Height Inequality of Algebraic Points on Curves over Functional Fields

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Introduction

In this paper, we shall give a linear and effective height inequality for algebraic points on curves over functional fields.

Let $f : S \longrightarrow C$ be a fibration of a smooth complex projective surface S over a curve C, and denote by g the genus of a general fiber of f. We assume that $g \geq 2$ and S is relatively minimal with respect to f, i.e., S has no (-1)-curves contained in a fiber of f. Let k be the functional field of C, and $\bar{k}$ its algebraic closure. For an algebraic point $P \in S(\bar{k})$, we let E_P be the corresponding horizontal curve on S. The geometric canonical height $h_K(P)$ and the geometric logarithmic discriminant $d(P)$ are defined as follows.

$$h_K(P) = \frac{K_{S/C}E_P}{[k(P):k]}, \qquad d(P) = \frac{2g(\widetilde{E}_P) - 2}{[k(P):k]},$$

where $\widetilde{E}_P$ is the normalization of E_P, and $[k(P);k] = FE_P$ is the degree of P. It is a fundamental problem to give an effective bound of height by the geometric discriminant. Up to now, many height inequalities have been obtained.

Szpiro,	$h_K(P) \leq 8 \cdot 3^{3g+1}(g-1)^2(d(P)/3^g + s + 1 + 1/3^{3g})$,
Vojta,	$h_K(P) \leq (8g-6)/3\ d(P) + O(1)$,
Parshin,	$h_K(P) \leq (20g-15)/6\ d(P) + O(1)$,
Esnault-Viehweg,	$h_K(P) < 2(2g-1)^2\ (d(P) + s)$,
Vojta,	$h_K(P) \leq (2+\epsilon)\ d(P) + O(1)$,
Moriwaki,	$h_K(P) \leq (2g-1)\ d(P) + O(1)$,

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where s is the number of singular fibers of f. These inequalities can be found respectively in [Sz], [Vo1], [Pa], [EV], [Vo2] and [Mo]. It is a problem to get an inequality linear in g with explicit $O(1)$. (cf. Lang's comments on this problem, [La], p.153). The purpose of this paper is to give such an inequality.

Theorem A. *Let $f : S \longrightarrow C$ be a non-trivial fibration of genus $g \geq 2$ with s singular fibers, and $P \in S(\bar{k})$ an algebraic point. If f is semistable, then*

$$h_K(P) \leq (2g-1)(d(P)+s) - K^2_{S/C},$$

and the equality holds only if f is smooth, i.e., $s = 0$.

If f is non-semistable, then

$$h_K(P) < (2g-1)(d(P)+3s) - K^2_{S/C}.$$

If we compare it with the canonical inequality, the term $3s$ in the second inequality seems to be natural. Vojta obtains a canonical class inequality for semistable fibrations:

$$K^2_{S/C} \leq (2g-2)(2g(C)-2+s).$$

Furthermore, we have shown that if the equality holds, then f is smooth (cf. [Ta2], Remark 3.6). In [Ta1], in a quite natural way, we generalized Vojta's inequality to the non-semistable case:

$$K^2_{S/C} < (2g-2)(2g(C)-2+3s).$$

The first step of the proof is to obtain the first inequality in Theorem A for rational points P, by using Miyaoka-Yau inequality. The ideal is motivated by Xiao's proof of Manin's Theorem (i.e., Modell conjecture over functional fields), (cf. [Xi], Corollary to Theorem 6.2.7). Then by using Kodaira-Parshin's trick, we can obtain the height inequality for the semistable case. The final step is the detailed study of the invariants of semistable reductions. Because the first step uses Miyaoka-Yau inequality, the proof is unlikely to translate into number fields case.

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1 Preliminaries

Let $f : S \longrightarrow C$ be a fibration of genus $g \geq 2$, let $F_1, \cdots, F_s$ be the singular fibers of f, and let $B = \sum_{i=1}^{s} F_i$. First of all, we consider the embedded resolution of the singularities of B_{red}. We denote by $K^2_{S/C}$, $\chi_f = \deg f_*\omega_{S/C}$ and $e_f = \sum_F(\chi_{\text{top}}(F) - (2-2g))$ the standard relative invariants of f.

Definition 1.1. The *embedded resolution* of the singularities of B is a sequence

$$(S,B) = (S_0,B_0) \overset{\sigma_1}{\leftarrow} (S_1,B_1) \overset{\sigma_2}{\leftarrow} \cdots \overset{\sigma_r}{\leftarrow} (S_r,B_r) = (S',B')$$

satisfying the following conditions.

1) σ_i is the blowing-up of S_{i-1} at a singular point $p_{i-1} \in B_{i-1,\mathrm{red}}$, which is not an ordinary double point.

2) $B_{r,\mathrm{red}}$ has at worst ordinary double points as its singularities.

3) B_i is the total transformation of B_{i-1}.

It is well-known that embedded resolution exists and is unique. We denote respectively by m_i and $\bar{m}_i$ the multiplicities of $(B_{i,\mathrm{red}}, p_i)$ and $(\bar{B}_{i,\mathrm{red}}, p_i)$, where $\bar{B}_{i,\mathrm{red}}$ is the strict transform of B_{red} in S_i. Then it is obvious that

$$\bar{m}_i \geq m_i - 2. \tag{1}$$

Now we let $\pi : \widetilde{C} \longrightarrow C$ be a base change of degree d. Let S_1 be the normalization of $S \times_C \widetilde{C}$. We can resolve the singularities of S_1 by using embedded resolution of B. It goes as follows.

$$\begin{array}{ccccc}
S_2 & \xrightarrow{\eta} & S_1' & \xrightarrow{\pi_r} & S' \\
{\scriptstyle \rho_2}\downarrow & & {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \sigma} \\
S_1 & = & S_1 & \xrightarrow[\rho_1]{} & S
\end{array}$$

where S_1' is the normalization of $S_1 \times_S S'$ (hence it is also the normalization of $S' \times_C \widetilde{C}$), and S_2 is the minimal resolution of the singularities of S_1'. All of the morphisms are induced naturally. So S_2 is also a resolution of S_1. We shall call such a ρ_2 the *embedded resolution* of the singularities of S_1.

Let $f_2 : S_2 \longrightarrow \widetilde{C}$ be the induced fibration, $\widetilde{\rho} : S_2 \longrightarrow \widetilde{S}$ the contraction of the (-1)-curves contained in the fibers of f_2. Then we have an induced fibration $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$, which is relatively minimal and is determined uniquely by f and π. We shall call $\widetilde{f}$ the *pullback fibration* of f under the base change π.

$$\begin{array}{ccccccc}
\widetilde{S} & \xleftarrow{\widetilde{\rho}} & S_2 & \xrightarrow{\rho_2} & S_1 & \xrightarrow{\rho_1} & S \\
\downarrow{\scriptstyle \widetilde{f}} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\
\widetilde{C} & = & \widetilde{C} & = & \widetilde{C} & \xrightarrow[\pi]{} & C
\end{array}$$

Let $\Pi_2 = \rho_1 \circ \rho_2 : S_2 \longrightarrow S$.

If $\widetilde{f}$ is semistable, then we say that π is a *semistable reduction* of f. We shall use Kodaira-Parshin's construction to construct some semistable reductions π.

Lemma 1.2. *There exist some semistable reductions $\pi : \widetilde{C} \longrightarrow C$ of f such that*

1) π is ramified uniformly over the s critic points of f, and the ramification index of any ramified point is exactly e.

2) e is divided by all of the multiplicities of the components of $\sigma^* B$, and it can be arbitrarily large.

In fact, a base change satisfying the above two conditions must be a semistable reduction. If $b = g(C) > 0$, then the existence follows from Kodaira-Parshin's construction. If $b = 0$ and f is non-trivial, then $s \geq 3$ (cf. [Be]). Hence we can construct a base change totally ramified over the s points. Then the existence is reduced to the case $b > 0$.

In Definition 1.1, we denote by $\mathcal{E}_i$ the total inverse image of the exceptional curve of σ_i in S'.

Lemma 1.3. *Let π be the semistable reduction constructed in Lemma 1.2. Then we have*

$$\widetilde{\rho}^* K_{\widetilde{S}/\widetilde{C}} = \Pi_2^* K_{S/C} - \Pi_2^* \left(\sum_{i=1}^{s} (F_i - F_{i,\mathrm{red}}) \right) + K_{\rho_2} - D'', \tag{2}$$

where $D'' = K_{S_2/\widetilde{C}} - \widetilde{\rho}^* K_{\widetilde{S}/\widetilde{C}}$ is an effective divisor supported on the exceptional set of $\widetilde{\rho}$, and K_{ρ_2} is the canonical rational divisor of the resolution ρ_2, i.e.,

$$-K_{\rho_2} = \eta^* \pi_r^* \left(\sum_{i=1}^{r} (m_{i-1} - 2) \mathcal{E}_i \right). \tag{3}$$

We refer to ([Ta1], §2.1 and §5) for the proof of this lemma. We only need to note that in this case, η is the resolution of rational double points of type A_n, so $K_\eta = 0$.

In [Ta1], for each (singular) fiber F of f, we associate to it three nonnegative rational numbers $c_1^2(F)$, $c_2(F)$ and χ_F.

Definition 1.4. Let $\pi : \widetilde{C} \longrightarrow C$ be a base change of degree d ramified over $f(F)$ and some non-critic points. If the fibers of $\widetilde{f}$ over F are semistable, then we define

$$c_1^2(F) = K_{S/C}^2 - \frac{1}{d} K_{\widetilde{S}/\widetilde{C}}, \quad c_2(F) = e_f - \frac{1}{d} e_{\widetilde{f}}, \quad \chi_F = \chi_f - \frac{1}{d} \chi_{\widetilde{f}}.$$

These three invariants are independent of the choice of π, and can be computed by embedded resolution of F. One of them is zero iff F is semistable. Let

$$I_K(f) = K_{S/C}^2 - \sum_F c_1^2(F), \quad I_\chi(f) = \chi_f - \sum_F \chi_F, \quad I_e(f) = e_f - \sum_F c_2(F).$$

where F runs over the singular fibers of f. Then $I_K(f)$, $I_\chi(f)$ and $I_e(f)$ are nonnegative invariants of f, and one of the first two invariants vanishes if and only if f is isotrivial, i.e., all of the nonsingular fibers are isomorphic. Note that if f is semistable, then these three invariants are nothing but the standard relative invariants of f.

Lemma 1.5. ([Ta1], Theorem A) *If $\widetilde{f}$ is the pullback fibration of f under a base change of degree d, then we have*

$$I_K(\widetilde{f}) = dI_K(f), \quad I_\chi(\widetilde{f}) = dI_\chi(f), \quad I_e(\widetilde{f}) = dI_e(f).$$

For later use, in what follows, we consider the computation of $c_1^2(F)$. For this, we have to introduce an invariant $c_{-1}(F)$ of F. In fact, we only need to note that if π is the semistable reduction as in Lemma 1.2, then we have

$$c_{-1}(F) = \frac{1}{\deg \pi} \#\{\text{curves over } F \text{ contracted by } \widetilde{\rho}\}.$$

Then we have (cf. [Ta1], Theorem 3.1)

$$c_1^2(F) = 4(g - p_a(F_{\mathrm{red}})) + F_{\mathrm{red}}^2 + \sum_{p \in F} \alpha_p - c_{-1}(F).$$

where $\alpha_p = \sum_i (m_i - 2)^2$, m_i come from the embedded resolution of the singular point (F, p). In fact, we have proved that

$$\sum_{p \in F} \alpha_p \leq 2p_a(F_{\rm red}),$$

with equality if and only if $p_a(F_{\rm red}) = 0$, i.e., F is a tree of nonsingular rational curves. (cf. [Ta1], Lemma 3.2). Hence we have

Lemma 1.6. *If F is a singular fiber of f, then*

$$c_1^2(F) + c_{-1}(F) \leq 4g - 3,$$

and if $p_a(F_{\rm red}) > 0$, then

$$c_1^2(F) + c_{-1}(F) \leq 4g - 4.$$

2 The proof of Theorem A for semistable curves

First of all, we give some notations. Let $f : S \longrightarrow C$ be a semistable fibration. We denote by $f^{\#} : S^{\#} \longrightarrow C$ the corresponding stable model, and by q a singular point of $S^{\#}$. Then q is a rational double point. Let μ_q be the Milnor number of $(S^{\#}, q)$, i.e., the number of (-2)-curves in the exceptional set E_q of the minimal resolution of q. Note that $\mu_q = 0$ means that q is a singular point of a fiber on the smooth part of $S^{\#}$.

Theorem 2.1. *If $f : S \longrightarrow C$ is non-trivial and semistable, and $P \in S(\bar{k})$ is an algebraic point, then*

$$h_K(P) \leq (2g - 1)(d(P) + s) - K_{S/C}^2,$$

and if the equality holds, then f is smooth.

Proof. *Case* I. P is a k rational point. Let E be the corresponding section of f.

If $b = g(C) > 0$, then we know

$$K_S \sim K_{S/C} + (2b - 2)F$$

is nef. Now we want to use Miyaoka's inequality ([Mi], Corollary 1.3). If $q \in E$, i.e., $E_q \cap E = x$, and E_x is the (-2)-curve in E_q passing through x, then

$$E_q - E_x = E_{q'} + E_{q''}.$$

In this case, we replace q by q' and q''. Note that $m(E_q) = 3(\mu_q + 1) - 3/(\mu_q + 1)$ (cf. [Hi]), and $\mu_q = \mu_{q'} + \mu_{q''} + 1$, hence

$$\varepsilon_q := m(E_q) - m(E_{q'}) - m(E_{q''}) = \frac{3}{\mu_{q'} + 1} + \frac{3}{\mu_{q''} + 1} - \frac{3}{\mu_q + 1}.$$

Then by using Miyaoka's inequality to E and

$$\{E_q \mid q \notin E\} \cup \{E_{q'}, E_{q''} \mid q \in E\},$$

we have

$$\sum_q m(E_q) + 3\chi_{\mathrm{top}}(E) \le 3c_2(S) - (K_S + E)^2 + \varepsilon \tag{4}$$

where $\varepsilon = \sum_{q \in E} \varepsilon_q$. Since $\sum_q (\mu_q + 1) = e_f$, and $h_K(P) = -E^2$, (4) implies that

$$h_K(P) \le \sum_q \frac{3}{\mu_q + 1} + (2g-1)(2b-2) - K^2_{S/C} + \varepsilon. \tag{5}$$

Now we consider the base change $\pi : \widetilde{C} \longrightarrow C$ constructed in Lemma 1.2. Let $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ be the pullback fibration of f, $\widetilde{P}$ the corresponding rational point of $\widetilde{f}$. It is easy to see that the corresponding objects of $\widetilde{f}$ satisfy

$$K^2_{\widetilde{S}/\widetilde{C}} = dK^2_{S/C}, \;\; \widetilde{s} = \frac{d}{e}s, \;\; \mu_{\widetilde{q}} + 1 = e(\mu_q + 1), \;\; \widetilde{\varepsilon} = \frac{d}{e^2}\varepsilon,$$
$$2g(\widetilde{C}) - 2 = d(2b-2) + d\left(1 - \frac{1}{e}\right)s, \;\; h_K(\widetilde{P}) = dh_K(P).$$

Applying (5) to $\widetilde{f}$, we have

$$dh_K(P) \le \frac{d}{e^2}\sum_q \frac{3}{\mu_q + 1} + (2g-1)\left((2b-2)d + d\left(1 - \frac{1}{e}\right)s\right) - dK^2_{S/C} + \frac{d}{e^2}\varepsilon,$$

i.e.,

$$h_K(P) - (2g-1)(d(P) + s) + K^2_{S/C} \le -\frac{(2g-1)s}{e} + \frac{1}{e^2}\left(\sum_q \frac{3}{\mu_q + 1} + \varepsilon\right).$$

Let e be large enough we can see that the lefthand side ≤ 0, or < 0 if $s > 0$.

Now we consider the case $b = 0$. Since f is non-trivial, we have $s \ge 5$ [Ta2]. Then we consider also the base change as given in Lemma 1.2. Since $g(\widetilde{C}) > 0$, so the height inequality for $\widetilde{P}$ holds, which implies the inequality for P.

Case II. P is an algebraic point of degree d_P. Let E_P be the corresponding reduced and irreducible horizontal curve on S, $\widetilde{C}$ the normalization of E_P, and $\pi : \widetilde{C} \longrightarrow C$ the morphism induced by f. Let $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$ be the pullback of f under π. Since f is semistable, we know that $\widetilde{\rho}$ is an isomorphism and

$$K_{\widetilde{S}/\widetilde{C}} = \Pi_2^*(K_{S/C}), \quad K^2_{\widetilde{S}/\widetilde{C}} = d_P K^2_{S/C}.$$

By the construction of $\widetilde{f} : \widetilde{S} \longrightarrow \widetilde{C}$, there is a section $\widetilde{E}$ of $\widetilde{f}$ such that $\Pi_{2*}(\widetilde{E}) = E_P$. Hence

$$\begin{aligned} h_K(P) &= \frac{1}{d_P} E_P K_{S/C} \\ &= \frac{1}{d_P} \widetilde{E} \cdot \Pi_2^*(K_{S/C}) \\ &= \frac{1}{d_P} \widetilde{E} K_{\widetilde{S}/\widetilde{C}} \\ &\leq (2g-1)\left(\frac{2g(\widetilde{C}) - 2}{d_P} + \frac{\widetilde{s}}{d_P} \right) - \frac{1}{d_P} K^2_{\widetilde{S}/\widetilde{C}} \\ &\leq (2g-1)(d(P) + s) - K^2_{S/C}. \end{aligned}$$

If $s > 0$, then the strict inequality holds. **Q.E.D.**

3 The proof of Theorem A for non-semistable curves

Let $f : S \longrightarrow C$ be a non-semistable fibration with s singular fibers. Let P be an algebraic point of degree d_P. We shall prove in this section that

$$h_K(P) < (2g-1)(d(P) + 3s) - K^2_{S/C}. \tag{6}$$

We let $\pi : \widetilde{C} \longrightarrow C$ be the semistable reduction of f as constructed in Lemma 1.2. If E_P is the corresponding horizontal curve on S, then we denote respectively by E_2 and $\widetilde{E}$ the strict transforms of E_P in S_2 and $\widetilde{S}$. Hence

$$\Pi_{2*}(E_2) = dE_P, \quad \widetilde{\rho}_*(E_2) = \widetilde{E}, \tag{7}$$

where $d = \deg \pi$.

Let C_P be the normalization of E_P, $\pi_P : C_P \longrightarrow C$ the morphism induced by f, and $f_P : S_P \longrightarrow C_P$ the pullback fibration of f under π_P. By the construction of f_P, there is a section of f_P whose image in S is E_P.

Now by considering the normalization of one component of the fiber product of C_P and $\widetilde{C}$ over C, we can obtain a curve $\hat{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} C_P & \xleftarrow{\ \psi\ } & \hat{C} \\ {\scriptstyle \pi_P}\downarrow & & \downarrow{\scriptstyle \phi} \\ C & \xleftarrow[\ \pi\ ]{} & \widetilde{C} \end{array}$$

Let $\hat{f} : \hat{S} \longrightarrow \hat{C}$ be the pullback fibration of $\widetilde{f}$ under ϕ. By the uniqueness of the relative minimal model (since $g > 0$) and the universal property of fiber product, we know that $\hat{f}$ is nothing but the pullback of f_P under ψ. Hence $\hat{f}$ has a section $\hat{E}$, which is induced by the above mentioned section of f_P. Therefore, we know that the image of $\hat{E}$ in $\widetilde{S}$ coincides with $\widetilde{E}$. Denote respectively by $\hat{p}$ and $\widetilde{P}$ the corresponding points of $\hat{E}$ and $\widetilde{E}$. Since $\widetilde{f}$ is semistable, by abusing notations, we have

$$K_{\hat{S}/\hat{C}} = \phi^* K_{\widetilde{S}/\widetilde{C}}, \quad \phi_* \hat{E} = \widetilde{E},$$

then from Lemma 1.3, $\widetilde{\rho}^* K_{\widetilde{S}/\widetilde{C}} = \Pi_2^* K_{S/C} - D_\pi$, hence we obtain

$$\begin{aligned} h_K(\hat{P}) &= K_{\hat{S}/\hat{C}}\hat{E} = \phi^* K_{\widetilde{S}/\widetilde{C}}\hat{E} \\ &= K_{\widetilde{S}/\widetilde{C}}\widetilde{E} = \widetilde{\rho}^* K_{\widetilde{S}/\widetilde{C}} E_2 \\ &= (\Pi_2^* K_{S/C} - D_\pi)E_2 \\ &= dK_{S/C}E_P - D_\pi E_2 \\ &= dd_P h_K(P) - D_\pi E_2, \end{aligned}$$

thus we have

$$h_K(P) = \frac{1}{dd_P} h(\hat{P}) + \frac{1}{dd_P} D_\pi E_2. \tag{8}$$

Note that

$$\frac{\deg\psi}{d} = \frac{\deg\phi}{d_P} \leq 1. \tag{9}$$

Lemma 3.1.

$$\frac{1}{dd_P} h(\hat{P}) \leq (2g-1)(d(P)+s) - I_K(f).$$

Proof. Since $\hat{f}$ is semistable, by Theorem 2.1, we have

$$\frac{1}{dd_P} h(\hat{P}) \leq (2g-1)\left(\frac{2g(\widetilde{C})-2}{dd_P} + \frac{\hat{s}}{dd_P}\right) - \frac{1}{dd_P} K_{\hat{S}/\hat{C}}^2, \tag{10}$$

where $\hat{s}$ is the number of singular fibers of $\hat{f}$. It is obvious that

$$\hat{s} \leq \frac{ds}{e} \deg\phi. \tag{11}$$

By Lemma 1.5, we have

$$\frac{1}{dd_P} K_{\hat{S}/\hat{C}}^2 = \frac{\deg\phi}{d_P} I_K(f). \tag{12}$$

By Hurwitz formula,

$$2g(\widetilde{C}) - 2 = \deg\psi(2g(C_P)-2) + r_\psi.$$

Then note that the ramification index of π at any ramified point is e, by the construction of ψ we can see that the index of ψ at any ramified point is at most e. Hence it is easy to know that the contribution of the ramified points of ψ over one branched point to $r_\psi/\deg\psi$ is at most $1 - 1/e$. Thus

$$\frac{r_\psi}{\deg\psi} \leq d_P \frac{r_\pi}{d},$$

it implies that

$$\begin{aligned}\frac{2g(\hat{C})-2}{dd_P} &\leq \frac{\deg\psi}{d}d(P)+\frac{\deg\phi}{d_P}\left(1-\frac{1}{e}\right)s \\ &= \frac{\deg\phi}{d_P}\left(d(P)+\left(1-\frac{1}{e}\right)s\right)\end{aligned} \tag{13}$$

Combining (9)–(13), we have

$$\begin{aligned}\frac{1}{dd_P}h(\hat{P}) &\leq \frac{\deg\phi}{d_P}((2g-1)(d(P)+s)-I_K(f)) \\ &\leq (2g-1)(d(P)+s)-I_K(f).\end{aligned}$$

Q.E.D.

Now we shall find the upper bound of $\frac{1}{dd_P}D_\pi E_2$. Note first that

$$D_\pi = \Pi_2^*\left(\sum_{i=1}^{s}(F_i - F_{i,\mathrm{red}})\right) - K_{\rho_2} + D''.$$

Since $\Pi_{2*}E_2 = dE_P$, and $E_P(F_i - F_{i,\mathrm{red}}) < d_P$, by project formula we have

Lemma 3.2.

$$\frac{1}{dd_P}\Pi_2^*\left(\sum_{i=1}^{s}(F_i - F_{i,\mathrm{red}})\right)E_2 < s.$$

Lemma 3.3.

$$-K_{\rho_2}E_2 \leq s - \#\{F_i \mid p_a(F_{i,\mathrm{red}})=0\}.$$

Proof. Since $p_a(F_{i,\mathrm{red}})=0$ implies that F_i is a tree of non-singular rational curves, it has no effect on $-K_{\rho_2}$ and $-K_{\rho_2}E_2$. For simplicity, we assume that $p_a(F_{i,\mathrm{red}}) \neq 0$ for all i.

By considering the embedded resolution, we let

$$\sigma^*E = \bar{E} + \sum_{i=1}^{r} a_{i-1}\mathcal{E}_i.$$

where $\bar{E}$ is the strict transform of E and $a_i \geq 0$ is the multiplicity of the strict transform of E at p_i. We have know that $\eta_*E_2 = \pi_r^*(\bar{E})$, and

$$-K_{\rho_2} = \eta^*\pi_r^*\left(\sum_{i=1}^{r}(m_{i-1}-2)\mathcal{E}_i,\right)$$

hence

$$\begin{aligned}-K_{\rho_2}E_2 &= \pi_r^*\left(\sum_{i=1}^{r}(m_{i-1}-2)\mathcal{E}_i\right)\eta_*E_2 \\ &= d\sum_{i=1}^{r}(m_{i-1}-2)\mathcal{E}_i\bar{E} \\ &= d\sum_{i=1}^{r}(m_{i-1}-2)a_{i-1}\end{aligned}$$

On the other hand,

$$\sigma^*\left(\sum_{i=1}^{s} F_i\right) = \sum_{i=1}^{s} \bar{F}_i + \sum_{i=1}^{r} \bar{m}_{i-1}\mathcal{E}_i,$$

where $\bar{F}_i$ is the strict transform of F_i, and $\bar{m}_{i-1}$ is the multiplicity of the strict transform of $\sum_i F_i$ at p_i. From $\sum_{i=1}^{s} \bar{F}_i\bar{E} \geq 0$, we have

$$\sum_{i=1}^{r} a_{i-1}\bar{m}_{i-1} \leq \sum_{i=1}^{s} F_i E = sd_P,$$

then from (1),

$$-K_{\rho_2}E_2 \leq sdd_P.$$

This completes the proof. **Q.E.D.**

Lemma 3.4.

$$\frac{D''E_2}{dd_P} \leq \sum_{i=1}^{s} c_{-1}(F_i).$$

Proof. Since $D'' = K_{S_2/\widetilde{C}} - \widetilde{\rho}^* K_{S/C}$, by induction on the number of the blowing-downs, we can see that the contribution of a curve in D'' to $D''E_2$ is at most d_P. On the other hand, the number of curves contracted by $\widetilde{\rho}$ is $d\sum_{i=1}^{s} c_{-1}(F_i)$. Hence we have the desired inequality. **Q.E.D.**

Proof of (6)

From the above lemmas, we have

$$\begin{aligned} h_K(P) <&(2g-1)(d(P)+s) - K_{S/C}^2 + \sum_{i=1}^{s}\left(c_1^2(F_i) + c_{-1}(F_i)\right) \\ &- \#\{F_i \mid p_a(F_{i,\mathrm{red}}) = 0\} + 2s. \end{aligned}$$

By Lemma 1.6,

$$\sum_{i=1}^{s}\left(c_1^2(F_i) + c_{-1}(F_i)\right) \leq (4g-4)s + \#\left\{F_i \mid p_a(F_{i,\mathrm{red}}) = 0\right\}.$$

Hence we have

$$h_K(P) < (2g-1)(d(P)+3s) - K_{S/C}^2.$$

Q.E.D.

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