gives us the local Picard-Fuchs system and the Gauss-Manin connexion associated with a bouquet of a line and the A_2-singularity with the characteristic polynomial

$$(\lambda - \frac{1}{2}k)^2(\lambda + \frac{1}{4}k).$$

References

[1] Aleksandrov A.G.: *Nonisolated hypersurface singularities.* In: Theory of singularities and its applications. Advances in Soviet Mathematics (Arnold V.I., ed.) **1**, 211-246. Amer. Math. Soc., Providence, RI, 1990.

[2] Aleksandrov A.G.: *Gauss-Manin connexion of curve singularities.* In: Partial differential and integral equations (H.Begehr et al.,.eds). Proceedings of the international congress of the International Society for Analysis, Applications and Computing, ISAAC '97. Dordrecht: Kluwer Academic Publishers. Int. Soc. Anal. Appl. Comput. **2** (1999), 1-22.

[3] Aleksandrov A.G.: *Generalized Picard-Fuchs systems in several complex variables.* Journal of Shanghai Jiaotong University **E-3** (1998), Sup. (I), 1-9.

[4] Aleksandrov A.G., Tanabé S.: *Computing Gauss-Manin systems for complete intersection singularities* S_μ, Georgian Math. J. **3** (1996), no. 5, 401-422.

[5] Aleksandrov A.G., Tanabé S.: *Gauss-Manin connexions, logarithmic forms and hypergeometric functions.* In: Geometry from the Pacific Rim (edited by J.Berrick et al.), 1-21. Walter de Gruyter, Berlin – New York, 1997.

[6] Aleksandrov A.G., Martin B.: *Gauss-Manin connexion associated with bouquets of curve singularities,* Preprint Reihe Mathematik M-05/1998, Brandenburgische Technische Universitat Cottbus, Germany, 1998, 1-24.

[7] Anosov D.V., Bolibruch A.A.: *The Riemann–Hilbert problem.* Aspects of Mathematics **E22**. Braunschweig; Wiesbaden: Vieweg, 1994.

[8] Aomoto K.: *Configurations and invariant Gauss-Manin connections of integrals I,* Tokyo J.Math. **5** (1982), 249-287

[9] Bolibruch A.A.: *On the fundamental matrix of a Pfaffian system of Fuchsian type.* Izv. AN SSSR, ser. matem. **41** (1977), no. 5, 1084-1109 (Russian).

[10] Brieskorn E.: *Die Monodromie der isolierten Singularitäten von Hyperflächen,* Manuscripta Math. **2** (1970), 103-161.

[11] Deligne P.: *Equations différentieles à points singuliers réguliers,* Lecture Notes in Mathematics **163**, Springer-Verlag, 1970.

[12] Gantmakher F.R.: *The theory of matrices.* Nauka, Moscow, 1967 (Russian).

[13] Gelfand I.M., Kapranov M.M., Zelevinski A.V.: *Generalized Euler integrals and A-hypergeometric functions,* Adv. in Math. **84** (1990), no.2, 255-271.

[14] Gérard R., Levelt A.H.M.: *Étude d'une classe particulière de système de Pfaff du type de Fuchs sur l'espace projectif complexe,* J. Math. Pures et Appl. **51** (1972), 189-217.

[15] Golubeva V.A.: *Fuchsian systems of differential equations on the complex projective space.* Diff. Uravneniya **13** (1977), no. 9, 1570-1580 (Russian).

[16] Greuel G.-M.: *Der Gauß-Manin-Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten,* Math. Ann. **214** (1975), no.3, 235-266.

[17] Hamm H.A.: *Zur analytischen und algebraishen Beschchreibung der Picard-Lefschetz-Monodromie*, Habilitationschrift. Göttingen, 1974.

[18] Helminck G.F.: *Integrable deformations of meromorphic equations on* $\mathbf{P^1(C)}$. In: Deformation theory and quantum groups with applications to mathematical physics, Contemp. Math. **134** (1992), 119-128.

[19] Ishiura S., Noumi M.: *A calculus of the Gauss-Manin system of type* A_ℓ*, I, II,* Proc. Japan Acad. Ser. A Math. Sci. **58** (1982), 13-16, 62-65.

[20] Leksin V.P.: *Meromorphic Pfaffian systems on complex projective spaces.* Matem. Sbornik, Nov. Ser. **129** (1986), no. 2, 201-217 (Russian).

[21] Looijenga E.: *Isolated singular points of complete intersections,* London Math. Soc. Lecture Notes Ser. **77**, 1984.

[22] Malikov H.M.: *Redefinition of the differential systems for versal integrals of type A, D, E,* Diff. Uravneniya **18** (1982), no.8, 1394-1400 (Russian).

[23] Manin Yu.I.: *Rational points on algebraic curves over function fields,* Izv. Akad. Nauk SSSR Ser. Mat. **27** (1963), 1395-1440 (Russian).

[24] Namba M.: *A class of differential equations of Fuchsian type.* Tôhoku Math. Jour. **39** (1987), 391-406.

[25] Noumi M.: *Expansion of the solutions of a Gauss-Manin system at a point of infinity,* Tokyo J. Math. **7** (1984), 1-60.

[26] Palamodov V.P.: *Deformations of complex spaces.* Current problems in mathematics. Fundamental directions **10**, 123-221, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1986 (Russian).

[27] PALAMODOV V.P.: *Special functions of several variables.* In: Linear topological spaces and complex analysis III (Aytuna A. et al., eds.) Papers dedicated to Prof. V.P.Zaharjuta on the occasion of his 60th birthday. Ankara: Scientific and Technical Research Council of Turkey (TUeBITAK), 1997, 120-137.

[28] PHAM F.: *Singularitiés des systèmes différentiels de Gauss-Manin,* Progress in Mathematics **2,** Birkhäuser Verlag, Basel, 1979.

[29] SAITO K.: *On the uniformization of complements of discriminant loci.* In: Hyperfunctions and Linear partial differential equations, RIMS Kōkyūroku **287** (1977), 117-137.

[30] SAITO K.: *Theory of logarithmic differential forms.* J. Fac. Sci. Univ. Tokyo, ser. IA **27** (1980), no. 2, 265-291.

[31] SAITO K.: *Regularity of Gauss-Manin connection of flat family of isolated singularities.* Introduction à la théorie des singularités, I. Singularités et monodromie. Travaux en Cours **36** (1988), 135-149. Hermann, Paris, 1988.

[32] SEKIGUCHI J.: *Some topics related with discriminant polynomials,* In: Algebraic analysis and number theory, RIMS Kōkyūroku **810** (1992), 85-94.

[33] VAN STRATEN D.: *A note on the discriminant of a space curve,* Manuscripta Math. **87** (1995), 167-177.

[34] TAKANO K.: *A reduction theorem for a linear Pfaffian system with regular singular points.* Arch. Math. **31** (1978), 310-316.

[35] VARCHENKO A.N.: *The Euler beta-function, the Vandermonde determinant, Legendre's equation, and critical values of linear functions on a configuration of hyperplanes. I,* Izv. Akad. Nauk SSSR, Ser. matem. **53** (1989), no.6, 1206-1235 (Russian).

[36] VASILEVICH N.D., GROMAK V.I.: *Integrability of complex Pfaffian equations.* Diff. Uravneniya **17** (1981), no. 4, 735-737 (Russian).

[37] YOSHIDA M., TAKANO K.: *On a linear system of Pfaffian equations with regular singular points.* Funkcial. Ekvac. **19** (1976), no. 2, 175-189.

Current address:

Max-Planck-Institut für Mathematik
Vivatsgasse 7
P.O.Box: 7280
D-53072, Bonn
Germany
E-mail: alexandr@mpim-bonn.mpg.de

Institute of Control Sciences
Russian Academy of Sciences
Profsojunaja ul. 65
GSP-7, Moscow, 117806
Russian Federation
E-mail: alexandr@ipu.rssi.ru